5th Zeuthen Workshop on Elementary Particle Theory

Loops and Legs in Quantum Field Theory

Bastei/Königstein, Germany
Sunday, April 09 - Friday, April 14

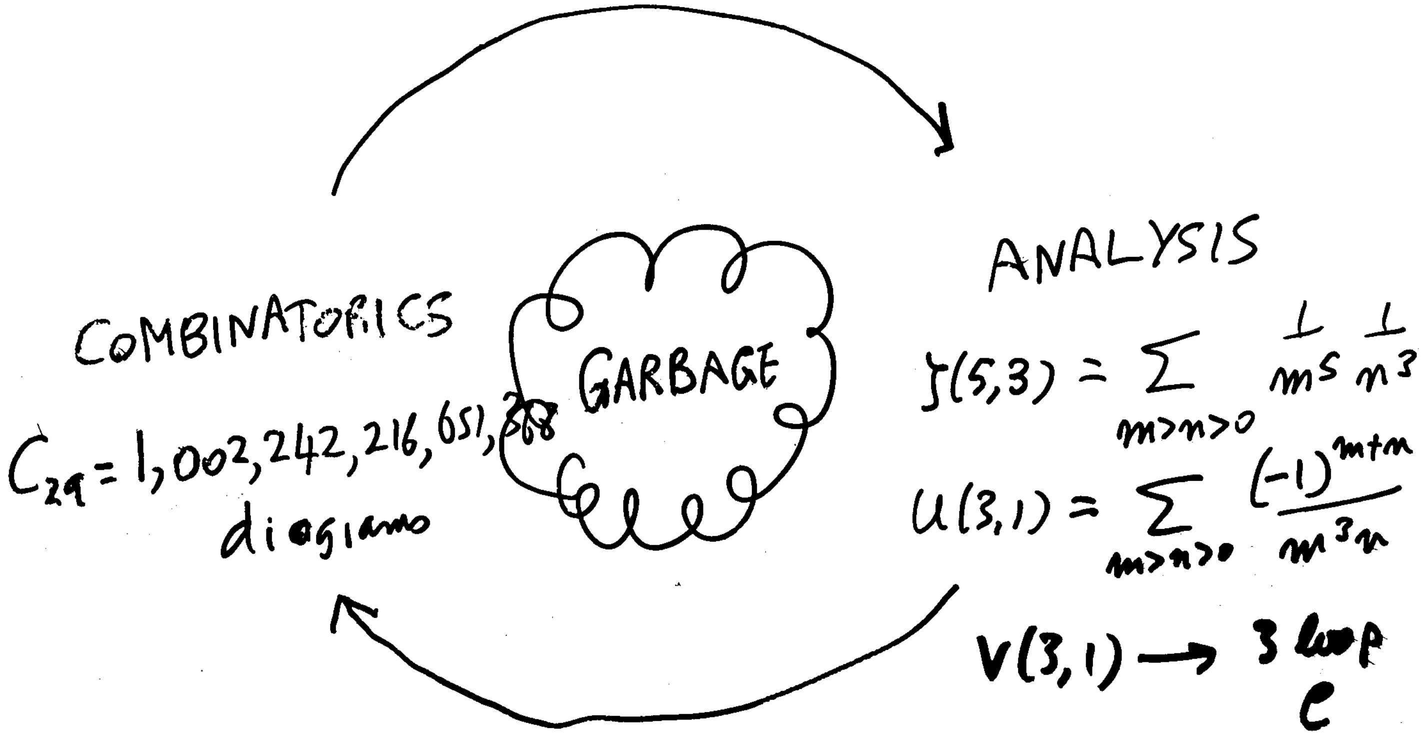
D. Broadhurst

Interplay of Combinatorics with Analysis in Quantum Field Theory

4.

INTERPLAY OF COMBINATORICS AND ANALYSIS IN Q.F.T.

DAVID BROADHURST d.broadhurst@open.ac.uk



COMBINATORICS

$C_{29} = 1,002,242,216,651,368$ diagrams

ANALYSIS

$$\zeta(5,3) = \sum_{m>n>0} \frac{1}{m^5}\frac{1}{n^3}$$

$$U(3,1) = \sum_{m>n>0} \frac{(-1)^{m+n}}{m^3 n}$$

$$V(3,1) \longrightarrow 3 \text{ loop } e$$

SQUEEZING THE PRECISE GARBAGE

PETERMANN 1957 / REMIDDI 2007 | 1. RECURRENCE / D.E. } ⇒ HYPERGEOM → POLYLOGS

DK 2. HOW TO RESUM 463,020,146,037,416,130,934 BPHZ subtractions

DK+BBBB 3. NAMING / COUNTING } THE TRANSCENPTAL ENEMY

ALK CJM 4. HYPERGEOM ⇒ LARGE ORDER BEHAVIOUR IN $H \to \bar{b}b$, m_s extraction

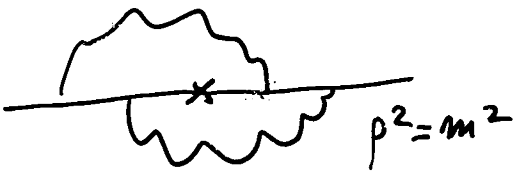
[hep-th / 9803091, 9805025, 9806174, 9810087, 9811173, 9909105, 9912098, 0001202 ..]

Konstantin Chetyrkin $\geqslant 5$
Ettore Remiddi HYPE[R]GEOM
Adrian Ghinculov GARBAGE SQUEEZING
Günter Quast TOOL

1. RECURRENCE $\sim$ D.E. $\Longrightarrow$ HYPEGEOM $\Longrightarrow$ POLYLOGS

KARPLUS+KROLL
g-2

PETERMANN 1957
SOMMERFELD 1958



WRONG!

RIGHT!

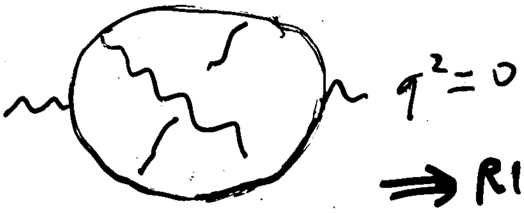
$\pi^2 \log 2 - \frac{3}{2}\zeta(3)$

GRAY, BROADHURST, GRAFE, SCHILCHER
1990

$$\frac{\bar{m}(M)}{M} = 1 + \frac{4}{3}\frac{\alpha_s(M)}{\pi} + \left[\quad\right]\frac{\alpha_s^2}{\pi^2}$$

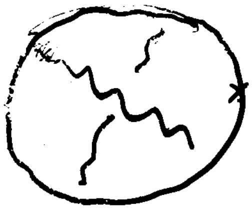
+ TIMO
+ROBIN

DJB 1992
on-shell 3-loop QED

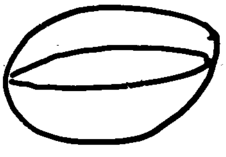


$\Longrightarrow$ RIWIAD 25σ
VEGAS 6σ

RECURRENCE
RELATIONS
LIKE (g-2) 2-LOOP



$\xrightarrow{\text{IBP}}$



$\longrightarrow \int [F_{21}]^2$

$\downarrow$

Saalschutz
Hardy
Bailey
Slater

$F_{32} \longleftarrow F_{43} \longleftarrow \int F_{32}$

Maths is $\displaystyle\sum_n \frac{(\frac{1}{2}+a\varepsilon)_n(\frac{1}{2}+b\varepsilon)_n}{(\frac{3}{2}+c\varepsilon)_n(\frac{3}{2}+d\varepsilon)_n}$ $D=4-2\varepsilon$

$\longrightarrow$ alternating Euler sums $\longrightarrow$ Fibonacci enumeration

1.1 LEONARDO DA PISA
(a.k.a FIBONACCI)

1 $\quad 1$

JS $\quad 1 \quad \underline{\log 2}$

2 $\quad \underline{\pi^2}, \log^2 2$

PETERMANN $\quad 3 \quad \underline{\zeta(3)}, \pi^2 \log 2, \log^3 2$

5 $\quad \underline{U_{3,1}}, \zeta(3)\log 2, \pi^4, \pi^2\log^2 2, \log^4 2$

L+R $\quad 8 \quad X_{3,1,1}, Y_{3,1,1}, \zeta(5), \pi^4 \log 2, \pi^2\zeta(3), \pi^2\log^3 2, \log^2 2\,\zeta(3), \log 2\, U_{3,1}$

(DHB+DJB)

weight=11 $\quad F_{12} = 144$

tested at 3,000 digits

$$U_{3,1} \equiv \sum_{m>n>0} \frac{(-1)^{m+n}}{m^3 n}$$

$$\Rightarrow \mathrm{Li}_4(\tfrac{1}{2}), \pi^4, \pi^2\log^2 2, \log^4 2$$

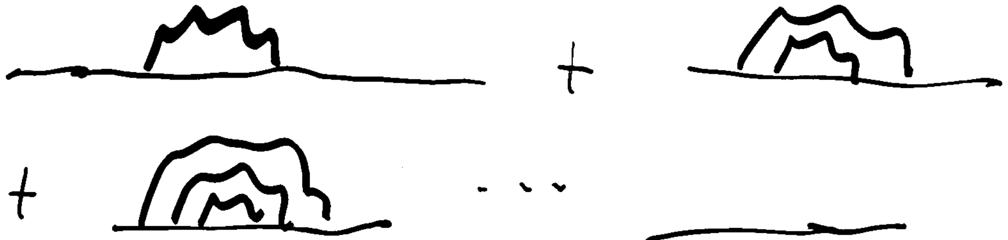
$$x \Rightarrow \frac{1-x}{1+x} \Rightarrow U_{4,2} = \frac{97}{96}\zeta(6) - \frac{3}{4}[\zeta(3)]^2$$

DJB 1994 > LE1777

$$\frac{1}{2}(g-2)_e \ni -\Big[\underset{m}{\frac{50}{3}} U_{3,1} + \underset{m}{\frac{13}{8}}\zeta(4)\Big]_{L+R}\left(\frac{e}{2\pi}\right)^6$$

2 RESUMMING 463,020,146,037,416,130,934 BPHZ SUBTRACTIONS OF 1,002,242,216,651,368 DIAGRAMS At 30 loops

ϕ^3_6

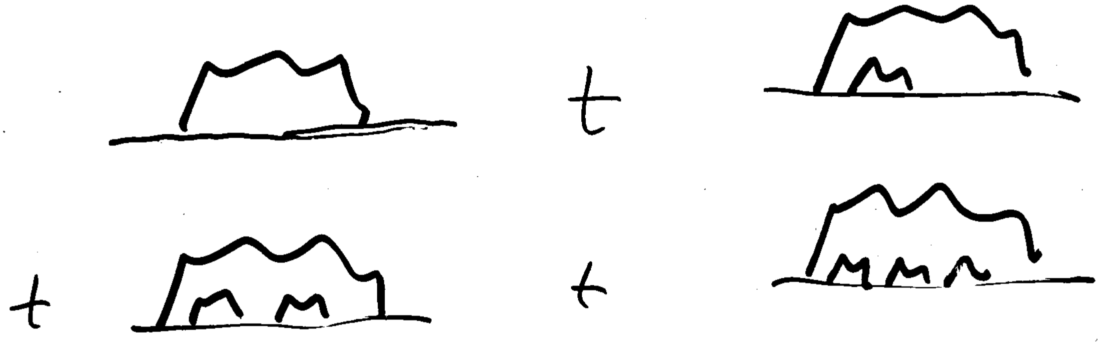


$$\gamma_2^{RAINBOW} = 3-\frac{\sqrt{5+4\sqrt{1+a}}}{2}$$

$$= -\frac{a}{6} + 11\frac{a^2}{6^3} - \underset{\sim\sim}{206}\,\frac{a^3}{6^5} \ldots,$$

$$a \equiv (g/4\pi)^2$$

[YUKAWA $1-\sqrt{1+a}$]



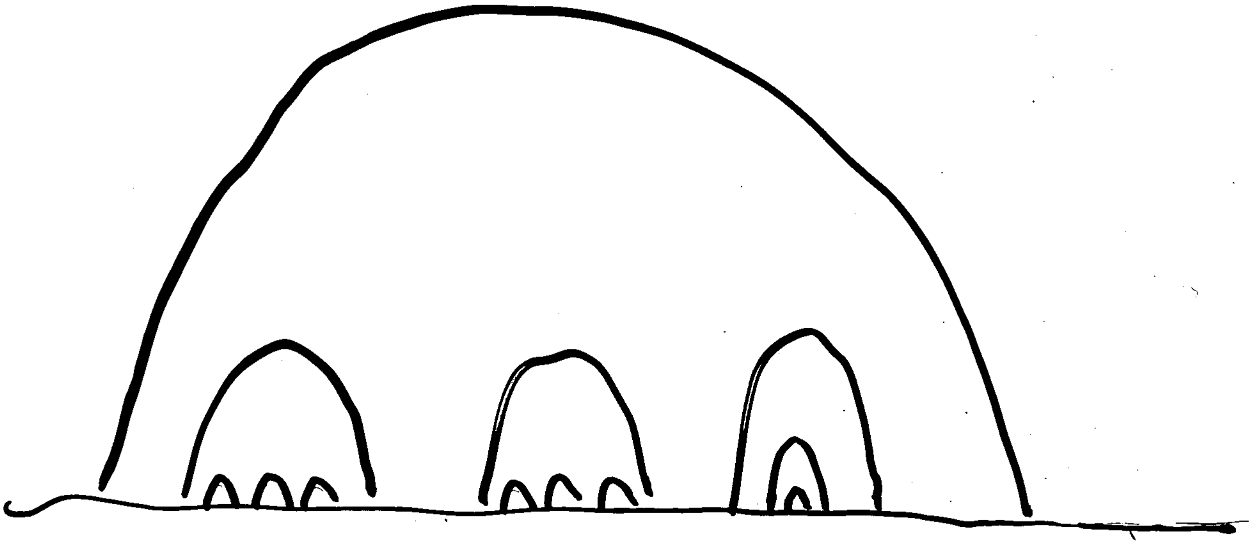
$$\gamma_2^{CHAIN} = -6\int_0^\infty \frac{\exp(-6x/a)dx}{(x+1)(x+2)(x+3)}$$

$$\simeq -\frac{a}{6} + 11\frac{a^2}{6^3} - \underset{\sim\sim}{170}\,\frac{a^3}{6^5} + \ldots$$

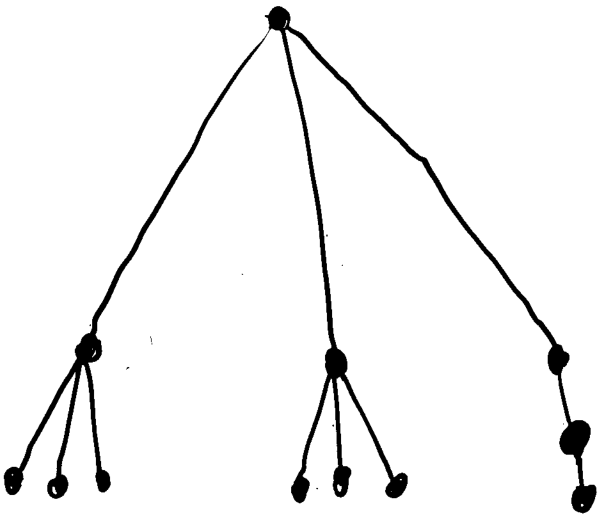
2.1 ANALYSIS G functions

$$G(a,b) = g(a)g(b)g(d-a-b)$$

$$g(a) = \Gamma(\tfrac{d}{2}-a)/\Gamma(a)$$



2.2



2.3

$$\gamma = \lim_{\varepsilon \to 0} \varepsilon S * Y$$

antipode

contraction of coproduct Δ

grading operator

$$\Delta : H \to H \otimes H$$

JOBLIST

extract subdivergences

subdivergences shrunken

$$S : H \to H$$

MOM $p^2 = \mu^2$

$$S * id = id * S = 0$$

$$\Gamma_R : H \to H$$

$$\Gamma_R = S_R * id \Big|_{\varepsilon = 0}$$

t Hooft-Veltman twisted antipode → R.H. CK

GT

2.4

Do it to 12 loops
using 4 lines of REDUCE
(J. Symb. Comp.)

0 loops would require handling 10^{24} bits of data by systematic BPHZ.

Smart Hopf recursion can be done in 750 MB and a few hours

$$\gamma_2^{\mathrm{HOPF}} = \underset{\curvearrowleft}{\overset{\curvearrowright}{\mathrm{CHAIN} + \mathrm{RAINBOW}}} \simeq \sum_n G_n \frac{(-a)^n}{6^{2n-1}}$$

$$G_{27} = 2^6 \times 5 \times 103 \times 184892457645068836717 \times 699431048506216812683294469624581$$

has 21 & **32 digit primes**

Best factorizer takes longer ~~than~~ 10^{20} BPHZ subtraction

NO Γ's !!!

2.5.

Padé–Borel resummation

Yukawa $\quad G_n \sim 2^n\,\Gamma(n+\tfrac{1}{2})$

$\phi^3_6 \quad G_n \sim 12^n\,\Gamma(n+2)$

empirically!

YUKAWA:

$$\gamma_2^{\mathrm{HOPF}} \approx -\frac{a}{\sqrt{2}}\int_0^\infty Q(ax/2)\,e^{-x}x^{\frac{1}{2}}\,dx$$

$$Q(y) = \frac{1}{1+y}\,\frac{P_{14}(y)}{\tilde{P}_{14}(y)} \quad [14\backslash 14] \text{ Pade}$$

reproduces 30 loops to 1 part in 10^{17}

at $g = 30 \qquad a = \left(\frac{g}{4\pi}\right)^2 \approx 5.7$

$\gamma_2 \approx -1.85202761$ [29 loops] $(14\backslash 14]$
$ -1.85202762$ [31 loops] $(15\backslash 15)$

PLB $\underline{470}$ (99) 168 for comparison with chains + rainbows

3 NAMING / COUNTING } THE TRANSCEDENTAL { ENEMIES / FRIENDS

MZV (QCD)	Euler sums (QED)	MCV (e param)	MGV
1	1	0	1
0	1	1	0
1	2	1 $Cl_2(\frac{\pi}{3})$	1 π^2
1 $\zeta(3)$	3	1	2
1 $\zeta(4)$	5 $u(3,1)$	2	3
2	8	4	4
2	13	7	6
3	21	11	10
4 $\boxed{\zeta(5,3)}$ $\zeta(5)\zeta(3)$ $(\zeta(3))^2\zeta(2)$ $\zeta(8)$	34	17	17
PADOVAN	FIB.	TWISTED	FIB
$\frac{dx}{x}$ $\frac{dx}{1-x}$	$\frac{dx}{x}$ $\frac{dx}{1-x}$ $\frac{dx}{1+x}$	$\frac{dx}{x}$ $\frac{dx}{1-x}$	$\frac{dx}{\lambda - x}$

$$\lambda = exp(2\pi i/6)$$

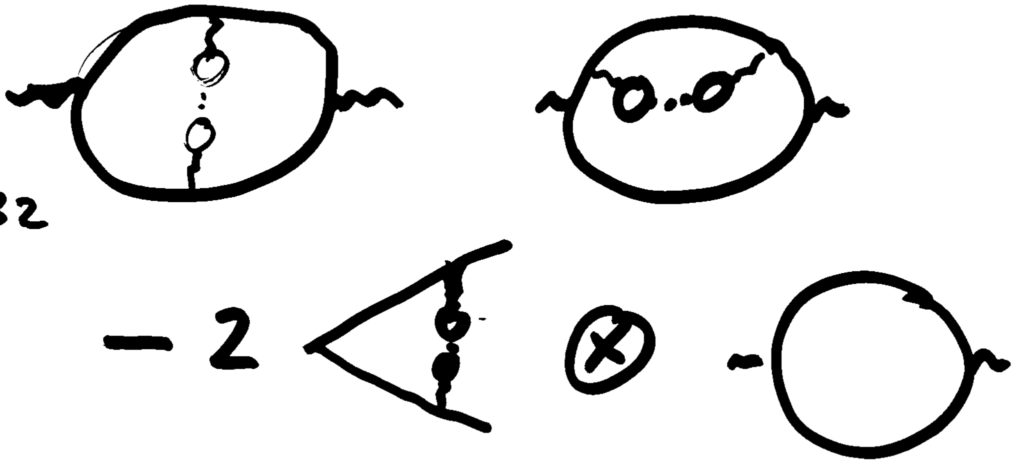
4. SCALAR CORRELATOR

$H \to \bar{b}b$ at $O(10^2 \text{GeV})$

m_s extraction at $O(1 \text{GeV})$

at large β_0 [Naive nonabelianiztion $N_f \to N_f - 33/2$]

$\frac{1}{\varepsilon^n} F_{32}$



to all orders

8 loops:

$$68850\zeta(9) - \frac{1323}{32}\zeta(8) + \frac{3967083}{56}\zeta(7) \ldots$$

11 terms

<u>NO</u> $\zeta(5,3)$

4 loops $N_f = 5$

$$\frac{\text{NNA}}{\text{CK}} = 1.08$$

$\downarrow$

renormalon growth

See hep-th soon for crucial difference between scalar + vector

Bastei noble prize LL2000
hereby awarded to

Tiny Veltman
Gerhart 't Hooft

Citation
- - - - - - - -
- - - - - - - - } see proceedings
- - - - - - - -

$$+ \quad \int \frac{d^D k}{k^2} = 0 \qquad \text{'t Hooft} + \text{Veltman}$$

⇒ no quad. divergences
⇒ no hierarchy "problem"
(cannot "fine tune" 0 or ∞)
⇒ no need for SUSY

SI MONUMENTUM
REQUIRIS —
CIRCUMSPICE!

if you want a monument —
read the proceedings