NNNLO correction to the toponium and bottomonium wave-functions at the origin *

M. Beneke[1], Y. Kiyo[2], A. Penin[3] and K. Schuller[1]

1- Institut für Theoretische Physik E, RWTH Aachen,
D-52056 Aachen, Germany
2- Institut für Theoretische Teilchenphysik, Universität Karlsruhe,
D-76128 Karlsruhe, Germany
3- Department of Physics, University Of Alberta,
Edmonton, AB T6G 2J1, Canada

We report new results on the NNNLO correction to the S-wave quarkonium wave-functions at the origin, which also provide an estimate of the resonance cross section in $t\bar{t}$ threshold production at the ILC.

1 Introduction

Top quark pair production near threshold will be an important process at the ILC to determine the top quark mass m_t, decay width Γ_t and the QCD coupling constant α_s. High precision is called for for these quantities, so the theoretical uncertainty for the cross section should be under control below the few percent level. For this purpose, the NNNLO QCD calculation of the cross section is mandatory.

Recently we computed the NNNLO correction [1, 2] to the quarkonium wave-functions at the origin, which governs height of the threshold cross section. In this proceedings we present an analysis of the combined result of the papers [1, 2]. For the details of the calculation we refer to the original papers.

The production cross section of a heavy quark pair $Q\bar{Q}$ is related to the two-point function of the vector current j^μ in QCD:

$$\left(q^\mu q^\nu - g^{\mu\nu} q^2\right) \Pi(q^2) = i \int d^d x e^{iqx} \langle\Omega|T\, j^\mu(x) j^\nu(0)|\Omega\rangle, \tag{1}$$

where $j^\mu = \bar{Q}\gamma^\mu Q$, $q^\mu \equiv (2m + E, \vec{0})$ in the center of mass frame of the $Q\bar{Q}$, and $d = 4 - 2\epsilon$. Near the $Q\bar{Q}$ threshold, the two-point function exhibits the bound-state contribution

$$\Pi(q^2) \overset{E\to E_n}{=} \frac{N_c}{2m^2} \frac{Z_n}{E_n - (E + i\,0)} + \text{non-pole}, \tag{2}$$

where E_n is energy of n-th resonance (n is principal quantum number of the quarkonium state, $i\,0$ specifies the physical sheet in the analytic continuation). The poles dominate the two-point function, therefore Z_n and E_n control the height and the pole position, respectively, of the threshold cross section.

The heavy quark threshold dynamics is non-relativistic (NR), so we utilize an effective field theory, non-relativistic QCD (NRQCD) for the quark (ψ) and anti-quark (χ). In

*Talk given by Y. Kiyo. SFB/CPP-07-68, TTP07-29, PITHA07/14.

NRQCD the vector current is mapped onto

$$j^i \;=\; c_v \psi^\dagger \sigma^i \chi + \frac{d_v}{6m^2}\psi^\dagger \sigma^i \mathbf{D}^2 \chi + \cdots\,, \tag{3}$$

where c_v, d_v are matching coefficients, having perturbative series expansion in α_s. Thus the two-point function reduces to the one in NRQCD, whose bound-state contribution is expressed by the quarkonium wave-function at the origin, $\psi_n(0)$,

$$i\int d^dx e^{iEt}\langle\Omega|T\,[\psi^\dagger\sigma^i\chi](x)[\chi^\dagger\sigma^i\psi](0)|\Omega\rangle \overset{E\to E_n}{=} 2N_c(d-1)\frac{|\psi_n(0)|^2}{E_n-(E+i\,0)} + \text{non-pole}. \tag{4}$$

The pre-factor $2N_c(d-1)$ is due to spin$\otimes$color$\otimes$space degrees of freedom. The relation between the residues of the QCD and NRQCD two-point functions is given by

$$Z_n \;=\; c_v\left[c_v - \frac{E_n}{m}\left(1+\frac{d_v}{3}\right)+\cdots\right]\times|\psi_n(0)|^2, \tag{5}$$

where the $\mathbf{D}^2$ term in eq.(3) was replaced by $-mE$ using the equations of motion of the NRQCD fields. The wave-function as well as the matching coefficients possess scale dependence because of their UV and IR divergences characteristic to the effective theory calculations which we treat according to the threshold expansion [3]. The physical quantity measured in experiments is Z_n, a scale-invariant combination of the matching coefficients and the NR wave-function. In the next section we present semi-analytical formulae for all the building blocks needed to get Z_1, and discuss the importance of the NNNLO correction for stabilizing the perturbative result for the quarkonium wave-functions at the origin against scale variation.

2 NNNLO corrections to the wave-function at the origin

The wave-function at the origin to NNNLO consists of the Coulomb contribution, the non-Coulomb potential contribution, and the ultra-soft correction in NRQCD. The Coulomb contribution is finite and calculated analytically in [4, 5]. The non-Coulomb [1] and ultra-soft [2] computations require regularization and renormalization prescriptions, so that they are scheme-dependent quantities. We computed them with conventional dimensional regularization and divergences are renormalized in $\overline{\text{MS}}$ scheme. Combining all corrections we obtain the following numerical formula for the ground-state wave-function:

$$\begin{aligned}\frac{|\psi_1(0)|^2}{|\psi_1^{(0)}(0)|^2} \;=\;& 1+\alpha_s(\mu)\Big[\big(5.25-0.32\,n_f\big)L+0.21-0.13\,n_f\Big]+\alpha_s^2(\mu)\Big[\big(18.39\\ &-2.23\,n_f+0.07\,n_f^2\big)L^2+\big(1.33-0.35\,n_f+0.02\,n_f^2\big)L+22.60-1.23\,n_f+0.02\,n_f^2\Big]\\ &+\alpha_s^3(\mu)\Big[\big(53.7-9.8\,n_f+0.6\,n_f^2-0.01\,n_f^3\big)\,L^3+\big(-6.7+0.6\,n_f-0.07\,n_f^2+0.002\,n_f^3\big)\,L^2\\ &\quad+\big(236.6-23.9\,n_f+0.8\,n_f^2-0.01\,n_f^3+15.0\,l_m\big)\,L-22.3\,L_{US}+3.0\,l_m-1.5\,l_m^2\\ &\quad+21.0+5.0\,n_f-0.3\,n_f^2+0.004\,n_f^3+0.0015\,a_3+\frac{\delta_\epsilon}{\pi}\Big],\end{aligned} \tag{6}$$

where $L = \ln(\mu/(mC_F\alpha_s(\mu)))$, $L_{US} = \ln\left(e^{5/6}\mu/(2m\alpha_s^2(\mu))\right)$, $l_m = \ln(\mu/m)$, and n_f is the number of light quark flavors, a_3[a] is the constant part of the three loop QCD potential, and δ_ϵ is a contribution from the $\mathcal{O}(\epsilon)$ terms of the non-Coulomb potentials given by

$$\delta_\epsilon = C_F^2\left(\frac{v_m^{(1,\epsilon)}}{8} + \frac{v_q^{(1,\epsilon)}}{12} + \frac{v_p^{(1,\epsilon)}}{8}\right) - \frac{C_F}{6}b_2^{(\epsilon)}. \tag{7}$$

The effect of δ_ϵ is estimated to be an order of magnitude smaller compared to other constant terms [1], so we neglect it in our phenomenological analysis. The $\ln^2\alpha_s$ [6, 7] and $\ln\alpha_s$ [8, 9] logarithmic terms in eq.(6) are already known.

From the divergent part of the wave-function calculation, the corresponding scale dependence of c_3 is extracted.[b] The matching coefficient c_v reads

$$\begin{aligned} c_v = 1 - \frac{8}{3\pi}\alpha_s(m) &+ \left[-\frac{35}{27}\ln\frac{\mu^2}{m^2} + \frac{11n_f}{27\pi^2} - \frac{125\,\zeta(3)}{9\pi^2} - \frac{14\ln 2}{9} - \frac{89}{54\pi^2} - \frac{511}{324}\right]\alpha_s(m)^2 \\ &+ \left[\left(\frac{43}{36\pi} - \frac{35n_f}{162\pi}\right)\ln^2\frac{\mu^2}{m^2} + \left(\frac{1399\,n_f}{1944\pi} - \frac{2818}{405\pi} - \frac{85\ln 2}{9\pi}\right)\ln\frac{\mu^2}{m^2} + \frac{\delta c_3}{\pi^3}\right]\alpha_s(m)^3. \end{aligned} \tag{8}$$

The constant part of δc_3 is not fully known up to now, but the fermionic correction was calculated in [10],

$$\delta c_{3,\,n_f} = n_f\,C_F\,T_F\left[39.6\,C_A + 46.7\,C_F - n_f\,T_F\left(\frac{163}{162} + \frac{4\pi^2}{27}\right) - T_F\left(\frac{557}{162} - \frac{26\pi^2}{81}\right)\right]. \tag{9}$$

The coefficient d_v is known from [11], and given by

$$d_v = 1 - \left[\frac{16}{9\pi}\left(1 + 3\ln\frac{\mu^2}{m^2}\right)\right]\alpha_s(\mu) + \cdots. \tag{10}$$

3 Residue of the QCD two-point function

Now we combine all pieces and show numerical formulae for the residue of the QCD two-point function. We use the same coupling $\alpha_s(\mu)$ [c] for the matching coefficient and the NRQCD wave-function to construct the scale-invariant physical residue Z_n.

---

[a]Only a Padé estimate [12] $a_{3,\,\mathrm{Pade}} = 6240$ (for $n_f = 4$), 3840 (for $n_f = 5$) is known.

[b]The result of [8] has been checked and one term (+ typos) of c_3 was corrected in [2].

[c]In eq.(8) $\alpha_s(m)$ is re-expressed by $\alpha_s(\mu)$ using $\alpha_s(m)/\alpha_s(\mu) = 1 + \frac{\alpha_s(\mu)}{4\pi}\beta_0\ln\frac{\mu^2}{m^2} + \left(\frac{\alpha_s(\mu)}{4\pi}\right)^2\left(\beta_0^2\ln^2\frac{\mu^2}{m^2} + \beta_1\ln\frac{\mu^2}{m^2}\right) + \cdots$ where β_i are the coefficients of the QCD β-function in $\overline{\mathrm{MS}}$-scheme, and $\alpha_s \equiv \alpha_s^{(n_f=4,5)}$ for the bottom and top quarks, respectively.

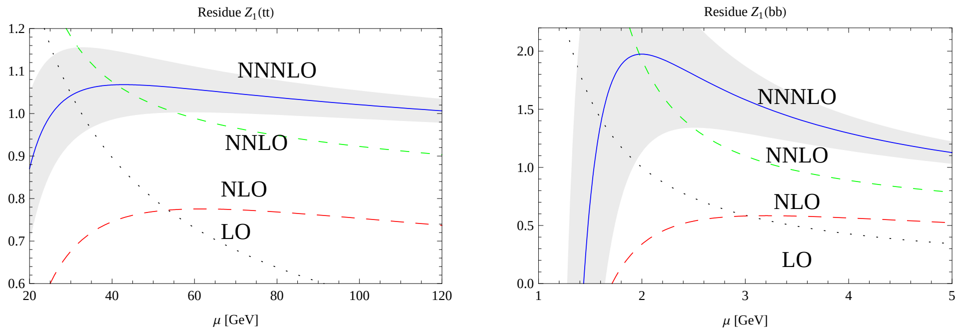

Figure 1: The scale dependence of the residue of the two-point function for the toponium (left) and bottomonium (right), normalized by its zeroth order value at $\mu = mC_F\alpha_s(\mu)$. The dotted line is LO and the solid line is NNNLO result.

For the ground state of top and bottom quarkonia, the residue is given by

$$
\begin{aligned}
Z_{1S(t\bar{t})} &= \Bigg\{1 + \Big[3.66\,L - 2.13\Big]\alpha_s(\mu) + \Big[8.93\,L^2 - 6.14\,L + 10.46 - 7.26\,l_m\Big]\,\alpha_s^2(\mu) \\
&\quad + \Big[18.17\,L^3 - 20.26\,L^2 + (110.82 - 11.57\,l_m)\,L - 22.27\,L_{US} - 16.35\,l_m^2 - 22.65\,l_m \\
&\quad + (22.60 + 0.0015\,a_3 + 0.32\,\delta_\epsilon + 0.0645\,\delta c_3)\Big]\,\alpha_s^3(\mu)\Bigg\} \times |\psi^{(0)}_{1S(t\bar{t})}(0)|^2\,, \qquad (11) \\
Z_{1S(b\bar{b})} &= \Bigg\{1 + \Big[3.98\,L - 2.00\Big]\alpha_s(\mu) + \Big[10.55\,L^2 - 6.51\,L + 11.19 - 7.44\,l_m\Big]\,\alpha_s^2(\mu) \\
&\quad + \Big[23.33\,L^3 - 23.12\,L^2 + (125.14 - 14.59\,l_m)\,L - 22.27\,L_{US} - 17.36\,l_m^2 - 26.61\,l_m \\
&\quad + (17.44 + 0.0015\,a_3 + 0.32\,\delta_\epsilon + 0.0645\,\delta c_3)\Big]\,\alpha_s^3(\mu)\Bigg\} \times |\psi^{(0)}_{1S(b\bar{b})}(0)|^2 \qquad (12)
\end{aligned}
$$

where $|\psi^{(0)}_{1S(Q\bar{Q})}(0)|^2 = (mC_F\alpha_s(\mu))^3/(8\pi)$ is the LO Coulomb wave-function. To see the numerical significance we plug the following values into the formulae: for the top quark, $m_t = 175$ GeV, $\mu = m_t\,C_F\,\alpha_s(\mu) = 32.62$ GeV; for the bottom quark, $m_b = 5$ GeV, $\mu = m_b\,C_F\,\alpha_s(\mu) = 2.02$ GeV. We use $a_3 = a_{3,\,\mathrm{Pade}}$, and the unknown $\mathcal{O}(\epsilon)$ potentials as well as non-n_f term of δc_3 are set to zero. We obtain the following numbers for the toponium and bottomonium ground state at $\mu = mC_F\alpha_s(\mu)$,

$$
Z_{1S(t\bar{t})} = \frac{(C_F\,m_t\,\alpha_s)^3}{8\pi}\Big[1 - 2.13\,\alpha_s + 22.7\,\alpha_s^2 + \Big(-38.8 + 5.8_{\,a3} + 37.6_{\,c3\,,nl}\Big)\,\alpha_s^3\Big], \qquad (13)
$$

$$
Z_{1S(b\bar{b})} = \frac{(C_F\,m_b\,\alpha_s)^3}{8\pi}\Big[1 - 2.00\,\alpha_s + 17.9\,\alpha_s^2 + \Big(-8.8 + 9.4_{\,a3} + 30.3_{\,c3\,,nl}\Big)\,\alpha_s^3\Big], \qquad (14)
$$

where the coupling constant is $\alpha_s = 0.14,\ 0.304$ for the top and bottom quarkonia, respectively.

In Fig.1 we show the scale dependence of the ground-state pole residue for toponium and bottomonium. For the NNNLO lines δc_3 is set to zero, while the gray band indicates the size of the contribution from the constant part of c_3; the upper/lower edge of the band is obtained by taking fermionic corrections $\delta c_{3,\,n_f}/-\delta c_{3,\,n_f}$ as an estimate of δc_3.[d] We observe that the scale dependence of the toponium wave-function is reduced significantly at NNNLO compared to NNLO as was also observed in renormalization group improved NNLO calculation [13, 14]. Its precise value will be fixed only once the third order matching coefficient is completely known. Since the threshold cross section is dominated by the ground-state contribution, we expect that the scale dependence of the $t\bar{t}$ threshold cross section will be also improved at NNNLO. For the bottomonium wave-function, strong scale dependence remains even at NNNLO and the perturbative expansion may be out of control. Only if the constant part of the matching coefficient δc_3 is negative in total, the scale dependence of the bottomonium wave-function at the origin might be acceptable. The complete knowledge of c_3 is thus mandatory to draw the final conclusion on the size of NNNLO correction.

Acknowledgments

This work was supported by the DFG Sonderforschungsbereich/Transregio 9 "Computer-gestützte Theoretische Teilchenphysik" and DFG Graduiertenkolleg "Elementarteilchen-physik an der TeV-Skala".

References

[1] M. Beneke, Y. Kiyo and K. Schuller, arXiv:0705.4518 [hep-ph].

[2] M. Beneke, Y. Kiyo and A. A. Penin, Phys. Lett. B **653** (2007) 53 [arXiv:0706.2733 [hep-ph]].

[3] M. Beneke and V. A. Smirnov, Nucl. Phys. B **522** (1998) 321 [arXiv:hep-ph/9711391].

[4] M. Beneke, Y. Kiyo and K. Schuller, Nucl. Phys. B **714** (2005) 67 [arXiv:hep-ph/0501289].

[5] A. A. Penin, V. A. Smirnov and M. Steinhauser, Nucl. Phys. B **716** (2005) 303 [arXiv:hep-ph/0501042].

[6] B. A. Kniehl and A. A. Penin, Nucl. Phys. B **577** (2000) 197 [arXiv:hep-ph/9911414].

[7] A. V. Manohar and I. W. Stewart, Phys. Rev. D **63** (2001) 054004 [arXiv:hep-ph/0003107].

[8] B. A. Kniehl, A. A. Penin, M. Steinhauser and V. A. Smirnov, Phys. Rev. Lett. **90** (2003) 212001; Erratum *ibid.* 91 (2003) 139903 [arXiv:hep-ph/0210161].

[9] A. H. Hoang, Phys. Rev. D **69** (2004) 034009 [arXiv:hep-ph/0307376].

[10] P. Marquard, J. H. Piclum, D. Seidel and M. Steinhauser, Nucl. Phys. B **758** (2006) 144 [arXiv:hep-ph/0607168].

[11] M. E. Luke and M. J. Savage, Phys. Rev. D **57** (1998) 413 [arXiv:hep-ph/9707313].

[12] F. A. Chishtie and V. Elias, Phys. Lett. B **521** (2001) 434 [arXiv:hep-ph/0107052].

[13] A. H. Hoang, A. V. Manohar, I. W. Stewart and T. Teubner, Phys. Rev. D **65** (2002) 014014 [arXiv:hep-ph/0107144].

[14] A. Pineda and A. Signer, Nucl. Phys. B **762** (2007) 67 [arXiv:hep-ph/0607239].

[d] By looking at constant part of the $c_v^{(2)}$, the non-fermionic correction is larger than the fermionic correction in magnitude and the sign is opposite. With this observation, a naive guess for c_3 is that the NNNLO line in the figure is most likely to be shifted down when the full constant part of c_3 is taken into account.