Four-Loop QCD Corrections and Master Integrals for the ρ-Parameter

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The calculation of the four-loop QCD corrections to the electroweak ρ-parameter arising from top- and bottom-quarks of the order $\mathcal{O}(G_F m_t^2 \alpha_s^3)$ is discussed. In particular the computation of the numerous master integrals in the standard and ε-finite basis is addressed.

1 Introduction

The ρ-parameter measures the relative strength of the charged and the neutral current and is equal one at lowest order perturbation theory in the Standard Model. Higher order corrections induce a shift in the lowest order value, which can be related to the transversal parts of the W- and Z-boson self-energies at zero momentum transfer

$$\delta\rho = \frac{\Pi_T^Z(0)}{M_Z^2} - \frac{\Pi_T^W(0)}{M_W^2}, \tag{1}$$

where M_Z and M_W are the $Z-$ and $W-$boson masses, respectively. The one-loop correction has first been evaluated in Ref. [2] and was used in order to establish a limit on the mass splitting within one fermion doublet. For a top-bottom fermion-doublet the dominant shift to the ρ-parameter is given by

$$\delta\rho_1 = 3\,\frac{G_F m_t^2}{8\sqrt{2}\,\pi^2} = 3\,x_t, \tag{2}$$

where the bottom quark mass has been neglected, hence it is quadratic in the top-quark mass m_t. The symbol G_F denotes the Fermi coupling constant.
The ρ-parameter enters in numerous physical quantities, e.g. it is related to the indirect prediction of the W-boson mass and to the weak mixing angle.
The perturbative expansion in the strong coupling constant α_s defined in the $\overline{\text{MS}}$-scheme for six flavors is given by:

$$\delta\rho^{\overline{\text{MS}}} = 3x_t \sum_{i=0}^{3} \left(\frac{\alpha_s}{\pi}\right)^i \delta\rho_i^{\overline{\text{MS}}}. \tag{3}$$

Here x_t is expressed in terms of the $\overline{\text{MS}}$ quark mass $m_t \equiv m_t(\mu)$ at the scale $\mu = m_t$ and α_s at the same scale. The two-loop QCD corrections [3–5] to the ρ-parameter have been calculated about 20 years ago and the three-loop QCD corrections [6, 7] more than ten years

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ago. Also important are two-loop [8–12] and three-loop electroweak effects proportional to x_t^2 and x_t^3, which have been determined in Refs. [13, 14] as well as three-loop mixed electroweak/QCD corrections of order $\alpha_s\, x_t^2$, which have been calculated in Ref. [14].
In a first step at four-loop order in perturbative QCD the singlet contributions have been computed in Ref. [15]. They are characterized by the fact that the external current couples to two different closed fermion loops, whereas for the non-singlet contributions the external current couples to the same closed fermion loop. Sample diagrams for both types of contributions are shown in Fig. 1.

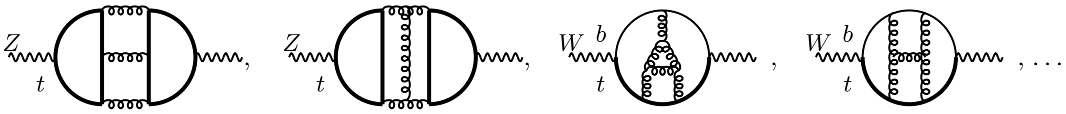

Figure 1: The first two diagrams are sample diagrams for the singlet contribution, whereas the third and fourth diagram are non-singlet type diagrams.

In the following Section 2 the calculation of the four-loop QCD corrections to the ρ-parameter from top- and bottom-quarks, in particular the non-singlet contribution, shall be discussed and we outline our computation of the appearing master integrals. In Section 3 we present the result and close with the summary and conclusions in Section 4.

2 Methods of calculation

The calculation of the transversal parts of the W- and Z-boson self-energies at zero momentum leads to the determination of four-loop tadpoles. In order to reduce all appearing integrals to a smaller set of master integrals the traditional Integration-by-parts(IBP) method has been employed in combination with Laporta's algorithm [16,17], which has been implemented in a FORM [18–20] based program, which uses Fermat [21] for the simplification of the polynomials in the space-time dimension d. The number of surviving master integrals is however with 63 quite sizable. The master integrals in the standard basis are shown in Fig. 2.

2.1 The master integrals in the standard basis

These master integrals can be classified into three groups. The first group of 13 integrals $\{T_{4,1}, T_{5,1}, T_{5,2}, T_{5,3}, T_{5,4}, T_{6,3}, T_{6,4}, T_{6,1}, T_{6,2}, T_{7,1}, T_{7,2}, T_{8,1}, T_{9,1}\}$ have already been used in previous calculations, e.g. in the determination of moments of the vacuum polarization function [22,23] or the decoupling relations [24–26]. All of them have been determined with the help of the method of difference equations [16, 17, 27–29] in Ref. [30] and subsequently in Ref. [31], where the method of ε-finite basis has been employed. Some of these master integrals or particular orders in the ε-expansion have also been found in Refs. [15,29,32–38] analytically or numerically.

The second set of 12 master integrals $\{T_{5,5}, T_{5,6}, T_{5,7}, T_{6,5}, T_{6,6}, T_{6,7}, T_{6,8}, T_{7,3}, T_{7,4}, T_{9,2}, T_{5,10}, T_{7,16}\}$ is "simple" in the sense, that they are factorized and can be found via a repeated application of the well-known analytical formulas for a one-loop massless propagator and a one-loop massive tadpole. Less simpler diagrams like $T_{7,4}$ can be extracted from [39–41] while the most complicated non-planar one $T_{9,2}$ can be obtained from [42,43].

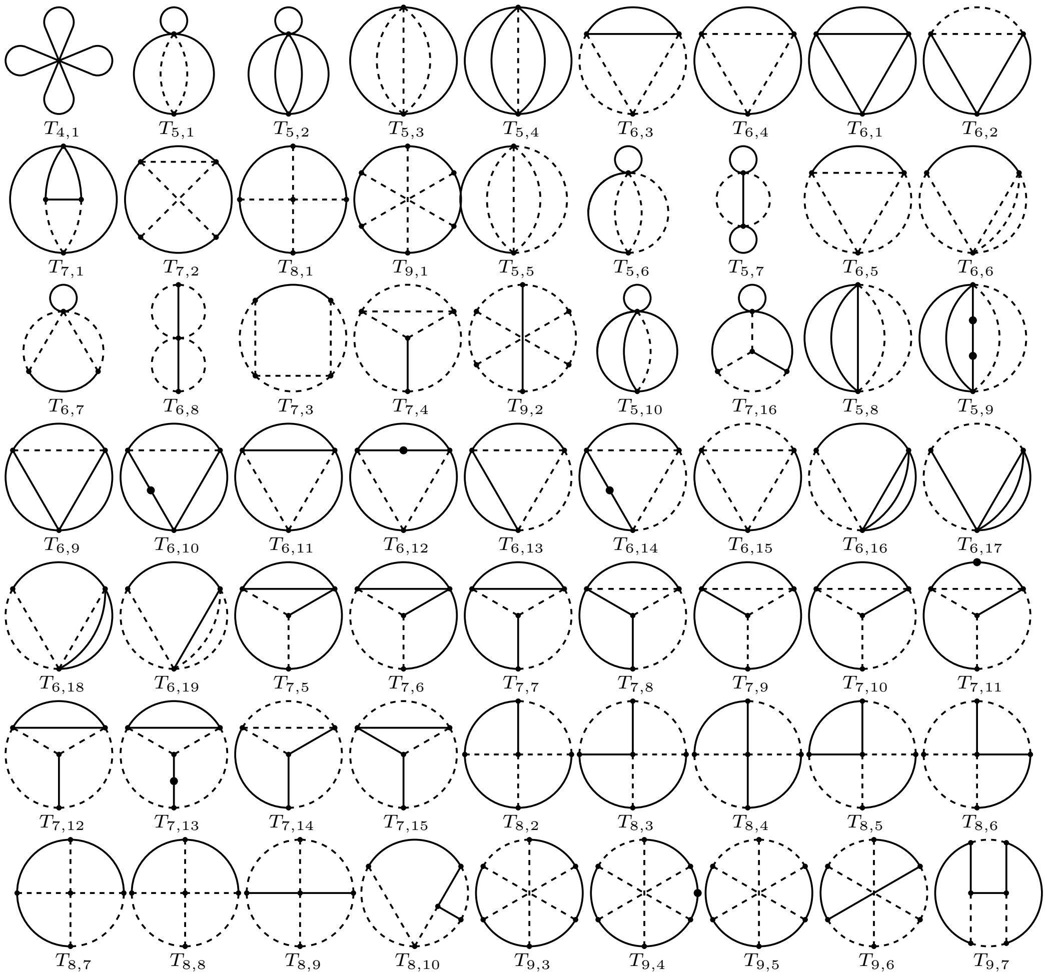

Figure 2: The master integrals in the standard basis, which come out naturally while solving the linear system of IBP equations with the help of Laporta's Algorithm and which have a minimal number of lines, a minimal number of dots and irreducible scalar products. Solid (dashed) lines denote massive (massless) propagators. The first index i of the topologies $T_{i,j}$ denotes the number of lines, whereas the second one j enumerates the topologies with the same number of lines.

The last group of remaining 38 master integrals we solved in Ref. [44] with the help of Padé approximations in the ε-finite basis or by means of the method of difference equations; results for the master integrals in the standard basis have also been determined in Ref. [50].

2.2 The master integrals in the ε-finite basis

One problem, which in general arises while solving the linear system of IBP-equations is, that a division by $(d-4)$ can occur. This can lead to spurious poles in front of a master integral. Each master integral which has a spurious pole as coefficient needs to be evaluated deeper in its ε-expansion, which is increasingly tedious. As a result of this it can be useful to select a new ε-finite basis of master integrals [31], whose coefficients in the space-time dimension $d = 4 - 2\,\varepsilon$ are finite in the limit $\varepsilon \to 0$. The ε-finite basis has the advantage, that the members only need to be evaluated up to the finite order in the ε-expansion.

Following the prescription of Ref. [31] an ε-finite basis has been constructed in Ref. [44] for the master integrals of Fig. 2, where the integrals $\{T_{4,1},\, T_{5,1},\, T_{5,3},\, T_{6,3},\, T_{5,5},\, T_{5,6},\, T_{5,7},\, T_{6,5},\, T_{6,6},\, T_{6,7},\, T_{6,8},\, T_{7,3},\, T_{7,4},\, T_{9,2}\}$ have been excluded from the construction, since they are known to sufficiently high order in the ε-expansion. In order to remove spurious poles in front of the remaining master integrals the integrals $\{T_{5,2},\, T_{5,4},\, T_{6,4},\, T_{6,1},\, T_{6,2},\, T_{7,1},\, T_{7,2},\, T_{5,10},\, T_{5,8},\, T_{5,9},\, T_{6,9},\, T_{6,10},\, T_{6,12},\, T_{6,13},\, T_{6,14},\, T_{6,15},\, T_{6,16},\, T_{6,17},\, T_{6,18},\, T_{6,19},\, T_{7,6},\, T_{7,11},\, T_{7,14},\, T_{7,15},\, T_{9,4}\}$ needed to be replaced. The master integrals in the ε-finite basis have been computed with a semi-numerical method based on Padé-approximations [31, 45–48]. The pole-part could be extracted completely analytically. The relations between the two bases can be used in order to compute master integrals in the standard basis from the results of the ε-finite one. This allows also to obtain analytical information for the master integrals in the standard basis from the ε-finite one. In addition one can also derive special relations among particular orders of different master integrals, e.g.

$$\frac{45\zeta_4}{2} - \frac{166\zeta_3}{9} + \frac{1685\zeta_2}{48} - \frac{9\sqrt{3}s_2}{2} + \frac{11561}{128} = T^{(0)}_{6,18} - T^{(0)}_{6,13}\,, \tag{4}$$

which are given in Ref. [44]. They are important, if one wants to compute further orders of the master integrals in the ε-expansion analytically. In Eq. (4) $T^{(0)}_{i,j}$ denotes the constant order in the ε-expansion, ζ_n is the Riemann zeta-function and s_2 is the Clausen-function $\mathrm{Cl}_2(\frac{\pi}{3})$.

3 Result

After having inserted the results for the master integrals into the parameter $\delta\rho$ and having performed renormalization in $\overline{\mathrm{MS}}$-scheme, one obtains the following result [49]:

$$\delta\rho^{\overline{\mathrm{MS}}} = 3x_t\Big(1 - \frac{\alpha_s}{\pi}0.19325 + \Big(\frac{\alpha_s}{\pi}\Big)^2(\underset{\sim\sim\sim}{-4.2072}\underline{+0.23764}) + \Big(\frac{\alpha_s}{\pi}\Big)^3(\underset{\sim\sim\sim}{-3.2866}\underline{+1.6067})\Big). \tag{5}$$

This result has been confirmed in the completely independent work of Ref. [50]. Starting from three-loop order there arise the singlet type diagrams, whose numerical value is shown separately in Eq. (5). The singlet contribution is underlined by the wavy line, whereas the non-singlet contribution is underlined by the solid line. At three-loop order the singlet-diagrams completely dominate the numerical correction, if the $\overline{\mathrm{MS}}$-definition is adopted for the top-quark mass. At four-loop order the dominance is less pronounced. If the result is expressed in terms of the top-quark pole-mass one obtains for the four-loop contribution $\delta\rho_3^{\mathrm{pole}} = -93.1501$, which corresponds to a small shift of around 2 MeV in the W-boson mass. This is well below the expected precision of future experiments and the result based on the three-loop calculation is stabilized.

4 Summary and conclusion

The four-loop QCD corrections from top- and bottom-quarks of order $\mathcal{O}(G_F m_t^2 \alpha_s^3)$ to the ρ-parameter have been computed. All appearing loop-integrals have been reduced to master integrals. These have been computed with the help of the method of difference equations in the standard basis or by means of Padé-approximations in the ε-finite basis. At least the pole-part of all the master integrals has been determined analytically. The four-loop contribution leads to a small shift in the W-boson mass of around 2 MeV, which is well below the anticipated precision of future experiments.

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