NLO Feynman integrals

Progress in tensor reduction of 1-loop Feynman integrals

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in cooperation with J. Fleischer, J. Gluza et al.

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http://indico.mppmu.mpg.de/indico/conferenceDisplay.py?confId=1369

Introduction

A long version of this talk was presented at:

5th Helmholtz International Summer School - Workshop
Dubna International Advanced School of Theoretical Physics - DIAS TH
Calculations for Modern and Future Colliders
July 23 - August 2, 2012, Dubna, Russia
http://theor.jinr.ru/ calc2012/

My first visit to West Germany was due to an invitation to the MPP Munich, see e.g. in spires:

On The Derivation Of Standard Model Parameters From The Z Peak
T. Riemann, M. Sachwitz (DESY, Zeuthen), D. Bardin, M. Bilenky (Dubna, JINR).
PHE-89-07, C89/04/03.1, Contribution to Conference: C89-04-03.1 Ringberg Workshop 1989

For the Fortran program package ZFITTER, which was used for that talk, see the private homepage:

http://zfitter.com
The webpage also reflects some recent experience of general interest.

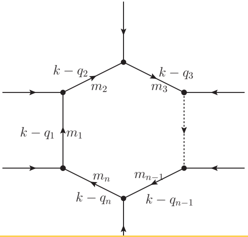
Definitions

n-point tensor integrals of rank *R*: (n,R)-integrals

$$I_n^{\mu_1 \cdots \mu_R} = \int \frac{d^d k}{i\pi^{d/2}} \frac{\prod_{r=1}^{R} k^{\mu_r}}{\prod_{j=1}^{n} c_j^{\nu_j}},$$

$d = 4 - 2\epsilon$ and denominators c_j have *indices* ν_j and *chords* q_j

$$c_j = (k - q_j)^2 - m_j^2 + i\varepsilon$$



tensor integrals due to, e.g.:

- fermion propagators
- three-gauge boson couplings

A simple example

1-loop self-energy:

$$I_2^\mu = \int \frac{d^d k}{i\pi^{d/2}} \frac{k^\mu}{[k^2 - M_1^2]\,[(k+p)^2 - M_2^2]} = p_\mu \cdot B_1$$

Solve:

$$\begin{aligned} p_\mu \cdot I_2^\mu &= p^2 \cdot B_1(p, M_1, M_2) \\ &= \int \frac{d^d k}{i\pi^{d/2}} \frac{pk}{[k^2 - M_1^2]\,[(k+p)^2 - M_2^2]} = \int \frac{d^d k}{i\pi^{d/2}} \frac{pk}{D_1\; D_2} \\ &= \int \frac{d^d k}{i\pi^{d/2}} \left[\frac{D_2 - (p^2 - M_2^2 - M_1^2) - D_1}{D_1\; D_2} \right], \\ B_1(p, M_1, M_2) &= \frac{1}{2p^2} \left[A_0(M_1) - A_0(M_2) - (p^2 - M_2^2 - M_1^2) B_0(p, M_1, M_2) \right] \end{aligned}$$

A tensor Feynman integral is expressed in terms of scalar Feynman integrals.

Passarino-Veltman algorithm

1. Contract n-point and R-rank Feynman integral with *external momenta* p_i^μ and with $g^{\mu\nu}$, and cancel propagators
2. Invert the resulting system of linear equations
3. The result consists of $(n-1)$-point and $(R-1)$-rank functions

Reducing tensor rank introduces inverse Gram determinant:

$$I_5^{\mu_1\dots\mu_{R-1}\mu_R} = \sum_{i=1}^{5} \frac{q_i^{\mu_R}}{\det(G_5)} \left[A_{0i}\; I_5^{\mu_1\dots\mu_{R-1}} - \sum_{s=1}^{5} A_{si}\; I_4^{\mu_1\dots\mu_{R-1},s} \right]$$

Gram determinant G_n:

$$G_n = |2q_iq_j|, \; i,j = 1,\dots n-1 \tag{1}$$

and A_{0i}, A_{si} are kinematic coefficients. The q_i are **internal** momenta.

Systematic approach to tensor reductions:

1,2,3,4-point functions:

- Passarino, Veltman 1978 [1]

Open source programs for 5,6-point reductions:

- LoopTools/FF ($n \leq 5$), T. Hahn [2, 3] 1998,1990.
- Golem95 T. Binoth et al. [4] 2008
- PJFry V. Yundin, PhD thesis 2012 [5] + Fleischer, T.R. [6] 2010

Need in addition a library of scalar functions:

- ’t Hooft, Veltman 1979 [7]
- QCDloop/FF K. Ellis and G. Zanderighi [8, 3] 2007,1990
- LoopTools/FF T. Hahn [2, 3] 1998,1990
- OneLOop (complex masses) van Hameren [9] 2010

This talk: Efficient reduction formulae in the algebraic Davydychev-Tarasov-Fleischer-Jegerlehner-TR approach

- Get $n > 4$ tensor reduction with $\cdots$:
- $\cdots$ arbitrary masses
- $\cdots$ killed pentagon Gram determinants
- $\cdots$ treatment of full kinematics, also with small sub-diagram Gram determinants
- **new:** $\cdots$ multiple sums over tensor coefficients made efficient by **contracting with external momenta**

  Fleischer, TR [10] PLB 701(2011)646
- **new:** $\cdots$ higher n point functions, $n \geq 7$

  Fleischer, TR [11] PLB 707(2012)375

History of the Approach - not a complete list of references

[12] Melrose 1965: Reduction of Feynman diagrams and Cayley determinants

[13] Davydychev 1991: Integrals in different space-time dimension.
See also Bern et al. (1993) [14]

[15] Tarasov 1996: Dimensional recurrence relations

[16] Fleischer,Jegerlehner,Tarasov 2000: 1-loop reductons and signed minors.

[4] Binoth,Guillet,Heinrich,Pilon,Schubert, 2005: Algebraic/numerical formalism for one-loop multi-leg amplitudes

[6] Fleischer and T.Riemann (since 2007) 2011: Complete reduction of 1-loop tensors.
See also Diakonidis et al. [17]

[18] Yundin's package PJFry 2010; https://github.com/Vayu/PJFry.
See also Fleischer,TR,Yundin [5, 19]

[10] Fleischer and T.Riemann 2011: Contracted tensor Feynman integrals.
See also Diakonidis et al. [20]

[21] Fleischer and T.Riemann 2012: A solution for tensor reduction of one-loop n-point functions with $n \geq 6$

Tensor integrals expressed in terms of scalar integrals in higher dimensions $D = d + 2l = 4 - 2\epsilon,\ 6 - 2\epsilon, \cdots$ [Davydychev:1991] , also [Fleischer et al.:2000]

$n_{ij} = \nu_{ij} = 1 + \delta_{ij}, n_{ijk} = \nu_{ij}\nu_{ijk}, \nu_{ijk} = 1 + \delta_{ik} + \delta_{jk}$

$$
\begin{aligned}
I_n^{\mu} &= \int^d k^{\mu} \prod_{r=1}^{n} c_r^{-1} = -\sum_{i=1}^{n} q_i^{\mu}\, I_{n,i}^{[d+]} \\
I_n^{\mu\,\nu} &= \int^d k^{\mu}\, k^{\nu} \prod_{r=1}^{n} c_r^{-1} = \sum_{i,j=1}^{n} q_i^{\mu}\, q_j^{\nu}\, n_{ij}\, I_{n,ij}^{[d+]^2} - \frac{1}{2} g^{\mu\nu}\, I_n^{[d+]} \\
I_n^{\mu\,\nu\,\lambda} &= \int^d k^{\mu}\, k^{\nu}\, k^{\lambda} \prod_{r=1}^{n} c_r^{-1} = -\sum_{i,j,k=1}^{n} q_i^{\mu}\, q_j^{\nu}\, q_k^{\lambda}\, n_{ijk}\, I_{n,ijk}^{[d+]^3} + \frac{1}{2}\sum_{i=1}^{n} g^{[\mu\nu} q_i^{\lambda]}\, I_{n,i}^{[d+]^2} \\
I_n^{\mu\,\nu\,\lambda\,\rho} &= \int^d k^{\mu}\, k^{\nu}\, k^{\lambda}\, k^{\rho} \prod_{r=1}^{n} c_r^{-1} = \sum_{i,j,k,l=1}^{n} q_i^{\mu}\, q_j^{\nu}\, q_k^{\lambda}\, q_l^{\rho}\, n_{ijkl}\, I_{n,ijkl}^{[d+]^4} \\
&\quad -\frac{1}{2}\sum_{i,j=1}^{n} g^{[\mu\nu} q_i^{\lambda} q_j^{\rho]}\, n_{ij}\, I_{n,ij}^{[d+]^3} + \frac{1}{4} g^{[\mu\nu} g^{\lambda\rho]}\, I_n^{[d+]^2} \qquad (2)
\end{aligned}
$$

Tensor integrals expressed in terms of scalar integrals in higher dimensions $D = d + 2l = 4 - 2\epsilon,\ 6 - 2\epsilon, \cdots$ [Davydychev:1991] , also [Fleischer et al.:2000] II

$$I_n^{\mu\,\nu\,\lambda\,\rho\,\sigma} = \int \frac{d^d k}{i\pi^{d/2}} k^\mu\, k^\nu\, k^\lambda\, k^\rho\, k^\sigma \prod_{j=1}^{n} c_j^{-1} = - \sum_{i,j,k,l,m=1}^{n} q_i^\mu\, q_j^\nu\, q_k^\lambda\, q_l^\rho\, q_m^\sigma\, n_{ijklm}\, I_{n,ijklm}^{[d+]^5}$$
$$+\frac{1}{2} \sum_{i,j,k=1}^{n} g^{[\mu\nu} q_i^\lambda q_j^\rho q_k^{\sigma]}\, n_{ijk}\, I_{n,ijk}^{[d+]^4} - \frac{1}{4} \sum_{i=1}^{n} g^{[\mu\nu} g^{\lambda\rho} q_i^{\sigma]}\, I_{n,i}^{[d+]^3}. \tag{3}$$

The integrals

$$I_{p,ijk\cdots}^{[d+]^l,stu\cdots} = \int^{[d+]^l} \prod_{r=1}^{n} \frac{1}{c_r^{1+\delta_{ri}+\delta_{rj}+\delta_{rk}+\cdots-\delta_{rs}-\delta_{rt}-\delta_{ru}-\cdots}}, \tag{4}$$

$$\int^d \equiv \int \frac{d^d k}{\pi^{d/2}},$$

where $[d+]^l = 4 + 2l - 2\epsilon$.

$$I_{n-1,ab}^{\{\mu_1,\cdots\},s}, \ a, b \neq s$$

is obtained from

$$I_n^{\{\mu_1,\cdots\}}$$

by

- shrinking line s
- raising the powers of inverse propagators a, b.

Dimensional shifts and recurrence relations for pentagons (II)

Direct approach – just perform Tarasov's dimensional recurrences

Following [Tarasov:1996,Fleischer:1999 [15, 16]]
apply recurrence relations, relating scalar integrals of different dimensions, in order to get rid of the dimensionalities $[d+]^l = 4 - 2\epsilon + 2l$:

shift dimension + index:

$$\nu_j (\mathbf{j}^+ I_5^{[d+]}) = \frac{1}{()_5} \left[-\binom{j}{0}_5 + \sum_{k=1}^{5} \binom{j}{k}_5 \mathbf{k}^- \right] I_5 \qquad (5)$$

shift dimension:

$$(d - \sum_{i=1}^{5} \nu_i + 1) I_5^{[d+]} = \frac{1}{()_5} \left[\binom{0}{0}_5 - \sum_{k=1}^{5} \binom{0}{k}_5 \mathbf{k}^- \right] I_5, \qquad (6)$$

also:

$$\nu_j \mathbf{j}^+ I_5 = \frac{1}{\binom{0}{0}_5} \sum_{k=1}^{5} \binom{0j}{0k}_5 \left[d - \sum_{i=1}^{5} \nu_i (\mathbf{k}^- \mathbf{i}^+ + 1) \right] I_5 \qquad (7)$$

where the operators $\mathbf{i}^\pm, \mathbf{j}^\pm, \mathbf{k}^\pm$ act by shifting the indices ν_i, ν_j, ν_k by ± 1.

Example

Example for a "scratched" integral ($\nu_{ij} = 1 + \delta_{ij}$):

$$\nu_{ij} I_{4,ij}^{[d+]^2,s} = -\frac{\binom{0s}{js}_5}{\binom{s}{s}_5} I_{4,i}^{[d+],s} + \frac{\binom{is}{js}_5}{\binom{s}{s}_5} I_4^{[d+],s} + \sum_{t=1}^{5} \frac{\binom{ts}{js}_5}{\binom{s}{s}_5} I_{3,i}^{[d+],st}.$$

The

$$\frac{\binom{0s}{js}_5}{\binom{s}{s}_5} \quad \text{and} \quad \frac{\binom{is}{js}_5}{\binom{s}{s}_5} \quad \text{and} \quad \frac{\binom{ts}{js}_5}{\binom{s}{s}_5}$$

etc. are ratios of *signed minors* of the modified Cayley determinant $()_n$, i.e. up to a sign, they are equal to *sub-determinants of* $()_n$.

An alternative to dimensional recurrences of scalars: **Recursions for tensors**

5-point tensor recursion:

Express any $(5, R)$ pentagon by a $(5, R-1)$ pentagon plus $(4, R-1)$ boxes

[Diakonidis,Fleischer,T.Riemann,Tausk: Phys.Lett. **B683** (2010)]

$$I_5^{\mu_1\cdots\mu_{R-1}\mu} = I_5^{\mu_1\cdots\mu_{R-1}} Q_0^{\mu} - \sum_{s=1}^{5} I_4^{\mu_1\cdots\mu_{R-1},s} Q_s^{\mu},$$

For $n = 6, 7, 8, \cdots$ things are close but differ a bit; see later.

auxiliary vectors with inverse Gram determinants

$$Q_s^{\mu} = \sum_{i=1}^{5} q_i^{\mu} \frac{\binom{s}{i}_5}{()_5}, \quad s = 0, \ldots, 5$$

For e.g. $R = 3$, again $[1/()_5]^3$ will occur.

Contractions

$$q_{i_1\mu_1} \cdots q_{i_R\mu_R} \quad I_5^{\mu_1\cdots\mu_R} = \int \frac{d^dk}{i\pi^{d/2}} \frac{\prod_{r=1}^{R}(q_{i_r}\cdot k)}{\prod_{j=1}^{5} c_j},$$

$$g_{\mu_1,\mu_2} q_{i_1\mu_3} \cdots q_{i_R\mu_R} \quad I_5^{\mu_1\cdots\mu_R} = \int \frac{k^2 d^dk}{i\pi^{d/2}} \frac{\prod_{r=3}^{R}(q_{i_r}\cdot k)}{\prod_{j=1}^{5} c_j}$$

One may arrange a one-loop calculation such that all the one-loop integrals appear only in such contractions.

Important:

The contraction with g_{μ_1,μ_2} is shown here in a symbolic form; in practice we work strictly 4-dimensional with g_{μ_1,μ_2}.

Notations: Gram and modified Cayley determinant, signed minors
[Melrose:1965]

Gram determinant G_n:

$$G_n \quad = \quad |2q_iq_j|, i,j = 1, \ldots n-1 \tag{8}$$

Modified Cayley determinant $()_N$ of a diagram with N internal lines and chords q_j:

$$()_N \equiv \begin{vmatrix} 0 & 1 & 1 & \ldots & 1 \\ 1 & Y_{11} & Y_{12} & \ldots & Y_{1N} \\ 1 & Y_{12} & Y_{22} & \ldots & Y_{2N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & Y_{1N} & Y_{2N} & \ldots & Y_{NN} \end{vmatrix} \tag{9}$$

with the matrix elements

$$Y_{ij} = -(q_i - q_j)^2 + m_i^2 + m_j^2, \quad (i,j = 1 \ldots N) \tag{10}$$

The propagators are: $D_i = (k - q_i)^2 - m_i^2$

For the choice $q_n = 0$, both determinants are related:

$$()_N = -G_N$$

$\Rightarrow$ The modfied Cayley determinant $()_N$ does not depend on masses.

Notations: signed minors [Melrose:1965]

signed minors of $()_N$ are constructed by deleting m rows and m columns from $()_N$, and multiplying with a sign factor:

$$\left(\begin{array}{cccc} j_1 & j_2 & \cdots & j_m \\ k_1 & k_2 & \cdots & k_m \end{array}\right)_N \equiv$$

$$\equiv (-1)^{\sum_l (j_l+k_l)} \mathrm{sgn}_{\{j\}}\, \mathrm{sgn}_{\{k\}} \left| \begin{array}{c} \text{rows } j_1 \cdots j_m \text{ deleted} \\ \text{columns } k_1 \cdots k_m \text{ deleted} \end{array} \right| \tag{11}$$

where $\mathrm{sgn}_{\{j\}}$ and $\mathrm{sgn}_{\{k\}}$ are the signs of permutations that sort the deleted rows $j_1 \cdots j_m$ and columns $k_1 \cdots k_m$ into ascending order.

Example:

$$\left(\begin{array}{c} 0 \\ 0 \end{array}\right)_N \equiv \left| \begin{array}{cccc} Y_{11} & Y_{12} & \ldots & Y_{1N} \\ Y_{12} & Y_{22} & \ldots & Y_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ Y_{1N} & Y_{2N} & \ldots & Y_{NN} \end{array} \right|, \tag{12}$$

Example: Getting a 4-point function from a six-point function

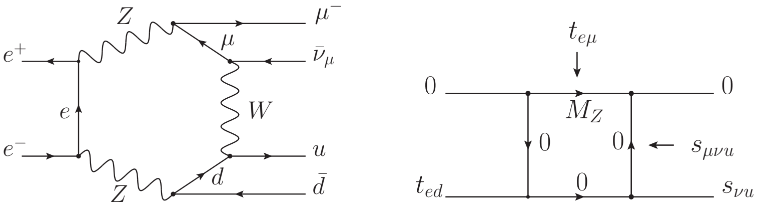


Figure: A six-point topology (a) leading to four-point functions (b) with realistically vanishing Gram determinants.

Example: Getting a 4-point function from a six-point function

The example is taken from a talk by A. Denner, [22].

The corresponding 4-point tensor integrals are, in LoopTools [2, 23] notation:

$$\mathrm{D0i}(\mathrm{id}, 0, 0, s_{\bar{\nu}u}, t_{ed}, t_{\bar{e}\mu}, s_{\mu\bar{\nu}u}, 0, M_Z^2, 0, 0). \tag{13}$$

The Gram determinant is:

$$()_4 \quad = \quad -2t_{\bar{e}\mu}[s^2_{\mu\bar{\nu}u} + s_{\bar{\nu}u}t_{ed} - s_{\mu\bar{\nu}u}(s_{\bar{\nu}u} + t_{ed} - t_{\bar{e}\mu})], \tag{14}$$

It vanishes if:

$$t_{ed} \to t_{ed,\mathrm{crit}} = \frac{s_{\mu\bar{\nu}u}(s_{\mu\bar{\nu}u} - s_{\bar{\nu}u} + t_{\bar{e}\mu})}{s_{\mu\bar{\nu}u} - s_{\bar{\nu}u}}. \tag{15}$$

In terms of a dimensionless scaling parameter x,

$$t_{ed} \quad = \quad (\mathbf{1} + \mathbf{x})t_{ed,\mathrm{crit}}, \tag{16}$$

The Gram determinant in terms of x:

$$()_4 = 2\, \mathbf{x}\, s_{\mu\bar{\nu}u} t_{\bar{e}\mu} (s_{\mu\bar{\nu}u} - s_{\bar{\nu}u} + t_{\bar{e}\mu}). \tag{17}$$

A minor of $()_4$:

$$\begin{aligned} \binom{0}{0}_4 &= \begin{pmatrix} 2M_Z^2 & M_Z^2 & M_Z^2 - s_{\mu\bar{\nu}u} & M_Z^2 \\ M_Z^2 & 0 & -s_{\bar{\nu}u} & M_Z^2 \\ M_Z^2 - s_{\mu\bar{\nu}u} & -s_{\bar{\nu}u} & 0 & -t_{ed} \\ M_Z^2 & -t_{\bar{e}\mu} & -t_{ed} & 0 \end{pmatrix} \\ &= s_{\mu\bar{\nu}u}^2 t_{\bar{e}\mu}^2 + 2\, M_Z^2 t_{\bar{e}\mu} [-2 s_{\bar{\nu}u} t_{ed} + s_{\mu\bar{\nu}u}(s_{\bar{\nu}u} + t_{ed} - t_{\bar{e}\mu})] \\ &\quad + M_Z^4 (s_{\bar{\nu}u}^2 + (t_{ed} - t_{\bar{e}\mu})^2 - 2 s_{\bar{\nu}u}(t_{ed} + t_{\bar{e}\mu})). \end{aligned} \tag{18}$$

We will need the ratio

$$R(x) = \frac{()_4}{\binom{0}{0}_4} \times (\mathrm{scale})^2 \sim \mathbf{x}$$

Following Davydychev [13], one gets

$$I_4^{\mu\,\nu\,\lambda\,\rho} = \int^d k^\mu\, k^\nu\, k^\lambda\, k^\rho \prod_{r=1}^{4} c_r^{-1} = \sum_{i,j,k,l=1}^{n} q_i^\mu\, q_j^\nu\, q_k^\lambda\, q_l^\rho\, n_{ijkl}\; I_{4,ijkl}^{[d+]^4}$$

$$-\frac{1}{2}\sum_{i,j=1}^{4} g^{[\mu\nu} q_i^\lambda q_j^{\rho]}\, n_{ij}\, I_{4,ij}^{[d+]^3} + \frac{1}{4} g^{[\mu\nu} g^{\lambda\rho]}\, I_4^{[d+]^2} \quad (19)$$

We identify the tensor coefficients $D_{11\ldots}$ a la LoopTools, e.g.:

$$I_{4,222}^{[d+]^3} = D_{111} \quad (20)$$

Similarly:

$$I_{4,2222}^{[d+]^4} = D_{1111} \quad (21)$$

Rank $R = 4$ tensor D_{1111} – Numerics with dimensional recurrences

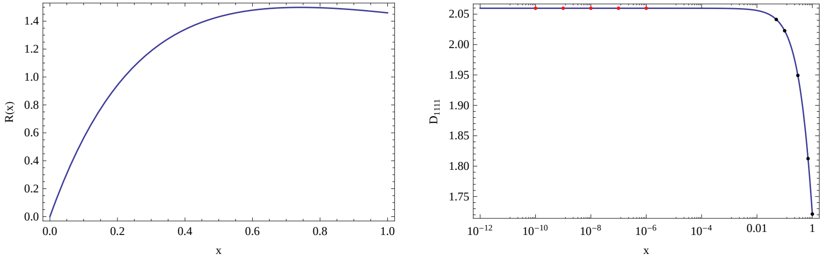
From (32) we see that a “small Gram determinant” expansion will be useful when the following dimensionless parameter becomes small:

$$R(x) \quad = \quad \frac{()_4}{\binom{0}{0}_4} \times s, \qquad (22)$$

where s is a typical scale of the process, e.g. we will choose $s = s_{\mu\bar{\nu}u}$.
Following [22], we further choose:

$$\begin{aligned} s_{\mu\bar{\nu}u} &= 2 \times 10^4 \mathrm{GeV}^2, \\ s_{\bar{\nu}u} &= 1 \times 10^4 \mathrm{GeV}^2, \\ t_{\bar{e}\mu} &= -4 \times 10^4 \mathrm{GeV}^2, \end{aligned}$$

and get $t_{ed,\mathrm{crit}} = -6 \times 10^4 \mathrm{GeV}^2$. For x=1, the Gram determinant becomes $()_4 = 4.8 \times 10^{13}$ GeV^3.
The small expansion parameter $R(x)$ and D_{1111} are shown in figure 2.



PJFry - an open source c++ program by V. Yundin

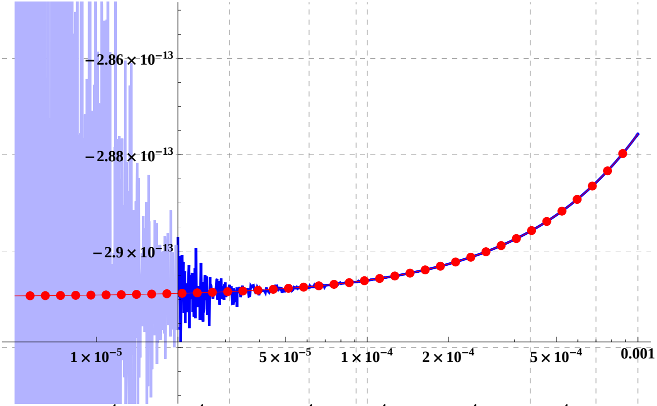
PJFry 1.0.0 - one loop tensor integral library

- More information and the latest source code:
  project page: https://github.com/Vayu/PJFry/
- $\rightarrow$ how to install
- $\rightarrow$ how to use
- $\rightarrow$ samples
- See also: Yundin's PhD thesis [5]

PJFry — small Gram region example

Example: E_{3333} coefficient in small Gram region ($x \to 0$) [from V.Y. Valencia 2011 [24]]

Comparison of Regular and Expansion formulae:



$x{=}0$: $E_{3333}(0, 0, -6\times10^4, 0, 0, 10^4, -3.5\times10^4, 2\times10^4, -4\times10^4, 1.5\times10^4, 0, 6550, 0, 0, 8315)$

Dimensional shifts and recurrence relations for pentagons

Following [Davydychev:1991 [13]]

Replace tensors by scalar integrals in higher dimensions:

Example $R = 3$:

$$
\begin{aligned}
I_5^{\mu\nu\lambda} &= \int \frac{d^{4-2\epsilon}k}{i\pi^{d/2}} \prod_{r=1}^{5} c_r^{-1} k^\mu k^\nu k^\lambda \\
&= -\sum_{i,j,k=1}^{4} q_i^\mu q_j^\nu q_k^\lambda n_{ijk} I_{5,ijk}^{[d+]^3} + \frac{1}{2}\sum_{i=1}^{n-1} (g^{\mu\nu} q_i^\lambda + g^{\mu\lambda} q_i^\nu + g^{\nu\lambda} q_i^\mu) I_{5,i}^{[d+]^2},
\end{aligned} \tag{23}
$$

and $n_{ijk} = (1+\delta_{ij})(1+\delta_{ik}+\delta_{jk})$.

$[d+]^l = 4 - 2\epsilon + 2l$

$I_{5,i}^{[d+]^2}$ – scratch the line i from $I_5^{[d+]^2}$.

The result of simplifying manipulations

... and collecting all contributions, our final result for e.g. the tensor of rank $R = 3$ can be written as follows:

$$I_5^{\mu\,\nu\,\lambda} = \sum_{i,j,k=1}^{4} q_i^{\mu}\, q_j^{\nu}\, q_k^{\lambda}\, E_{ijk} + \sum_{k=1}^{4} g^{[\mu\nu}\, q_k^{\lambda]}\, E_{00k}, \tag{24}$$

with:

$$E_{00j} = \sum_{s=1}^{5} \frac{1}{\binom{0}{0}_5} \left[\frac{1}{2} \binom{0s}{0j}_5 I_4^{[d+],s} - \frac{d-1}{3} \binom{s}{j}_5 I_4^{[d+]^2,s} \right], \tag{25}$$

$$E_{ijk} = -\sum_{s=1}^{5} \frac{1}{\binom{0}{0}_5} \left\{ \left[\binom{0j}{sk}_5 I_{4,i}^{[d+]^2,s} + (i \leftrightarrow j) \right] + \binom{0s}{0k}_5 \nu_{ij} I_{4,ij}^{[d+]^2,s} \right\}. \tag{26}$$

✓ no scalar 5-point integrals in higher dimensions
✓ no inverse Gram det. $()_5$

We have yet:

† scalar 4-point integrals in higher dimensions: $I_{4,ij}^{[d+]^2,s}$ etc.

† inverse Gram det. $\binom{0}{0}_5 \equiv ()_4$

Reduce $I_{4,ij\cdots}^{[d+]^l,s}$ to $I_4^{[d+]^L,s}$ plus simpler objects I

By nontrivial manipulations we get e.g.:

$$I_{4,i}^{[d+]^2,s} = \frac{1}{\binom{0s}{0s}_5}\left[-\binom{0s}{is}_5 (d-3)\, I_4^{[d+],s} + \sum_{t=1}^{5}\binom{0st}{0si}_5 I_3^{st}\right] \qquad (27)$$

$$\begin{aligned}\nu_{ij} I_{4,ij}^{[d+]^2,s} = {} & \frac{\binom{0}{i}_4}{\binom{0}{0}_4}\frac{\binom{0}{j}_4}{\binom{0}{0}_4}(d-2)(d-1)\, I_4^{[d+]^2} + \frac{\binom{0i}{0j}_4}{\binom{0}{0}_4} I_4^{[d+]} \\ & - \frac{\binom{0}{j}_4}{\binom{0}{0}_4}\frac{d-2}{\binom{0}{0}_4}\sum_{t=1}^{4}\binom{0t}{0i}_4 I_3^{[d+],t} + \frac{1}{\binom{0}{0}_4}\sum_{t=1}^{4}\binom{0t}{0j}_4 I_{3,i}^{[d+],t} \qquad (28)\end{aligned}$$

These equations are free of inverse Gram determinants $()_4$.
But they contain yet the generic 4-point and (partly indexed) 3-point functions in higher dimensions, $I_4^{[d+],s}$, $I_3^{[d+],t}$, etc.

Last step: evaluate the $I_4^{[d+],s}$, $I_3^{[d+],t}$, etc. I

Several strategies are now possible:

- Just evaluate them analytically in $d + 2l - 2\epsilon$ dimensions – if you may do that → Fleischer,Jerlehner,Tarasov 2003 [25]
- Just evaluate them numerically in $d + 2l - 2\epsilon$ dimensions

- Reduce them further by recurrences – buy the towers of $1/()_4$ → apply (6)

- Make a small Gram determinant expansion → apply (6) another way round

Last two items are done here.

Reduction of scalars I_4^D to the generic dimension $\rightarrow I_4^d = D_0, I_3^d = C_0$ I

Non-small 4-point Gram determinants:

Direct, iterative use of (6) yields e.g.:

$$I_4^{[d+]^l} = \left[\frac{\binom{0}{0}_4}{()_4} I_4^{[d+]^{l-1}} - \sum_{t=1}^{4} \frac{\binom{t}{0}_4}{()_4} I_3^{[d+]^{l-1},t} \right] \frac{1}{d+2l-5} \quad (29)$$

$$I_3^{[d+]^l,t} = \left[\frac{\binom{0t}{0t}_4}{\binom{t}{t}_4} I_3^{[d+]^{l-1},t} - \sum_{u=1,u\neq t}^{4} \frac{\binom{ut}{0t}_4}{\binom{t}{t}_4} I_2^{[d+]^{l-1},tu} \right] \frac{1}{d+2l-4} \quad (30)$$

And we are done.

This works fine if $()_4$ is not small [and also the $\binom{t}{t}_4$].

Make a small Gram expansion I

Again use (6):

$$()_4 (d - \sum_{i=1}^{4} \nu_i + 1) I_4^{[d+]} = \left[\binom{0}{0}_4 I_4 - \sum_{k=1}^{4} \binom{0}{k}_4 I_3^k \right]$$

If $()_4 = 0$, then it follows ($n = 4$):

$$I_n^D = \sum_k^n \frac{\binom{0}{k}_n}{\binom{0}{0}_n} I_{n-1}^{D,k} \qquad (31)$$

If $()_4 \ll 1$, re-write (6), as follows:

$$I_n^D = \sum_k^n \frac{\binom{0}{k}_n}{\binom{0}{0}_n} I_{n-1}^{D,k} - \frac{()_n}{\binom{0}{0}_n} [(D+1) - \sum_i^n \nu_i] I_n^{D+2}. \qquad (32)$$

Effectively we may evaluate I_n^D in terms of simpler functions $I_{n-1}^{D,k}$ with a small correction depending on I_n^{D+2}.

We may go a step further, and insert into (32) for I_n^{D+2} the rhs. of (31), taken now at $D' = D + 2$:

$$
\begin{aligned}
I_n^D &= \sum_k^n \frac{\binom{0}{k}_n}{\binom{0}{0}_n} I_{n-1}^{D,k} \\
&- \frac{()_n}{\binom{0}{0}_n} [(D+1) - \sum_i^n \nu_i] \\
&\times \left[\sum_k^n \frac{\binom{0}{k}_n}{\binom{0}{0}_n} I_{n-1}^{D+2,k} - \frac{()_n}{\binom{0}{0}_n} [(D+3) - \sum_i^n \nu_i] I_n^{D+4} \right].
\end{aligned}
$$

The terms proportional to $[()_n/\binom{0}{0}_n]^a$, $a = 0, 1$ may be evaluated at the correct kinematics. They depend on three-point functions, and their reduction by normal recurrences will not introduce the unwanted powers of $1/()_4$. The last term, suppressed by the factor $[()_n/\binom{0}{0}_n]^2$, depends on I_n^{D+4}. It may either be taken approximately at $()_n = 0$, where it can also be represented by 3-point functions (and their reductions), or it may be evaluated more correctly by another iteration based on (31).

And so on and so on ...

In the tables with numerical examples D_{111}, D_{1111} we worked out up to 10 stable iterations.

Contractions with external momenta p_i (or with internal momenta q_i) I

We expect strong improvements of efficiency by using **contracted tensor integrals**

[Fleischer,TR: PLB 2011 [10]]

After having tensor reductions with basis functions I_n^D,
which are independent of the indices $i, j, k, ...$,
one may use contractions with external momenta in order to perform all the sums over $i, j, k, ...$.
This leads to a significant simplification and shortening of calculations.
Reminder:
One option was to avoid the appearance of inverse Gram determinants $1/()_5$.

Contractions with external momenta p_i (or with internal momenta q_i) II

For rank $R = 5$, e.g.:

$$I_5^{\mu\,\nu\,\lambda\rho\sigma} = \sum_{s=1}^{5} \Bigg[\sum_{i,j,k,l,m=1}^{5} q_i^{\mu} q_j^{\nu} q_k^{\lambda} q_l^{\rho} q_m^{\sigma} E_{ijklm}^{s} + \sum_{i,j,k=1}^{5} g^{[\mu\nu} q_i^{\lambda} q_j^{\rho} q_k^{\sigma]} E_{00ijk}^{s} + \sum_{i=1}^{5} g^{[\mu\nu} g^{\lambda\rho} q_i^{\sigma]} E_{0000i}^{s} \Bigg] \tag{33}$$

Contractions with external momenta I

The tensor coefficients are expressed in terms of integrals $I_{4,i\cdots}^{[d+]^l,s}$, e.g.:

$$E_{ijklm}^{s} = -\frac{1}{\binom{0}{0}_5}\left\{\left[\binom{0l}{sm}_5 n_{ijk} I_{4,ijk}^{[d+]^4,s} + (i \leftrightarrow l) + (j \leftrightarrow l) + (k \leftrightarrow l)\right] + \binom{0s}{0m}_5 n_{ijkl} I_{4,ijkl}^{[d+]^4,s}\right\}.$$

Now, in a next step, one may avoid the appearance of inverse sub-Gram determinants $()_4$.
The complete dependence on the indices i of the tensor coefficients is contained now in the pre-factors with signed minors. One can say that **the indices *decouple* from the integrals**.
As an example, we reproduce the 4-point part of

$$\begin{aligned} n_{ijkl} I_{4,ijkl}^{[d+]^4} &= \frac{\binom{0}{i}}{\binom{0}{0}}\frac{\binom{0}{j}}{\binom{0}{0}}\frac{\binom{0}{k}}{\binom{0}{0}}\frac{\binom{0}{l}}{\binom{0}{0}} d(d+1)(d+2)(d+3) I_4^{[d+]^4} \\ &+ \frac{\binom{0i}{0j}\binom{0}{k}\binom{0}{l} + \binom{0i}{0k}\binom{0}{j}\binom{0}{l} + \binom{0j}{0k}\binom{0}{i}\binom{0}{l} + \binom{0i}{0l}\binom{0}{j}\binom{0}{k} + \binom{0j}{0l}\binom{0}{i}\binom{0}{k} + \binom{0k}{0l}\binom{0}{i}\binom{0}{j}}{\binom{0}{0}^3} d(d+1) I_4^{[d+]^3} \\ &+ \frac{\binom{0i}{0l}\binom{0j}{0k} + \binom{0j}{0l}\binom{0i}{0k} + \binom{0k}{0l}\binom{0i}{0j}}{\binom{0}{0}^2} I_4^{[d+]^2} + \cdots \end{aligned} \tag{34}$$

Contractions with external momenta II

In (34), one has to understand the 4-point integrals to carry the corresponding index s and the signed minors are $\binom{0}{k} \to \binom{0s}{ks}_5$ etc.

Contractions with external momenta I

A chord is the momentum shift of an internal line due to external momenta, $D_i = (k - q_i)^2 - m_i^2 + i\epsilon$, and $q_i = (p_1 + p_2 + \cdots + p_i)$, with $q_n = 0$.

The tensor 5-point integral of rank $R = 1$ is ([6], eq. (4.6)):

$$I_5^\mu = -\sum_{i=1}^{5} q_i^\mu I_{5,i}^{[d+]} \tag{35}$$

$$= -\sum_{i=1}^{4} q_i^\mu \sum_{s=1}^{5} \frac{\binom{0i}{0s}_5}{\binom{0}{0}_5} I_4^s \tag{36}$$

This yields, when contracted with a chord,

$$q_{a\mu} I_5^\mu = -\frac{1}{\binom{0}{0}_5} \sum_{s=1}^{5} \left[\sum_{i=1}^{4} (q_a \cdot q_i) \binom{0i}{0s}_5 \right] I_4^s. \tag{37}$$

In fact, the sum over i may be performed explicitly:

Contractions with external momenta II

$$\Sigma_a^{1,s} \equiv \sum_{i=1}^{4} (q_a \cdot q_i) \binom{0s}{0i}_5 = +\frac{1}{2} \left\{ \binom{s}{0}_5 (Y_{a5} - Y_{55}) + \binom{0}{0}_5 (\delta_{as} - \delta_{5s}) \right\},$$

Contractions with external momenta I

We get immediately

$$q_{a\mu} I_5^{\mu} = -\frac{1}{\binom{0}{0}_5} \sum_{s=1}^{5} \Sigma_a^{1,s} \quad I_4^s. \tag{38}$$

Contractions with external momenta I

The tensor 5-point integral of rank $R = 2$

$$I_5^{\mu\,\nu} = \sum_{i,j=1}^{4} q_i^{\mu}\, q_j^{\nu}\, E_{ij} + g^{\mu\nu}\, E_{00}, \tag{39}$$

has the following tensor coefficients free of $1/()_5$:

$$E_{00} \;=\; -\sum_{s=1}^{5} \frac{1}{2} \frac{1}{\binom{0}{0}_5} \binom{s}{0}_5 I_4^{[d+],s}, \tag{40}$$

$$E_{ij} \;=\; \sum_{s=1}^{5} \frac{1}{\binom{0}{0}_5} \left[\binom{0i}{sj}_5 I_4^{[d+],s} + \binom{0s}{0j}_5 I_{4,i}^{[d+],s} \right]. \tag{41}$$

Contractions with external momenta I

Equation (39) yields for the contractions with chords:

$$q_{a\mu} q_{b\nu} I_5^{\mu\,\nu} = \sum_{i,j=1}^{4} (q_a \cdot q_i)(q_b \cdot q_j) E_{ij} + (q_a \cdot q_b) E_{00}. \qquad (42)$$

and finally (42) simply reads

$$\begin{aligned} q_{a\mu} q_{b\nu} I_5^{\mu\,\nu} \quad = \quad & \frac{1}{4} \sum_{s=1}^{5} \Bigg\{ \frac{\binom{s}{0}_5}{\binom{0s}{0s}_5} (\delta_{ab}\delta_{as} + \delta_{5s}) + \frac{\binom{s}{s}_5}{\binom{0s}{0s}_5} \Big[(\delta_{as} - \delta_{5s}) \left(Y_{b5} - Y_{55} \right) \\ & + \left(\delta_{bs} - \delta_{5s} \right) \left(Y_{a5} - Y_{55} \right) + \frac{\binom{s}{0}_5}{\binom{0}{0}_5} \left(Y_{a5} - Y_{55} \right) \left(Y_{b5} - Y_{55} \right) \Big] \Bigg\} I_4^{[d+],s} \\ & + \frac{1}{\binom{0}{0}_5} \sum_{s=1}^{5} \frac{\Sigma_b^{1,s}}{\binom{0s}{0s}_5} \sum_{t=1}^{5} \Sigma_a^{2,st} I_3^{st}, \end{aligned}$$

Contractions with external momenta

with

$$\Sigma_a^{2,st} \equiv \sum_{i=1}^{4} (q_a \cdot q_i) \begin{pmatrix} 0st \\ 0si \end{pmatrix}_5$$
$$= \frac{1}{2}(1-\delta_{st}) \left\{ \begin{pmatrix} ts \\ 0s \end{pmatrix}_5 (Y_{a5} - Y_{55}) + \begin{pmatrix} 0s \\ 0s \end{pmatrix}_5 (\delta_{at} - \delta_{5t}) - \begin{pmatrix} 0s \\ 0t \end{pmatrix}_5 (\delta_{as} - \delta_{5s}) \right\}$$

This has been extended also to higher ranks.
We need at most double sums, e.g.:

$$\Sigma_{ab}^{2,s} \equiv \sum_{i,j=1}^{4} (q_a \cdot q_i)(q_b \cdot q_j) \begin{pmatrix} si \\ sj \end{pmatrix}_5$$
$$= \frac{1}{2}(q_a \cdot q_b) \begin{pmatrix} s \\ s \end{pmatrix}_5 - \frac{1}{4}()_5 (\delta_{ab}\delta_{as} + \delta_{5s}), \quad (43)$$

Contractions with external momenta I

Many of the sums over signed minors, weighted with scalar products of chords are given in PLB 2011 [[10]], and an almost complete list may be obtained on request from J. Fleischer, T.R.

Modifications for 7- and higher point functions I

$$n = 6, 7, 8, \cdots$$

For details see:
Fleischer, T.Riemann PLB 2012 [21],
Fleischer, T.Riemann, Yundin, 2011 [19, 26]
Here, the Gram determinant vanishes, and also further determinants:

$$\left(\right)_n = 0 \quad , \quad n > 5 \tag{44}$$

$$\binom{0}{k}_7 = 0$$

etc.

Modifications for 7- and higher point functions II

As a result, one has to reorganize the reductions, avoiding the $1/()_n$ completely.
This may be done, and we are following here:

T. Binoth, J. Guillet, G. Heinrich, E. Pilon, C. Schubert 2005 [27]

In [27], the formalism was not worked out until numerics, and for the solutions no analytical expressions are given.

For the approach, see also in Z. Bern, L. Dixon, D. Kosower 1994 [28].)

Two examples: $n = 7, R = 2, 3$ I

In [11] we solve analytically the generalized recursions for $n \geq 6$, derived in [27]:

$$I_n^{\mu_1 \mu_2 \cdots \mu_R} = - \sum_{r=1}^{n} C_r^{\mu_1}(n) \; I_{n-1}^{\mu_2 \cdots \mu_R, r}, \tag{45}$$

where in $I_{n-1}^{\mu, \cdots, r}$ the line r is scratched.
Equation (61) of [27] will be our starting point; it contains an implicit solution for the coefficients C_j^{μ}:

$$\sum_{j=1}^{N} C_j^{\mu}(n) \; q_j^{\nu} = \frac{1}{2} g_{[4]}^{\mu\nu}. \tag{46}$$

Two examples: $n = 7, R = 2, 3$ II

The subscript [4], indicating explicitly the 4-dimensional metric tensor, will be skipped in the following.
An additional requirement according to eq. (62) in [27] has to be fulfilled by the $C_r^{\mu_1}(n)$:

$$\sum_{j=1}^{N} C_j^{\mu}(n) = 0, \tag{47}$$

The coefficients for 6-point functions are:

$$C_r^{s,\mu}(6) = \sum_{i=1}^{5} \frac{1}{\binom{0}{s}_6} \binom{0r}{si}_6 q_i^{\mu_1}, \quad s = 0 \ldots 6, \tag{48}$$

where the $\binom{0r}{si}_6$ etc. are signed minors with arbitrary s.

Two examples: $n = 7, R = 2, 3$ III

For the 7-point and 8-point functions, we found several representations, among them

$$C_r^{st,\mu}(7) = \sum_{i=1}^{6} \frac{1}{\binom{st}{st}_7} \binom{sti}{str}_7 q_i^\mu \tag{49}$$

and

$$C_r^{stu,\mu}(8) = \sum_{i=1}^{7} \frac{1}{\binom{stu}{stu}_8} \binom{stui}{stur}_8 q_i^\mu \tag{50}$$

The upper indices s, t and u stand for the redundancy of the solutions and can be freely chosen.

Contractions:

We reproduce here two 7-point examples.
The rank $R = 2, 3$ integrals become by contraction

$$q_{a,\mu} q_{b,\nu} \, I_7^{\mu\nu} = \sum_{r,t=1}^{7} K^{ab,rt} I_5^{rt}, \tag{51}$$

$$q_{a,\mu} q_{b,\nu} q_{c,\lambda} \, I_7^{\mu\nu\lambda} = \sum_{r,t,u=1}^{7} K^{abc,rtu} I_4^{rtu}, \tag{52}$$

where I_5^{rt} and I_4^{rtu} are scalar 5- and 4-point functions, arising from the 7-point function by scratching lines $r, t, \ldots$ In the general case, we have at this stage higher-dimensional integrals $I_n^{d+2l}, n = 2, \ldots, 5$, to be further reduced following the known scheme, if needed. Here, the I_5^{rt} have to be expressed by 4-point functions.

The expansion coefficients are factorizing here,

$$K^{ab,rt} = K^{a,r}\, K^{b,rt}, \tag{53}$$

$$K^{abc,rtu} = -K^{a,r}\, K^{b,rt}\, K^{c,rtu}, \tag{54}$$

and the sums over signed minors have been performed analytically:

$$K^{a,r} = \frac{1}{2}\left(\delta_{ar} - \delta_{7r}\right), \tag{55}$$

$$K^{b,rt} = \sum_{j=1}^{6} (q_b q_j) \frac{\binom{rst}{rsj}_7}{\binom{rs}{rs}_7} \equiv \frac{\Sigma_b^{1,stu}}{\binom{rs}{rs}_7} = \frac{1}{2}\left(\delta_{bt} - \delta_{7t}\right) - \frac{1}{2}\frac{\binom{rs}{ts}_7}{\binom{rs}{rs}_7}\left(\delta_{br} - \right.$$

$$K^{a,stu} = \sum_{i=1}^{6} (q_a q_i) \binom{0stu}{0sti}_7 \equiv \Sigma_a^{2,stu}$$
$$= \frac{1}{2}\left\{ \binom{stu}{st0}_7 (Y_{a7} - Y_{77}) + \binom{0st}{0st}_7 (\delta_{au} - \delta_{7u}) - \binom{0st}{0su}_7 (\delta_a\right.$$

with

$$Y_{jk} = -(q_j - q_k)^2 + m_j^2 + m_k^2. \quad (58)$$

Conventionally, $q_7 = 0$.
The sums may be found in eqns. (A.15) and (A.16) of [10]. The s is redundant and fulfils $s \neq r, b, 7$ in $K^{b,rt}$. In $K_0^{a,stu}$ it is $s, t, u = 1, \ldots 7$ with $s \neq u, t \neq u$.

Summary

- Recursive treatment of **heptagon, hexagon and pentagon tensor integrals** of rank R in terms of pentagons and boxes of rank $R-1$
- Systematic derivation of expressions which are explicitely free of inverse Gram determinants $()_5$ until pentagons of rank $R=5$
- Proper isolation of inverse Gram determinants of subdiagrams of the type $\binom{s}{s}_4$; they cannot be completely avoided
- Numerical C++ package PJFry (V. Yundin, open source) for C, C++, Mathematica, Fortran
- Perform multiple sums with signed minors and scalar products after contractions with chords or external momenta

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