Feynman Integrals
and
Mellin-Barnes representations

Computer Algebra and Particle Physics
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For longer versions of my lectures and of the exercises worked out by Janusz Gluza, with much more material included, see:

http://www-zeuthen.desy.de/~riemann/
http://prac.us.edu.pl/~gluza/ambre/
and, of course, http://www.us.edu.pl/~gluza/capp2011/

We profited much from collaborations with Krzysztof Kajda and Valery Yundin.

See also for this and related software:

http://projects.hepforge.org/mbtools/

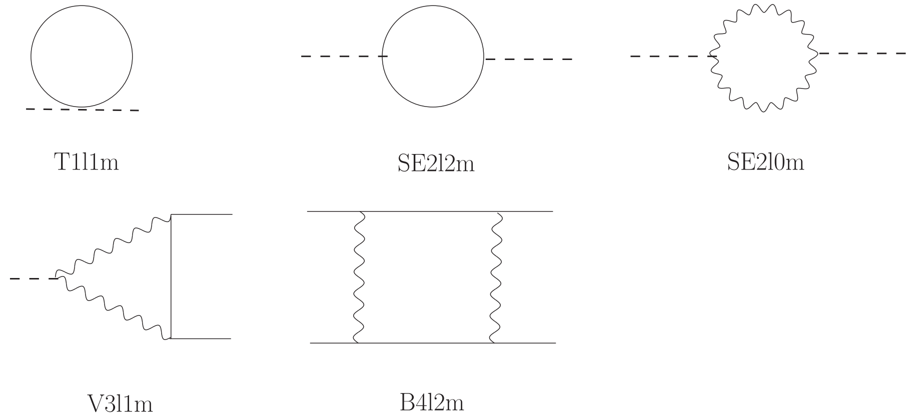
Introductory remarks

For many problems of the past, a relatively simple approach to the evaluation of Feynman integrals was sufficient:

★ Tensor reduction a la Passarino/Veltmann

★ Evaluate Feynman parameter integrals by direct integration

Typically 1-loop (massless: 2-loop), typically $2 \longrightarrow 2$ scattering (plus bremsstrahlung)
Feynman parameters may be used and by direct integration over them one gets objects like: $\frac{23}{57}$, $\zeta(3)$, $\ln(\frac{t}{s})$, $\ln(\frac{t}{s}) \cdot \ln(\frac{s}{m^2})$, $\mathrm{Li}_2(\frac{t}{s+i\epsilon})$ etc.
With more complexity of the reaction (more legs) and more perturbative accuracy (more loops), this approach appears to be not sufficiently sophisticated.



$$\mathtt{T1l1m} = \frac{1}{\epsilon} + 1 + (1 + \frac{\zeta_2}{2})\epsilon + (1 + \frac{\zeta_2}{2} - \frac{\zeta_3}{3})\epsilon^2 +$$

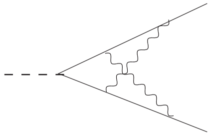
$$\mathtt{B4l2m} = [-\frac{1}{\epsilon} + \ln(-s)]\frac{2y\ln(y)}{s(1-y^2)} + c_1\epsilon + \cdots$$

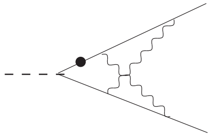
with $d = 4 - 2\epsilon$ and $m = 1$ and

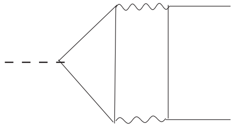
$$y = \frac{\sqrt{1-4/t}-1}{\sqrt{1-4/t}+1}$$

Figure shows so-called master integrals.

More loops

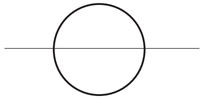

V6l4m1

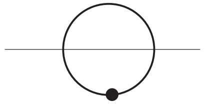

V6l4m1d

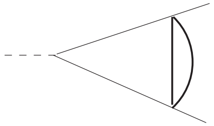

V6l4m2

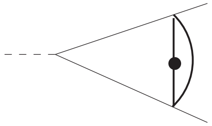
Two-loop vertex integrals with six internal lines

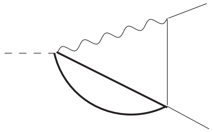
massless case: only fixed numbers and one scale factor

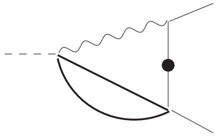

SE3l2M1m


SE3l2M1md


V4l2M2m


V4l2M2md


V4l2M1m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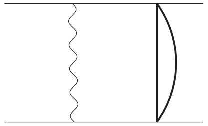
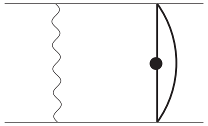
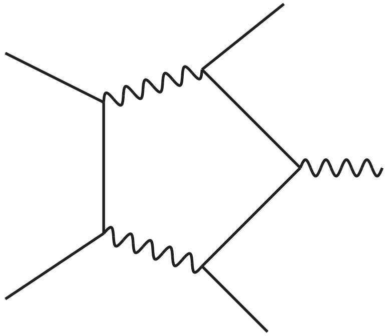

B5l2M2md

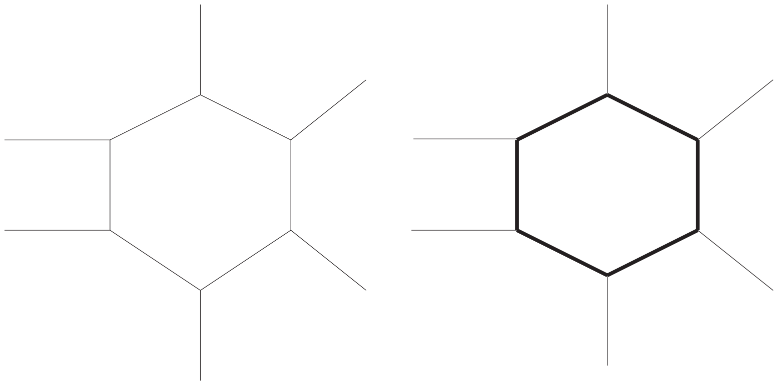

B5l2M2m

Integrals with two different mass scales m and M

More legs



Massive pentagon: 5 kinematic variables + several masses



Massless and massive hexagons: 8 kinematic variables + several masses

Variables for $2 \to 2$ scattering: 2 – i.e. box diagrams: s, t **or** s **and** $\cos\theta$

Variables for $2 \to 3$ scattering: 5 = 2 + 3 (three additional momenta of a particle)

Variables for $2 \to 4$ scattering: 8 = 5 + 3 (another three additional momenta)

What we will do here (i)

Want to evaluate some Feynman integral in momentum space:

$$\boxed{I \;=\; \frac{e^{\epsilon\gamma_E L}}{(i\pi^{d/2})^L}\int \frac{d^dk_1\ldots d^dk_L \quad k_1^{\mu}\cdots k_R^{\nu}}{D_1^{\nu_1}\ldots D_i^{\nu_i}\ldots D_N^{\nu_N}}.}$$

L $\cdots$ number of loops

n $\cdots$ external lines with momenta p_e

N $\cdots$ internal lines with momenta q_i

$$D_i \;=\; q_i^2 - m_i^2 \;=\; \left[\sum_{l=1}^{L} c_i^l\, k_l + \sum_{e=1}^{n} d_i^e\, p_e\right]^2$$

ν_i $\cdots$ index of propagator/line

μ, ν $\cdots$ tensor degrees of integral

Instrumentarium

- Just tackle the integrals needed directly by e.g. Feynman parameter integration
- Perform tensor reduction, get scalar master integrals, solve the latter
- Use integration-by-parts and related methods to determine master integrals, solve the latter
- Combine all this with solving of (a system of) differential equation(s) for the integrals
- Rewrite the (master) integrals by using Mellin-Barnes representations and try to solve the latter
  May be combined with other methods
  Advantage: one has to deal with single objects, not with systems of them

There are few masters of using Mellin-Barnes representations, among them:

- V. Smirnov – massless planar double box and many other
- B. Tausk – massless non-planar double box and others
- M. Czakon – some massive non-planar double box and many others

Using tools lets you come close to the masters.

Some of the tools are:

- MB – M. Czakon
- Ambre – J. Gluza, K. Kajda, T.R., V. Yundin
- MBresolve – 2 Smirnovs
- maybe others

What we will do here (ii)

Solve the momentum integral, get a Feynman parameter integral (a la textbook)

$$I = \frac{e^{\epsilon \gamma_E L}}{(-1)^{N_\nu}} \frac{\Gamma\left(N_\nu - \frac{d}{2}L\right)}{\Gamma(\nu_1)\ldots\Gamma(\nu_N)} \int_0^1 \prod_{j=1}^{N} dx_j \; x_j^{\nu_j-1} \delta\left(1 - \sum_{i=1}^{N} x_i\right) \frac{U(x)^{N_\nu - d(L+1)/}}{F(x)^{N_\nu - dL/2}}$$

with

$$N_\nu = \text{sum of indices}$$

$$U(x) = (\text{det } M) \; (\rightarrow 1 \text{ for } L = 1)$$

$$F(x) = (\text{det } M)\; \mu^2 = -(\text{det } M)\; J + Q\; \tilde{M}\; Q \; (\rightarrow -J + Q^2 \text{ for } L = 1)$$

M is a matrix in terms of the x_i and J, Q_μ depend on external momenta, masses and x_i

What we will do here (iii)

Mellin-Barnes formula transforms sums into products:

$$\frac{1}{[A(s)x_1^{a_1}+B(s)x_1^{b_1}x_2^{b_2}]^a} = \frac{1}{2\pi i}\int\limits_{-i\infty}^{i\infty} d\sigma [A(s)x_1^{a_1}]^\sigma [B(s)x_1^{b_1}x_2^{b_2}]^{a+\sigma}\ \frac{\Gamma(a+\sigma)\Gamma(-\sigma)}{\Gamma(a)}$$

Depending on complexity, get multi-dimensional complex path integral, and the x-dependence can be integrated out – generalization of the Beta-function:

$$\int_0^1 \prod_{j=1}^N dx_j\ x_j^{\alpha_j-1}\ \delta\left(1-\sum x_i\right) \quad = \quad \frac{\Gamma(\alpha_1)\Gamma(\alpha_2)\cdots\Gamma(\alpha_7 N)}{\Gamma\left(\alpha_1+\alpha_2+\cdots+\alpha_N\right)}$$

with coefficients α_i dependent on ν_i and on the structure of the F

Some Mathematical Preparations

We will often use, for $d = 4 - 2\epsilon$:

$$a^{\epsilon} \;=\; e^{\epsilon \ln(a)} = 1 + \ln(a)\,\epsilon + \frac{1}{2}\ln^2(a)\,\epsilon^2 + \ldots$$

The Γ-function

The Γ-function may be defined by a difference equation:

$$z\Gamma(z) - \Gamma(z+1) = 0$$

$$\begin{aligned}
\Gamma(z) &= \int_0^\infty t^{z-1} e^{-t} dt \\
\Gamma(0) &= \infty \\
\Gamma(1) &= 1 \\
\Gamma(n) &= (n-1)!, \quad n = 2, 3, \cdots
\end{aligned}$$

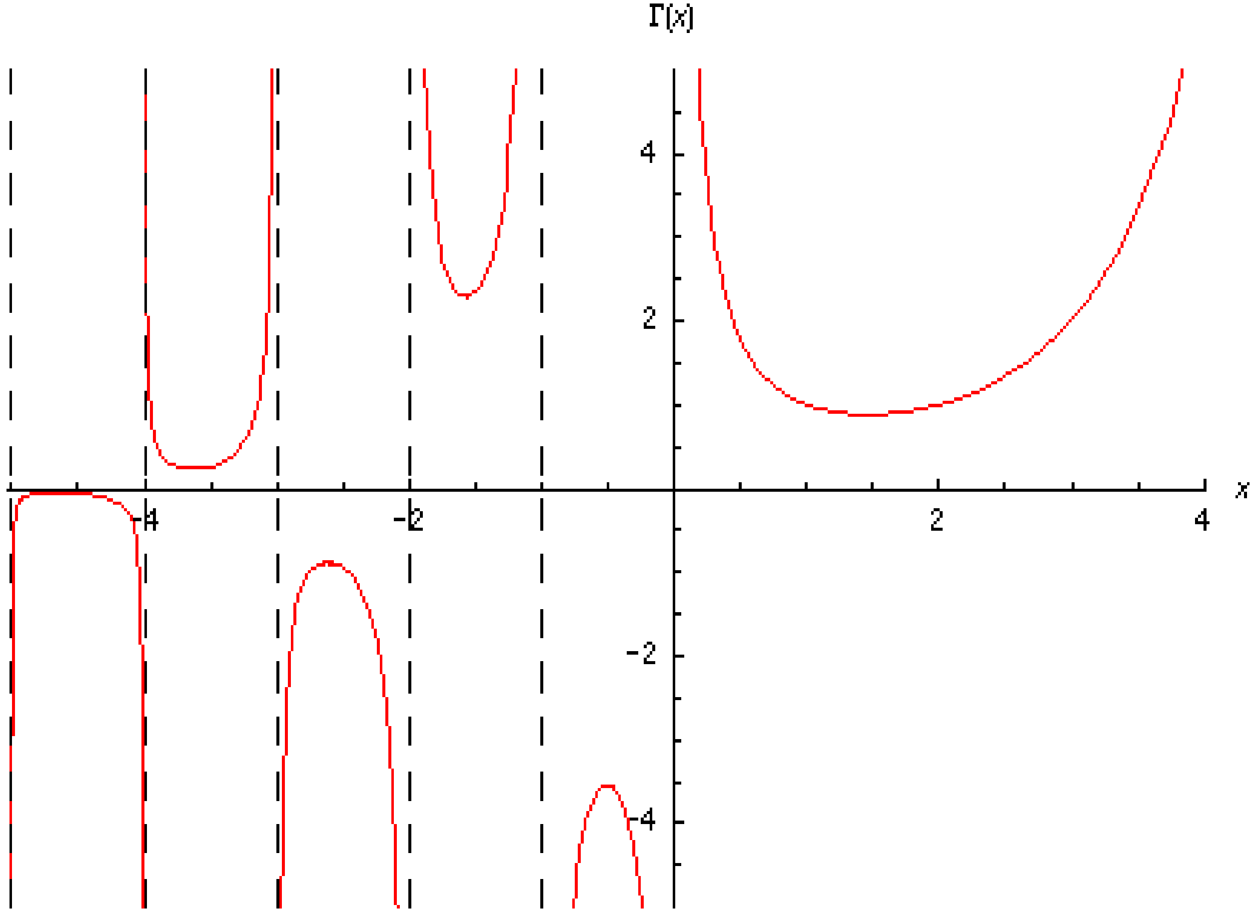
You remember that $\Gamma(z)$ has poles at $z = -n, n = 0, 1, 2, 3, \cdots$, and it is

$$\begin{aligned}
\Gamma[\epsilon] &= \frac{1}{\epsilon} - \gamma_E + \frac{1}{2}\left[{\gamma_E}^2 + \zeta(2)\right]\epsilon + \frac{1}{6}\left[-{\gamma_E}^3 - 3{\gamma_E}^2\zeta(2) - 2\zeta(3)\right]\epsilon^2 + \cdots \\
e^{\epsilon\gamma_E}\ \Gamma[\epsilon] &= \frac{1}{\epsilon} + \frac{1}{2}\zeta(2)\epsilon - \frac{1}{3}\zeta(3)\epsilon^2 + \cdots
\end{aligned}$$

For definitions of Riemann's zeta-numbers $\zeta(N)$ and the Euler constant γ_E see next slides.

Look at the singularities in the complex plane.

Figure shows the real part of Γ:



$$\text{Gamma}[-1 \pm 10i] = -4.9974\ 10^{-9} \pm 1.07847\ 10^{-8} i$$

$$\text{Gamma}[-1 \pm 100i] = 1.51438\ 10^{-71} \pm 1.27644\ 10^{-73} i \qquad (1)$$

$$\text{Gamma}[\pm 100.1] \approx \pm 10^{\pm 157} \qquad (2)$$

Just to remind you:

$$\text{Riemann zeta numbers} \quad \zeta(a) = \sum_{k=1}^{\infty} \frac{1}{k^a} \qquad \zeta(2) = \pi^2/6, \quad \zeta(3) = 1.20206, \quad \zeta(4) = \pi^4/90$$

$$\text{HarmonicNumber}[N, a] = S_a(N) = \sum_{k=1}^{N} \frac{1}{k^a} = H_{N,a} \tag{3}$$

$$\zeta(1) \to \gamma_E = \lim_{N\to\infty} \left[\sum_{k=1}^{N} \frac{1}{k^1} - \ln(N) \right] = 0.57721 \cdots \tag{4}$$

$$\text{HarmonicNumber}[N] = S_1(N) = \sum_{k=1}^{N} \frac{1}{k^1} = H_N \tag{5}$$

We will also need derivatives of $\Gamma(z)$:

$$\text{PolyGamma [z]} = \Psi(z) \equiv \text{PolyGamma [0,z]} = \frac{1}{\Gamma(z)} \frac{d}{dz} \Gamma(z)$$

At integer values:

$$\Psi(N+1) = \sum_{k=1}^{N} \frac{1}{k} - \gamma_E = S_1(N) - \gamma_E$$

Cauchy Theorem and Residues

An integral over an anti-clockwise directed closed path C is:

$$\oint F(z)dz \quad = \quad 2\pi i \sum_{z=z_i} \mathrm{Res}[F(z)]$$

where the residues $\mathrm{Res}[F(z)]|_{z=z_i}$ are coefficients a^i_{-1} of the Laurent series of $F(z)$ around z_i:

$$\begin{aligned} F(z) = \sum_{n=-N}^{\infty} a^i_n (z-z_i)^n \quad &= \quad \frac{a^i_{-N}}{(z-z_i)^N} + \cdots + \frac{a^i_{-1}}{(z-z_i)} + a^i_0 + \cdots \\ \mathrm{Res}[F(z)]|_{z=z_i} \quad &= \quad a^i_{-1} \end{aligned} \tag{6}$$

If $G(z)$ has a Taylor expansion around z_0 , then it is:

$$\mathrm{Res}[G(z)\ F(z)]|_{z=z_i} = \sum_{n=1}^{N} \frac{a^i_{-n}}{k!} \frac{d^n}{dz^n} G(z)|_{z=z_i}$$

Due to this property, we need for applications not only $\Gamma(z)$, but also its derivatives.

Some residues with $\Gamma(z)$

$$\begin{aligned}
\Psi(z) &= \text{PolyGamma}[z] = \text{PolyGamma}[0, z] \\
\text{Residue}[F[z]\Gamma[z], \{z, -n\}] &= \frac{(-1)^n}{n!} F[-n] \qquad (7) \\
\text{Residue}[F[z]\Gamma[z]^2, \{z, -n\}] &= \frac{2\text{PolyGamma}[n+1]F[-n] + F'[-n]}{(n!)^2} \qquad (8)
\end{aligned}$$

Integrals + Some sums `Mathematica` can do

$$\begin{aligned} & \oint_{-1/3-9i}^{-1/3+9i} dz\Gamma[z] &=& \quad (-i)\ 3.97173 \\ \text{close path to the left :} \quad & 2\pi \mathrm{i} \sum_{\mathrm{n}=1}^{\infty} \frac{(-1)^{\mathrm{n}}}{\mathrm{n}!} = (2\pi \mathrm{i}) \frac{1-\mathrm{e}}{\mathrm{e}} &=& \quad (-i)\ 3.97173 \end{aligned} \tag{9}$$

while

$$\text{close path to the right :} \quad (-1) * 2\pi \mathrm{i} \sum_{\mathrm{n}=0}^{0} \frac{(-1)^{\mathrm{n}}}{\mathrm{n}!} = (2\pi \mathrm{i}) \quad \neq \quad (-i)\ 3.97173$$

Sum[s^(n) Gamma[n + 1]^3/(n!Gamma[2 + 2n]), n, 0, Infinity] =

. (4*ArcSin[Sqrt[s]/2])/(Sqrt[4 - s]*Sqrt[s])

Sum[s^(n) PolyGamma[0, n + 1], n, 0, Infinity] =

. (EulerGamma + Log[1 - s])/(-1 + s)

The above sums were done with Mathematica 5.2.

Later Mathematica versions are much more powerful.

Now come back to L-loop n-point Feynman Integrals of tensor rank R with N internal lines

- Internal loop momenta are $k_l, \quad l = 1 \cdots L$
- Propagators have mass m_i and momentum $q_i, \quad i = 1 \cdots N$ and indices ν_i – see $G(X)$
- External legs have momentum $p_e, \quad e = 1 \cdots n$, with $p_e^2 = M_e^2$

Feynman integrals have the following general form:

$$\boxed{G(X) \quad = \quad \frac{e^{\epsilon \gamma_E L}}{(i\pi^{d/2})^L} \int \frac{d^d k_1 \ldots d^d k_L \quad X(k_{l_1}, \cdots, k_{l_R})}{D_1^{\nu_1} \ldots D_i^{\nu_i} \ldots D_N^{\nu_N}}.}$$

The N propagators are: $$\boxed{D_i \quad = \quad q_i^2 - m_i^2 \quad = [\sum_{l=1}^{L} c_i^l k_l + \sum_{e=1}^{n} d_i^e p_e]^2 - m_i^2}$$

The numerator X may contain a tensor structure (see later for more on that):

$$X(k_{l_1}, \cdots, k_{l_R}) \quad = \quad (k_{l_1} P_{e_1}) \cdots (k_{l_R} P_{e_R}) \quad = \quad (P_{e_1}^{\alpha_1} \cdots P_{e_R}^{\alpha_R}) \, (k_{l_1}^{\alpha_1} \cdots k_{l_R}^{\alpha_R})$$

Tensor integrals

Tensor integrals appear naturally in Feynman diagrams, due to

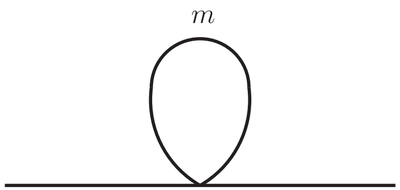
- **fermion propagators**
- **non-abelian triple-boson vertices**
- **boson propagators in R_ξ gauges and unitary gauge**

Example: Fermionic vacuum polarization

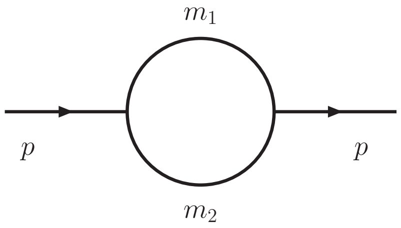
$$\begin{aligned}\Pi^{\alpha\beta} &\sim \frac{1}{(i\pi^{d/2})}\int d^d k \,\mathrm{Tr}\left[\frac{[\gamma k + m_1]}{D_1}\gamma^\beta \frac{[\gamma(k+p_1)+m_2]}{D_2}\gamma^\alpha\right] \\ &\sim \frac{1}{(i\pi^{d/2})}\int \frac{d^d k}{D_1 D_2}\left[(m_1 m_2 - k^2 - kp_1)g^{\alpha\beta} + 2k^\alpha k^\beta + k^\alpha p_1^\beta + p_1^\alpha k^\beta\right]\end{aligned}$$

So, one needs also efficient ways to evaluate tensor integrals – see later

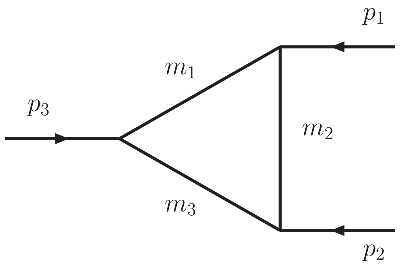
Simple examples of scalar integrals



$$A_0 = \frac{1}{(i\pi^{d/2})} \int \frac{d^d k}{D_1} \quad \to \mathrm{UV-divergent}: \quad \sim \frac{d^4k}{k^2}$$



$$B_0 = \frac{1}{(i\pi^{d/2})} \int \frac{d^d k}{D_1 D_2} \quad \to \mathrm{UV-divergent} \quad \sim \frac{d^4k}{k^4}$$



$$C_0 = \frac{1}{(i\pi^{d/2})} \int \frac{d^d k}{D_1 D_2 D_3} \quad \to \mathrm{UV-finite} \quad \sim \frac{d^4k}{k^6}$$

Dependent on conventions, where k starts to run in the loop, it is:

$$\begin{aligned} D_1 &= k^2 - m_1^2 \\ D_2 &= (k+p_1)^2 - m_2^2 \\ D_3 &= (k+p_1+p_2)^2 - m_3^2 \end{aligned}$$

Evaluate Feynman integrals

There are two strategies to solve a Feynman integral:

- Reduction

  Express the integral with the aid of recurrence relations by other, known integrals.
  These are then the Master Integrals.

- Direct evaluation

Introduce Feynman parameters

$$\frac{1}{D_1^{\nu_1} D_2^{\nu_2} \dots D_N^{\nu_N}} = \frac{\Gamma(\nu_1 + \dots + \nu_N)}{\Gamma(\nu_1) \dots \Gamma(\nu_N)} \int_0^1 dx_1 \dots \int_0^1 dx_N \frac{x_1^{\nu_1 - 1} \dots x_N^{\nu_N - 1} \delta(1 - x_1 \dots - x_N)}{(x_1 D_1 + \dots + x_N D_N)^{N_\nu}},$$

with $N_\nu = \nu_1 + \dots \nu_N$.

The denominator of G contains, after introduction of Feynman parameters x_i, the momentum dependent function m^2 with index-exponent N_ν:

$$(m^2)^{-(\nu_1 + \dots + \nu_N)} = (x_1 D_1 + \dots + x_N D_N)^{-N_\nu} = (k_i M_{ij} k_j - 2 Q_j k_j + J)^{-N_\nu}$$

Here M is an (LxL)-matrix, $Q = Q(x_i, p_e)$ an L-vector and $J = J(x_i, m_i^2, p_{e_j} p_{e_l})$.
M, Q, J are linear in x_i. The momentum integration is now simple:
Shift the momenta k such that m^2 has no linear term in $\bar{k}$:

$$\begin{aligned} k &= \bar{k} + (M^{-1}) Q, \\ m^2 &= \bar{k} M \bar{k} - Q M^{-1} Q + J. \end{aligned}$$

Remember: $M_{1-\text{loop}} = 1$, in general:

$$M^{-1} = \frac{1}{(\text{det } M)} \tilde{M},$$

where $\tilde{M}$ is the transposed matrix to M. The shift leaves the integral unchanged.

The shift leaves the integral unchanged (rename $\bar{k} \to k$):

$$G(1) \quad = \quad \int \frac{Dk_1 \ldots Dk_L}{(kMk + J - QM^{-1}Q)^{N_\nu}}.$$

Go Euclidean: Rotate now the $k^0 \to iK_E^0$ with $k^2 \to -k_E^2$ (and again rename $k^E \to k$):

$$G(1) \to (i)^L \int \frac{Dk_1^E \ldots Dk_L^E}{(-k^E M k^E + J - QM^{-1}Q)^{N_\nu}} \quad = \quad (-1)^{N_\nu} (i)^L \int \frac{Dk_1 \ldots Dk_L}{[kMk - (J - QM^{-1}Q)]^{N_\nu}}.$$

Call

$$\mu^2(x) \quad = \quad -(J - QM^{-1}Q)$$

and get

$$G(1) \quad = \quad (-1)^{N_\nu} (i)^L \int \frac{Dk_1 \ldots Dk_L}{(kMk + \mu^2)^{N_\nu}}.$$

For 1-loop integrals it is $L = 1, M = 1$ - and we will use nearly only those - we are ready to do the k-integration.

Additional step for L-loop integrals

For L-loops go on and now diagonalize the matrix M by a rotation:

$$\begin{aligned} k \to k'(x) &= V(x)\ k, \\ kMk &= k' M_{diag} k' \\ &\to \sum \alpha_i(x) k_i^2(x), \\ M_{diag}(x) &= (V^{-1})^+ M V^{-1} \quad = (\alpha_1, \ldots, \alpha_L). \end{aligned}$$

This leaves both the integration measure and the integral invariant:

$$G(1) \quad = \quad (-1)^{N_\nu} (i)^L \int \frac{Dk_1 \ldots Dk_L}{(\sum_i \alpha_i k_i^2 + \mu^2)^{N_\nu}}.$$

Rescale now the k_i,

$$\bar{k}_i \quad = \quad \sqrt{\alpha_i} k_i,$$

with

$$\begin{aligned} d^d k_i &= (\alpha_i)^{-d/2} d^d \bar{k}_i, \qquad (10) \\ \prod_{i=1}^{L} \alpha_i &= \text{det } M, \end{aligned}$$

and get the Euclidean integral to be calculated (and rename $\bar{k} \to k$):

$$G(1) \quad = \quad (-1)^{N_\nu} (i)^L (\text{det } M)^{-d/2} \int \frac{Dk_1 \ldots Dk_L}{(k_1^2 + \ldots + k_L^2 + \mu^2)^{N_\nu}} .$$

Use now (remembering that $Dk = dk/(i\pi^{d/2})$):

$$\begin{aligned} i^L \int \frac{Dk_1 \ldots Dk_L}{(k_1^2 + \ldots + k_L^2 + \mu^2)^{N_\nu}} &= \frac{\Gamma\left(N_\nu - \frac{d}{2} L\right)}{\Gamma\left(N_\nu\right)} \frac{1}{(\mu^2)^{N_\nu - dL/2}}, \\ i^L \int \frac{Dk_1 \ldots Dk_L \; k_1^2}{(k_1^2 + \ldots + k_L^2 + \mu^2)^{N_\nu}} &= \frac{d}{2} \frac{\Gamma\left(N_\nu - \frac{d}{2} L - 1\right)}{\Gamma\left(N_\nu\right)} \frac{1}{(\mu^2)^{N_\nu - dL/2 - 1}}. \end{aligned} \tag{11}$$

These formulae follow for $L = 1$ immediately from any textbook.

See 'Mathematical Interlude'.

For $L > 1$, get it iteratively, with setting $(k_1^2 + k_2^2 + m^2)^N = (k_1^2 + M^2)^N$, $M^2 = k_2^2 + m^2$, etc.

Finally, one gets for **Scalar integrals:**

$$G(1) \quad = \quad (-1)^{N_\nu} \frac{\Gamma\left(N_\nu - \frac{d}{2}L\right)}{\Gamma(\nu_1)\dots\Gamma(\nu_N)} \int_0^1 \prod_{j=1}^{N} dx_j \; x_j^{\nu_j - 1} \delta\left(1 - \sum_{i=1}^{N} x_i\right) \frac{(\text{det } M)^{-d/2}}{(\mu^2)^{N_\nu - dL/2}},$$

or

$$G(1) \quad = \quad (-1)^{N_\nu} \frac{\Gamma\left(N_\nu - \frac{d}{2}L\right)}{\Gamma(\nu_1)\dots\Gamma(\nu_N)} \int_0^1 \prod_{j=1}^{N} dx_j \; x_j^{\nu_j - 1} \delta\left(1 - \sum_{i=1}^{N} x_i\right) \frac{U(x)^{N_\nu - d(L+1)/2}}{F(x)^{N_\nu - dL/2}}$$

with

$$\begin{aligned} U(x) \quad &= \quad (\text{det } M) \; (\to 1 \text{ for } L = 1) \\ F(x) \quad &= \quad (\text{det } M)\; \mu^2 \quad = \quad -\,(\text{det } M)\; J + Q\; \tilde{M}\; Q \; (\to -J + Q^2 \text{ for } L = 1) \end{aligned}$$

Trick for one-loop functions:
$U = \text{det } M = 1 = \sum x_i$ and so U 'disappears' and the construct $F_1(x)$ is bilinear in $x_i x_j$:

$$F_1(x) = -J(\sum x_i) + Q^2 = \sum A_{ij} x_i x_j .$$

The vector integrals differ by some numerator $k_i p_e$ and thus there is a single shift in the integrand

$$k \to \bar{k} + U(x)^{-1} \tilde{M} Q$$

the $\int d^d\bar{k}\ \bar{k}/(\bar{k}^2 + \mu^2) \to 0$, and no further changes:

$$G(k_{1\alpha}) = (-1)^{N_\nu} \frac{\Gamma\left(N_\nu - \frac{d}{2}L\right)}{\Gamma(\nu_1)\ldots\Gamma(\nu_N)} \int_0^1 \prod_{j=1}^{N} dx_j\ x_j^{\nu_j - 1} \delta\left(1 - \sum_{i=1}^{N} x_i\right) \frac{U(x)^{N_\nu - d(L+1)/2 - 1}}{F(x)^{N_\nu - dL/2}} \left[\sum_l \tilde{M}_{1l} Q_l\right]_\alpha ,$$

Tensor integrals also follow this scheme: The x-integrals have the same struckture like the scalar ones.

Often, the 1-loop presentation may be used for a sequential treatment of an L-loop integral by iterating the basic representation L times.

Beware: This does NOT work for all cases. Keyword: Non-planar diagrams

Examples for one-loop F-polynomials

One-loop vertex:

$$F(t,m^2) = m^2(x_1+x_2)^2 + [-t]x_1x_2$$

one-loop box:

$$F(s,t,m^2) = m^2(x_1+x_2)^2 + [-t]x_1x_2 + [-s]x_3x_4$$

one-loop pentagon:

$$F(s,t,t',v_1,v_2,m^2) = m^2(x_1+x_3+x_4)^2 + [-t]x_1x_3 + [-t']x_1x_4 + [-s]x_2x_5 + [-v_1]x_3x_5 + [-v_2]x_2x_4$$

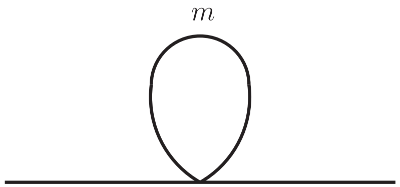
Euclidean kinematics:

s,t,t',v_1,v_2 are negative, so that F is positive semi-definite.

Otherwise, analytic continuations have to be carefully performed if needed, having in mind e.g. that

$$s \to s + i\epsilon$$

The Tadpole $A_0(m)$



$$T1l1m[a] \;=\; A_0 \;=\; \frac{e^{\epsilon\gamma_E}}{(i\pi^{d/2})}\int \frac{d^dk}{(k^2-m^2)^a} \qquad \to \mathrm{UV-divergent}$$

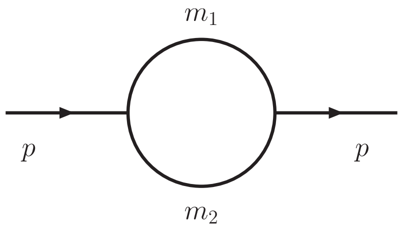
With our general formulae we get, in the 1-dimensional Feynman parameter integral, for the numerator

$$\begin{aligned} N &= (k^2-m^2)x_1 \equiv k^2+J \\ F &= m^2x_1 \equiv m^2x_1^2 \end{aligned} \tag{12}$$

and thus

$$\begin{aligned} T1l1m[a] &= (-1)^a e^{\epsilon\gamma_E}\frac{\Gamma[a-d/2]}{\Gamma[a]}\int_0^1 dx x^{a-1}\delta[1-x]\frac{1}{F^{a-d/2}} \\ &= (-1)^a e^{\epsilon\gamma_E}(m^2)^{2-a-\epsilon}\frac{\Gamma[a-2+\epsilon]}{\Gamma[a]} \\ &\to -e^{\epsilon\gamma_E}\Gamma[-1+\epsilon] \quad \mathrm{for\ a=1,\ \ m=1} \\ &= \frac{1}{\epsilon}+1+\left(1+\frac{\zeta_2}{2}\right)\epsilon+\left(1+\frac{\zeta_2}{2}-\frac{\zeta_3}{3}\right)\epsilon^2+\cdots \end{aligned} \tag{13}$$

The Self-energy $B_0(s, m_1, m_2)$



$$SE2l = B_0[s, m_1, m_2] = (2\sqrt{\pi}\mu)^{4-d} \frac{e^{\epsilon\gamma_E}}{(i\pi^{d/2})} \int \frac{d^d k}{[k^2 - m^2][(k+p)^2 - m_2^2]}$$

The $SE2l$ is UV-divergent and the corresponding F-function is:

$$F[s, m_1, m_2] = m_1^2 x_1^2 + m_2^2 x_2^2 + [-s + m_1^2 + m_2^2] x_1 x_2$$

and for special cases:

$$\begin{aligned} F[s, m_1, 0] &= m_1^2 x_1^2 + [-s + m_1^2] x_1 x_2 \\ F[s, m_1, m_1] &= m_1^2 (x_1 + x_2)^2 + [-s] x_1 x_2 \\ F[s, 0, 0] &= [-s] x_1 x_2 \end{aligned} \tag{14}$$

The 'conventional' Feynman parameter integral is 1-dimensional because $x_2 \equiv 1 - x_1$:

$$F(x) = -sx(1-x) + m_2^2(1-x) + m_1^2 x \equiv -s(x - x_a)(x - x_b)$$

The result is of logarithmic type for the constant term in ϵ:

$$\begin{aligned} B_0[s,m_1,m_2] &= (4\pi\mu^2)^\epsilon \; e^{\epsilon\gamma_E} \; \frac{\Gamma(1+\epsilon)}{\epsilon} \int_0^1 \frac{dx}{F(x)^\epsilon} \\ &= \frac{1}{\epsilon} - \int_0^1 dx \ln\left(\frac{F(x)}{4\pi\mu^2}\right) \\ &\quad + \epsilon \; \left\{ \frac{\zeta_2}{2} + \frac{1}{2}\int_0^1 dx \ln^2\left(\frac{F(x)}{4\pi\mu^2}\right) \right\} + \mathcal{O}(\epsilon^2). \end{aligned} \tag{15}$$

Here we used the expansion:

$$e^{\epsilon\gamma_E}\Gamma(1+\epsilon) = 1 + \frac{\zeta_2}{2}\epsilon^2 - \frac{\zeta_3}{3}\epsilon^3 \cdots$$

When using `LoopTools`**, the corresponding call returns exactly the constant term of B_0 in ϵ (with use of $e^{\epsilon\gamma_E} = 1 + \epsilon\gamma_E + \cdots \to 1$):**

$$B_0^{(0)}(s, m_1^2, m_2^2) \quad = \quad \texttt{b0}(\texttt{s}, \texttt{am12}, \texttt{am22})$$

For $4\pi\mu^2 \to 1$ B_0 looks quite compact:

$$B_0(s,m_1,m_2) \quad = \quad \frac{1}{\epsilon} - \int_0^1 dx \;\; \ln[F(x)] + \frac{\epsilon}{2}\left[\zeta_2 + \int_0^1 dx \ln^2[F(x)]\right] + \cdots$$

Explicitly, one has to integrate

$$\begin{aligned} \ln[F(x)] &= \ln[-s(x-x_a)(x-x_b)] \\ \ln^2[F(x)] &= \ln^2[-s(x-x_a)(x-x_b)] \end{aligned} \tag{16}$$

So we will need the integrals:

$$\int dx_0^1 \{\ln(x-x_a),\ ln(x-x_a)ln(x-x_b)\}$$

which is trivial, together with some complex algebra rules how to handle complex arguments of logarithms with

$$s \to s + i\epsilon$$

wherever needed.

For the case $m_1 = m_2 = 1$, one gets for the first terms in ϵ:

$$\begin{aligned} B_0[s,1,1] &= \frac{1}{\epsilon} + 2 + \frac{1+y}{1-y} H(0,y), \\ H(0,y) &= \ln(y). \end{aligned} \tag{17}$$

The $H(0,y)$ is a harmonic polylogarithmic function, and

$$\begin{aligned} y &= \frac{\sqrt{-s+4} - \sqrt{-s}}{\sqrt{-s+4} + \sqrt{-s}} \\ s &= -\frac{(1-x)^2}{x} \end{aligned} \tag{18}$$

The other case treated later again is $m_1 = 0, m_2 = m$:

$$B_0[s, m^2, 0] = \frac{1}{\epsilon} + 2 + \frac{1 - s/m^2}{s/m^2} \ln(1 - s/m^2)$$

Now using Mellin-Barnes Representations

Perform the x-integrations

Find an as-general-as-possible general formula

Make it ready for algorithmic analytical and/or numerical evaluation

Computer codes:

- Ambre.m – Derive Mellin-Barnes representations for Feynman integrals *[Gluza:2007rt]*
- MB.m – Find an ϵ-expansion and evaluate numerically in Euclidean region *[Czakon:2005rk]*

Integrating the Feynman parameters – get MB-Integrals

We derived:

$$\begin{aligned} SE2l1m = B_0(s,m,0) &= e^{\epsilon\gamma_E}\ \Gamma(\epsilon)\int_0^1 dx_1 dx_2 \frac{\delta(1-x_1-x_2)}{F(x)^{\epsilon}} \\ V3l2m = C_0(s,m,m,0) &= e^{\epsilon\gamma_E}\ \Gamma(1+\epsilon)\int_0^1 dx_1 dx_2 dx_3 \frac{\delta(1-x_1-x_2-x_3)}{F(x)^{1+\epsilon}} \qquad (19) \end{aligned}$$

and

$$\begin{aligned} F_{SE2l1m} &= m^2x_1^2 + (-s+m^2)x_1x_2 \\ F_{V3l2m} &= m^2(x_1+x_2)^2 + (-s)x_1x_2 \qquad (20) \end{aligned}$$

We want to apply now:

$$\int_0^1 \prod_{j=1}^{N} dx_j\ x_j^{\alpha_j-1}\ \delta\left(1-\sum x_i\right) = \frac{\Gamma(\alpha_1)\Gamma(\alpha_2)\cdots\Gamma(\alpha_7N)}{\Gamma\left(\alpha_1+\alpha_2+\cdots+\alpha_N\right)}$$

with coefficients α_i dependent on ν_i and on the structure of the F

See in a minute:
For this, we have to apply one or several MB-integrals here.

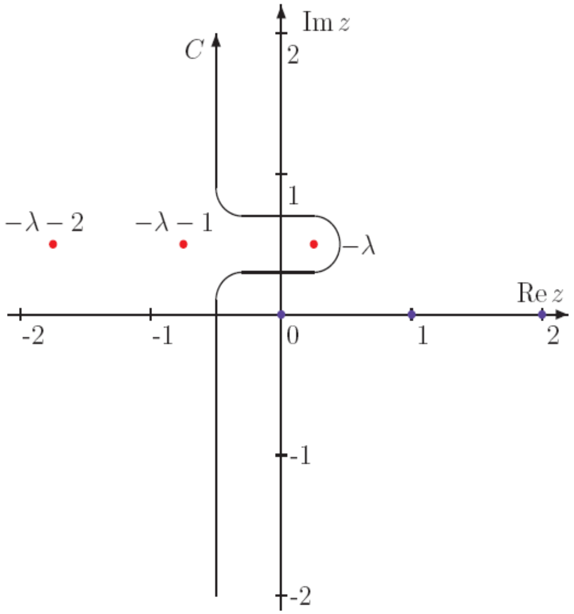
$$\int_0^1 \prod_{j=1}^{N} dx_j \; x_j^{\alpha_j-1} \; \delta\left(1-\sum_{i=1}^{N} x_i\right) \quad = \quad \frac{\prod_{i=1}^{N}\Gamma(\alpha_i)}{\Gamma\left(\sum_{i=1}^{N}\alpha_i\right)}$$

Simplest cases:

$$\begin{aligned} \int_0^1 dx_1 \; x_1^{\alpha_1-1} \; \delta\left(1-x_1\right) \quad &= \quad 1 \\ \int_0^1 \prod_{j=1}^{2} dx_j \; x_j^{\alpha_j-1} \; \delta\left(1-\sum_{i=1}^{N} x_i\right) = \int_0^1 dx_1 x_1^{\alpha_1-1}(1-x_1)^{\alpha_2-1} \quad &= \quad B(\alpha_1,\alpha_2) \\ &= \quad \frac{\Gamma(\alpha_1)\Gamma(\alpha_2)}{\Gamma\left(\alpha_1+\alpha_2\right)} \end{aligned}$$

Here we want to go:

$$\frac{1}{(A+B)^{\lambda}} = \frac{1}{\Gamma(\lambda)}\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} dz \Gamma(\lambda+z)\Gamma(-z)\frac{B^{z}}{A^{\lambda+z}}$$

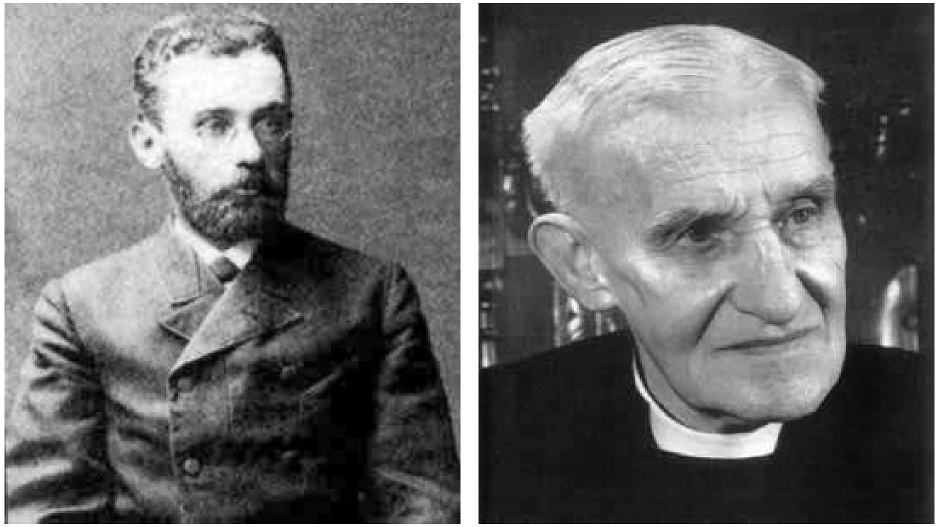


The integration path separates poles of $\Gamma[\lambda+z]$ and $\Gamma[-z]$.

The formula looks a bit unusual to loop people, but for persons with a mathematical background it is common knowledge.

One might well assume that these two gentlemen did not dream of so heavy use of their results in basic research $\cdots$

Mellin, Robert, Hjalmar, 1854-1933
Barnes, Ernest, William, 1874-1953



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Barnes' contour integrals for the hypergeometric function

Exact proof and further reading: Whittaker & Watson (CUP 1965) 14.5 - 14.52, pp. 286-290

Consider

$$F(z) \quad = \quad \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} d\sigma (-z)^{\sigma} \frac{\Gamma(a+\sigma)\Gamma(b+\sigma)\Gamma(-\sigma)}{\Gamma(c+\sigma)}$$

where $|\mathrm{arg}(-\mathrm{z})| < \pi$ (i.e. $(-z)$ is not on the neg. real axis) and the path is such that it separates the poles of $\Gamma(a+\sigma)\Gamma(b+\sigma)$ from the poles of $\Gamma(-\sigma)$.
$1/\Gamma(c+\sigma)$ **has no pole.**
Assume $a \neq -n$ and $b \neq -n, n = 0, 1, 2, \cdots$ so that the contour can be drawn.

The poles of $\Gamma(\sigma)$ are at $\sigma = -n, n = 1, 2, \cdots$, and it is:

```
Residue[ F[s] Gamma[-s] , {s,n} ]  =  (-1)^n/n!  F(n)
```

Closing the path to the right gives then, by Cauchy's theorem, for $|z| < 1$ the

hypergeometric function $_2F_1(a,b,c,z)$ (for proof see textbook):

$$
\begin{aligned}
\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} d\sigma(-z)^{\sigma}\frac{\Gamma(a+\sigma)\Gamma(b+\sigma)\Gamma(-\sigma)}{\Gamma(c+\sigma)} &= \sum_{n=0}^{N\to\infty}\frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)}\frac{z^n}{n!} \\
&= \frac{\Gamma(a)\Gamma(b)}{\Gamma(c)} \quad {}_2F_1(a,b,c,z)
\end{aligned}
$$

The continuation of the hypergeometric series for $|z|>1$ is made using the intermediate formula

$$
\begin{aligned}
F(z) &= \sum_{n=0}^{\infty}\frac{\Gamma(a+n)\Gamma(1-c+a+n)\sin[(c-a-n)\pi]}{\Gamma(1+n)\Gamma(1-a+b+n)\cos(n\pi)\sin[(b-a-n)\pi]}(-z)^{-a-n} \\
&+\sum_{n=0}^{\infty}\frac{\Gamma(b+n)\Gamma(1-c+b+n)\sin[(c-b-n)\pi]}{\Gamma(1+n)\Gamma(1-a+b+n)\cos(n\pi)\sin[(a-b-n)\pi]}(-z)^{-b-n}
\end{aligned}
$$

and yields

$$
\begin{aligned}
\frac{\Gamma(a)\Gamma(b)}{\Gamma(c)} \quad {}_2F_1(a,b,c,z) &= \frac{\Gamma(a)\Gamma(a-b)}{\Gamma(a-c)}(-z)^{-a} \quad {}_2F_1(a,1-c+a,1-b+ac,z^{-1}) \\
&+\frac{\Gamma(b)\Gamma(b-a)}{\Gamma(b-c)}(-z)^{-b} \quad {}_2F_1(b,1-c+b,1-a+b,z^{-1})
\end{aligned}
$$

Corollary I

Putting $b = c$, we see that

$$
\begin{aligned}
{}_2F_1(a,b,b,z) &= \sum_{n=0}^{\infty} \frac{\Gamma(a+n)}{\Gamma(a)} \frac{z^n}{n!} \\
&= \frac{1}{(1-z)^a} = \frac{1}{2\pi i\ \Gamma(a)} \int_{-i\infty}^{+i\infty} d\sigma\ (-z)^{\sigma}\ \Gamma(a+\sigma)\Gamma(-\sigma)
\end{aligned}
$$

This allows to replace sum by product:

$$
\frac{1}{(A+B)^a} = \frac{1}{B^a[1-(-A/B)]^a} = \frac{1}{2\pi i \Gamma(a)} \int\limits_{-i\infty}^{i\infty} d\sigma A^{\sigma}\ B^{-\sigma-a}\ \Gamma(a+\sigma)\Gamma(-\sigma)
$$

Barnes' lemma

If the path of integration is curved so that the poles of $\Gamma(c-\sigma)\Gamma(d-\sigma)$ lie on the right of the path and the poles of $\Gamma(a+\sigma)\Gamma(b+\sigma)$ lie on the left, then

$$\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} d\sigma \Gamma(a+\sigma)\Gamma(b+\sigma)\Gamma(c-\sigma)\Gamma(d-\sigma) \quad = \quad \frac{\Gamma(a+c)\Gamma(a+d)\Gamma(b+c)\Gamma(b+d)}{\Gamma(a+b+c+d)}$$

It is supposed that a, b, c, d are such that no pole of the first set coincides with any pole of the second set.

Scetch of proof: Close contour by semicircle C to the right of imaginary axis. The integral exists and $\int_C$ vanishes when $\Re(a+b+c+d-1)<0$. Take sum of residues of the integrand at poles of $\Gamma(c-\sigma)\Gamma(d-\sigma)$. The double sum leads to two hypergeometric functions, expressible by ratios of Γ-functions, this in turn by combinations of sin, may be simplifies finally to the r.h.s.

Analytical continuation: The relation is proved when $\Re(a+b+c+d-1)<0$. Both sides are analytical functions of e.g. a. So the relation remains true for all values of a, b, c, d for which none of the poles of $\Gamma(a+\sigma)\Gamma(b+\sigma)$, as a function of σ, coincide with any of the poles of $\Gamma(c-\sigma)\Gamma(d-\sigma)$.

Corollary II Any real shift k: $\sigma+k, a-k, b-k, c+k, d+k$ together with $\int_{-k-i\infty}^{-k+i\infty}$ leaves the result true.

How can the Mellin-Barnes formula be made useful in the context of Feynman integrals?

- Apply corollary I to propagators and get:

$$\frac{1}{(p^2-m^2)^a} = \frac{1}{2\pi i\ \Gamma(a)} \int\limits_{-i\infty}^{i\infty} d\sigma \frac{(-m^2)^\sigma}{(p^2)^{a+\sigma}}\ \Gamma(a+\sigma)\Gamma(-\sigma)$$

  which transforms a massive propagator to a massless one (with index a of the line changed to $(a+\sigma)$).

- Apply corollary I after introduction of Feynman parameters and after the momentum integration to the resulting F- and U-forms, in order to get a single monomial in the x_i, which allows the integration over the x_i:

$$\frac{1}{[A(s)x_1^{a_1}+B(s)x_1^{b_1}x_2^{b_2}]^a} = \frac{1}{2\pi i\ \Gamma(a)} \int\limits_{-i\infty}^{i\infty} d\sigma [A(s)x_1^{a_1}]^\sigma [B(s)x_1^{b_1}x_2^{b_2}]^{a+\sigma}\ \Gamma(a+\sigma)\Gamma(-\sigma)$$

Both methods leave Mellin-Barnes (MB-) integrals to be performed afterwards.

A short remark on history

- N. Usyukina, 1975: "ON A REPRESENTATION FOR THREE POINT FUNCTION", Teor. Mat. Fiz. 22;
  a finite massless off-shell 3-point 1-loop function represented by 2-dimensional MB-integral
- E. Boos, A. Davydychev, 1990: "A Method of evaluating massive Feynman integrals", Theor. Math. Phys. 89 (1991);
  N-point 1-loop functions represented by n-dimensional MB-integral
- V. Smirnov, 1999: "Analytical result for dimensionally regularized massless on-shell double box", Phys. Lett. B460 (1999);
  treat UV and IR divergencies by analytical continuation: shifting contours and taking residues 'in an appropriate way'
- B. Tausk, 1999: "Non-planar massless two-loop Feynman diagrams with four on-shell legs", Phys. Lett. B469 (1999);
  nice algorithmic approach to that, starting from search for some unphysical space-time dimension d for which the MB-integral is finite and well-defined
- M. Czakon, 2005 (with experience from common work with J. Gluza and TR): "Automated analytic continuation of Mellin-Barnes integrals", Comput. Phys. Commun. (2006);
  Tausk's approach realized in Mathematica program MB.m, published and available for use

A self-energy: SE2l1m

This is a nice example, being simple but showing [nearly] all essentials in a nutshell.

We get for this $F(x) = m^2 x_1^2 + [-s + m^2] x_1 x_2$ the following representation:

$$SE2l1m = \frac{e^{\epsilon\gamma_E}}{2\pi i} \frac{(m^2)^{-\epsilon}}{\Gamma[2-2\epsilon]} \int_{\Re z=-1/8} dz \left[\frac{-s+m^2}{m^2}\right]^{-\epsilon-z} \Gamma_1[1-\epsilon-z]\Gamma_2[-z]\Gamma_3[1-\epsilon+z]\Gamma_4[\epsilon+z]$$

Tausk approach:

Seek a configuration (i.e. values of z here) where all arguments of Γ-functions have positive real part. Then the $SE2l1m$ is well-defined and finite.

For small ϵ this is - here - evidently impossible; set $\epsilon \to 0$ and look at $\Gamma_2[-z]\Gamma_4[+z]$:

$$\Gamma_1[1-z]\Gamma_2[-z]\Gamma_3[1+z]\Gamma_4[+z]$$

What to do ????

Tausk: Set ϵ such that all arguments of Γ-functions get positive real parts, e.g. with the choice:

$$\Re(z) = -\frac{1}{8} \quad \text{and also} \quad \epsilon = 3/8$$

To make physics we have now to deform the integrand or the path such that $\epsilon \to 0$; when crossing a residue, take it and add it up.

Varying $\epsilon \to 0$ from $3/8$ makes crossing in $\Gamma_4[\epsilon + z]$ a pole at $\epsilon = -z = +1/8$; there is $\epsilon + z = 0$:

$$\textbf{Residue}[\textbf{SE2l1m}, \{z, -\epsilon\}] = e^{\epsilon\gamma_E} \frac{(m^2)^{-\epsilon}}{\Gamma[2-2\epsilon]} \Gamma_3[1-2\epsilon]\Gamma_2[\epsilon]$$

Here we 'loose' one integration (easier term!) and catch the IR-singularity in $\Gamma_2[\epsilon] \sim 1/\epsilon$!

The function becomes now, for small ϵ:

$$\begin{aligned} \textbf{SE2l1m} &= \frac{e^{\epsilon\gamma_E}}{2\pi i} \frac{(m^2)^{-\epsilon}}{\Gamma[2-2\epsilon]} \int_{\Re z = -1/8} dz \left[\frac{-s+m^2}{m^2}\right]^{-\epsilon - z} \Gamma_1[1-\epsilon-z]\Gamma_2[-z]\Gamma_3[1-\epsilon+z]\Gamma_4[\epsilon+z] \\ &+ e^{\epsilon\gamma_E} \frac{(m^2)^{-\epsilon}}{\Gamma[2-2\epsilon]} \Gamma_1[1-2\epsilon]\Gamma_2[\epsilon] \end{aligned} \tag{21}$$

Now we may take the limit of small ϵ because the integral will stay finite and well-defined:

$$\textbf{SE2l1m} = \frac{e^{\epsilon\gamma_E}}{2\pi i} \int_{\Re z = -\frac{1}{8}} dz \left[\frac{-s+m^2}{m^2}\right]^{-z} \Gamma[1-z]\Gamma[-z]\Gamma[z]\Gamma[1+z] + e^{\epsilon\gamma_E}\left(2 + \frac{1}{\epsilon} - \ln[m^2]\right) + O(\epsilon)$$

Now we close the integration path to the left, catch all residues from $\Gamma[z]\Gamma[1+z]$ for $\Re z < -1/8$, i.e. at $z = -n, \; n = 1, 2, \ldots$:

$$\textbf{Res}\left\{\left[\frac{-s+m^2}{m^2}\right]^{-z} \Gamma_1[1-\epsilon-z]\Gamma_2[-z]\Gamma_3[1-\epsilon+z]\Gamma_4[\epsilon+z], \{z, -n\}\right\} = (-s+m^2)^n \ln(-s+m^2)$$

The sum to be done is trivial (in this trivial case!!):

$$\sum_{n=1}^{\infty}\left[\frac{-s+m^2}{m^2}\right]^n = \frac{1}{1-\frac{-s+m^2}{m^2}} - 1$$

and we end up with:

$$\textbf{SE2l1m} \quad = \quad \frac{1}{\epsilon} + 2 + \left[\frac{1-s/m^2}{s/m^2}\ln(1-s/m^2)\right]$$

This is what we had also from the direct Feynman parameter integration above

Expansion in a small parameter: vertex V3l2m for m^2/s

Use as an example for determining the small mass expansion:

$$\begin{aligned}\text{V3coefm1} &= \text{Coefficient}[\text{V3l2m}[[1,1]],\epsilon,-1] \\ &= -\frac{1}{2s}\frac{1}{2\pi i}\int_{-i\infty-1/2}^{+i\infty-1/2} dz \left(-\frac{m^2}{s}\right)^z \frac{\Gamma_1[-z]^3\Gamma_2[1+z]}{\Gamma_3[-2z]}\end{aligned} \tag{22}$$

If $|m^2/s| << 1$, then the smallest [positive] power of it gives the biggest contribution: its exponent has to be positive and small.

So, close the contour to the right (positive $\Re z$), and leading terms come from the residua expansion of $\Gamma_1[-z]^3/\Gamma_3[-2z]$ at $-z = -1, -2, \cdots$. The residues are terms of a binomial sum:

$$Residue = -\frac{1}{s}\left(\frac{m^2}{s}\right)^n \frac{(2n)!}{(n!)^2}\left[2HarmonicNumber[n] - 2HarmonicNumber[2n] - \ln\left(-\frac{m^2}{s}\right)\right]$$

with first terms equal to (-1)*Residua:

$$V3l2m = \frac{1}{s}\ln\left(-\frac{m^2}{s}\right) + \frac{m^2}{s^2}\left[\ln\left(2+2\frac{m^2}{s}\right)\right] + \frac{m^4}{s^3}\left[\ln\left(7+6\frac{m^2}{s}\right)\right] + O(m^6/s^3)$$

Sector decomposition

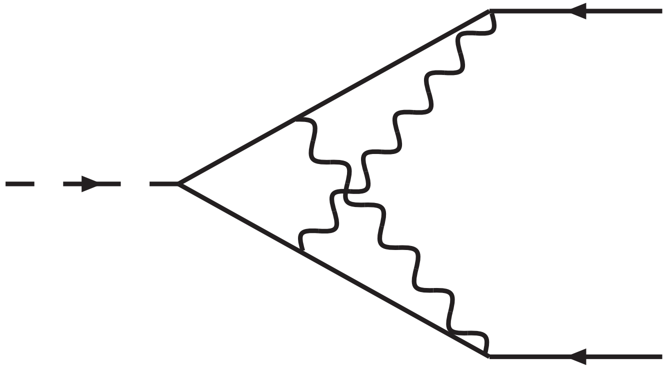
For Euclidean kinematics, the integrand for the multi-dimensional x-integrations is positive semi-definite. Proof: See examples

In numerical integrations, one has to separate the poles in $(d-4)$, and in doing so one has to avoid overlapping singularities.

A method for that is sector decomposition.

There are quite a few recent papers on that, and also nice reviews are given Binoth:2000ps,Denner:2004iz,Bogner:2007cr,Heinrich:2008si,Smirnov:2008aw The intention is to separate singular regions in different variables from each other, as is nicely demonstrated by an example borrowed from Heinrich:2008si:

$$\begin{aligned} I &= \int_0^1 dx \int_0^1 dy \frac{1}{x^{1+a\epsilon} y^{b\epsilon}[x+(1-x)y]} \\ &= \int_0^1 \frac{dx}{x^{1+(a+b)\epsilon}} \int_0^1 \frac{dt}{t^{b\epsilon}[1+(1-x)t]} + \int_0^1 \frac{dy}{y^{1+(a+b)\epsilon}} \int_0^1 \frac{dt}{t^{1+a\epsilon}[1+(1-y)t]}. \end{aligned} \tag{23}$$



The master integral `V6l4m1`

At several occasions, we used for cross checks the package `sector_decomposition`

[Bogner:2007cr]

built on the C++ library GINAC

[Bauer:2000cp]

For that reason, the interface `CSectors` was written; Gluza, Kajda, Yundin, T.R., Eur.Phys.J. C71 (2011) 1516

The syntax is similar to that of AMBRE.

Example:

The program input for the evaluation of the integral `V6l4m1` is simple; we choose $m = 1, s = -11$, and the topology may be read from the arguments of propagator functions PR:

```
<< CSectors.m

Options[DoSectors]
SetOptions[DoSectors, TempFileDelete -> False, SetStrategy -> C]

n1 = n2 = n3 = n4 = n5 = n6 = n7 = 1;
m = 1; s = -11;
invariants = {p1^2 -> m^2, p2^2 -> m^2, p1 p2 -> (s - 2 m^2)/2};

DoSectors[{1},
  {PR[k1,0,n1]       PR[k2,0,n2]       PR[k1+p1,m,n3]
   PR[k1+k2+p1,m,n5] PR[k1+k2-p2,m,n6] PR[k2-p2,m,n7]},
   {k2, k1}, invariants][-4, 2]
```

Here, the numerator is 1 (see the first argument $\{1\}$ of `DoSectors`), and the output contains the functions U_2 and F_2:

```
Using strategy C
U = x3 x4+x3 x5+x4 x5+x3 x6+x5 x6+x2 (x3+x4+x6)+x1 (x2+x4+x5+x6)

F = x1 x4^2+13 x1 x4 x5+x4^2 x5+x1 x5^2+x4 x5^2+13 x1 x4 x6
+2 x1 x5 x6+13 x4 x5 x6+x5^2 x6+x1 x6^2+x5 x6^2+x3^2 (x4+x5+x6)
```

```
+x2(x3^2+x4^2+13 x4 x6+x6^2+x3 (2 x4+13 x6))+x3 (x4^2+(x5+x6)^2
+x4 (2x5+13 x6))
```

Notice the presence of a U-function and the complexity of the F-function (compared to $U = 1$ and `f1` and `f2` in the loop-by-loop MB-approach) due to the non-sequential, direct performance of both momentum integrals at once. Both U and F are evidently positive semi-definite. The numerical result for the Feynman integral is:

$$\mathtt{V614m1}(-\mathtt{s})^{2\epsilon} = -0.052210\ \frac{1}{\epsilon} - 0.17004 + 0.24634\ \epsilon + 4.8773\ \epsilon^2 + \mathcal{O}(\epsilon^3). \tag{24}$$

The numbers may be compared to (27). We obtained a third numerical result, also by sector decomposition, with the Mathematica package `FIESTA`

[Smirnov:2008py]

$$\mathtt{V614m1}(-\mathtt{s})^{2\epsilon} = -0.052208\ \frac{1}{\epsilon} - 0.17002 + 0.24622\ \epsilon + 4.8746\ \epsilon^2 + \mathcal{O}(\epsilon^3). \tag{25}$$

Most accurate result: obtained with an analytical representation based on harmonic polylogarithmic functions obtained by solving a system of differential equations

[Gluza, TR, unpubl.;Remiddi:1999ew,Maitre:2005uu]

$$\mathtt{V614m1}(-\mathtt{s})^{2\epsilon} = -0.0522082\ \frac{1}{\epsilon} - 0.170013 + 0.246253\ \epsilon + 4.87500\ \epsilon^2 + \mathcal{O}(\epsilon^3). \tag{26}$$

All displayed digits are accurate here.

MB-integrals – Try V6l4m1 with the loop-by-loop approach

In a loop-by-loop approach, after the first momentum integration one gets here $U = 1$ and a first F-function, which depends yet on one internal momentum k_1:

```
f1 = m^2 [X[2]+X[3]+X[4]]^2 - s X[2]X[4] - PR[k1+p1,m] X[1]X[2]
               - PR[k1+p1+p2,0] X[2]X[3] - PR[k1-p2,m] X[1]X[4]
                     - PR[k1,0] X[3]X[4] ,
```

leading to a 7-dimensional MB-representation; after the second momentum integration, one has:

```
f2 = m^2 [X[2]+X[3]]^2 - s X[2]X[3] - s X[1]X[4] - 2s X[3]X[4],
```

leading to another 4-dimensional integral.

After several applications of Barnes' first lemma, an 8-dimensional integral has to be treated.

We made no attempt here to simplify the situation by any of the numerous tricks and reformulations etc. known to experts.

at $s = -11$:

$$\texttt{V6l4m}\ (-\texttt{s})^{2\epsilon} = -0.0522082\ \frac{1}{\epsilon} - 0.17002 + 0.25606\ \epsilon + 4.67\ \epsilon^2 + \mathcal{O}(\epsilon^3). \tag{27}$$

.

A nice massive 2-loop box with numerator is e.g. B5l3m($p_e.k_1$)

We determined its expansion in a small mass Czakon, Gluza, T.R., NPB 751 (2006).

$$\texttt{B5l3m}(\texttt{p}_\texttt{e}\cdot\texttt{k}_1) = \frac{m^{4\epsilon}(-1)^{a_{12345}}\ e^{2\epsilon\gamma_E}}{\prod_{j=1}^{5}\Gamma[a_i]\Gamma[5-2\epsilon-a_{123}](2\pi i)^4}\int_{-i\infty}^{+i\infty}d\alpha\int_{-i\infty}^{+i\infty}d\beta\int_{-i\infty}^{+i\infty}d\gamma\int_{-i\infty}^{+i\infty}d\delta$$

$$(-s)^{(4-2\ \epsilon)-a_{12345}-\alpha-\beta-\delta}\ (-t)^{\delta}$$

$$\frac{\Gamma[-4+2\ \epsilon+a_{12345}+\alpha+\beta+\delta]}{\Gamma[6-3\ \epsilon-a_{12345}-\alpha]}\frac{\Gamma[-\alpha]\ \Gamma[-\beta]}{\Gamma[7-3\ \epsilon-a_{12345}-\alpha]\ \Gamma[5-2\ \epsilon-a_{123}]}\frac{\Gamma[-\delta]}{\Gamma[4-2\ \epsilon-a_{1123}-2\ \alpha-\gamma]\ \Gamma[5-2\ \epsilon-a_{1123}-2}$$

$$\frac{\Gamma[2-\epsilon-a_{13}-\alpha-\gamma]}{\Gamma[8-4\ \epsilon-a_{112233445}-2\ \alpha-2\ \beta-2\ \delta-\gamma]}\frac{\Gamma[4-2\ \epsilon-a_{12345}-\alpha-\beta-\delta-\gamma]}{\Gamma[9-4\ \epsilon-a_{112233445}-2\ \alpha-2\ \beta-2\ \delta-\gamma]}\Big\{(p_e\cdot p_3)\ \Gamma[1+a_4+\delta]\ \Gamma[6-3\ \epsilon-$$

$$\Gamma[4-2\ \epsilon-a_{1234}-\alpha-\beta-\delta]\ \Gamma[3-\epsilon-a_{12}-\alpha]\ \Gamma[8-4\ \epsilon-a_{112233445}-2\ \alpha-2\ \delta-\gamma]\Gamma[9-4\ \epsilon-a_{112233445}-2\ \alpha-2\ \beta-2$$

$$\Gamma[5-2\ \epsilon-a_{1123}-\gamma]\ \Gamma[4-2\ \epsilon-a_{1123}-2\ \alpha-\gamma]\ \Gamma[a_1+\gamma]\ \Gamma[-2+\epsilon+a_{123}+\alpha+\delta+\gamma]+\Gamma[a_4+\delta]\Big[-(p_e\cdot p_1)\ \Gamma[7-3\ \epsilon-$$

$$\Gamma[4-2\ \epsilon-a_{1234}-\alpha-\beta-\delta]\Gamma[8-4\ \epsilon-a_{112233445}-2\ \alpha-2\ \delta-\gamma]\Gamma[9-4\ \epsilon-a_{112233445}-2\ \alpha-2\ \beta-2\ \delta-\gamma]$$

$$\Big[\Gamma[3-\epsilon-a_{12}-\alpha]\ \Gamma[5-2\ \epsilon-a_{1123}-\gamma]\Gamma[4-2\ \epsilon-a_{1123}-2\ \alpha-\gamma]\ \Gamma[a_1+\gamma]+\Gamma[2-\epsilon-a_{12}-\alpha]\ \Gamma[4-2\ \epsilon-a_{1123}-\gamma]$$

$$\Gamma[5-2\ \epsilon-a_{1123}-2\ \alpha-\gamma]\ \Gamma[1+a_1+\gamma]\Big]\ \Gamma[-2+\epsilon+a_{123}+\alpha+\delta+\gamma]+\Gamma[6-3\ \epsilon-a_{12345}-\alpha]\ \Gamma[3-\epsilon-a_{12}+\alpha]$$

$$\Gamma[5-2\ \epsilon-a_{1123}-\gamma]\ \Gamma[4-2\ \epsilon-a_{1123}-2\ \alpha-\gamma]\Gamma[a_1+\gamma]\Big[((p_e\cdot(p_1+p_2))\ \Gamma[5-2\ \epsilon-a_{1234}-\alpha-\beta-\delta]\Gamma[9-4\ \epsilon-a_{11223}$$

$$\Gamma[8-4\ \epsilon-a_{112233445}-2\ \alpha-2\ \beta-2\ \delta-\gamma]\Gamma[-2+\epsilon+a_{123}+\alpha+\delta+\gamma]+(p_e\cdot p_1)\ \Gamma[4-2\ \epsilon-a_{1234}-\alpha-\beta-\delta]$$

$$\Gamma[8-4\ \epsilon-a_{112233445}-2\ \alpha-2\ \delta-\gamma]\Gamma[9-4\ \epsilon-a_{112233445}-2\ \alpha-2\ \beta-2\ \delta-\gamma]\Gamma[-1+\epsilon+a_{123}+\alpha+\delta+\gamma]\Big]\Big\}$$

Summary

- We have introduced the MB-representations of L-loop N-point Feynman integrals of general type
- The determination of the ϵ-poles is generally solved
- The remaining problem is the evaluation of the multi-dimensional, finite MB-Integrals
- This is unsolved in the general case, . . . so you have something to do if you like to . . .

Not discussed, although there is some recent activity also in that field of research:
Phase space integrals

- Angular integrals in d dimensions
  G. Somogyi (DESY, Zeuthen) arXiv:1101.3557
- A subtraction scheme for ... QCD ...cross sections at NNLO: integrating the iterated singly-unresolved subtraction terms
  P. Bolzoni, G. Somogyi, Z. Trocsanyi JHEP 1101 (2011), arXiv:1011.1909