.

Status of Massive Twoloop Boxes

for Bhabha Scattering

Tord Riemann, DESY, Zeuthen

7. Arbeitstreffen des SFB/TR 9: Computational Particle Physics
DESY, Zeuthen, 20. und 21. März 2006

A project in collaboration with
Michal Czakon Univ. Würzburg (and Katowice)
Janusz Gluza Katowice
Stefano Actis Zeuthen (new)

See also:
- **PRD 71 (2005), hep-ph/0412164, Ustron 2005, Radcor 2005**
- **http://www-zeuthen.desy.de/theory/research/bhabha/**

- **Introduction: Why twoloop Bhabha scattering?**
- **Evaluation of massive twoloop boxes**
- **Outlook**

The Physics Needs

ILC/GigaZ – Need Bhabha cross-sections with 3–4 significant digits.

- **ILC:** $e^+e^- \to W^+W^-, f\bar{f}$ **with** $O(10^6)$ **events** $\to 10^{-3}$
- **ILC:** $e^+e^- \to e^+e^-$**, a probe for New Physics with** $O(10^5)$ **events/year** $\to 10^{-3}$
- **GigaZ: relevant physics derived from** $Z \to$ **hadrons**$, l^+l^-$**, the latter with** $O(10^8)$ **events** $\to 10^{-4}$**, the systematic errors (luminosity!) influence this**

Conclude: will need $\Delta\mathcal{L}/\mathcal{L} \approx 2 \times 10^{-4}$

The luminosity comes from very forward Bhabha scattering.

October 20, 2003

MEMO
Luminosity Measurement via Bhabha Scattering: Requirements on Position Reconstruction to Achieve a 10^{-4} Precision

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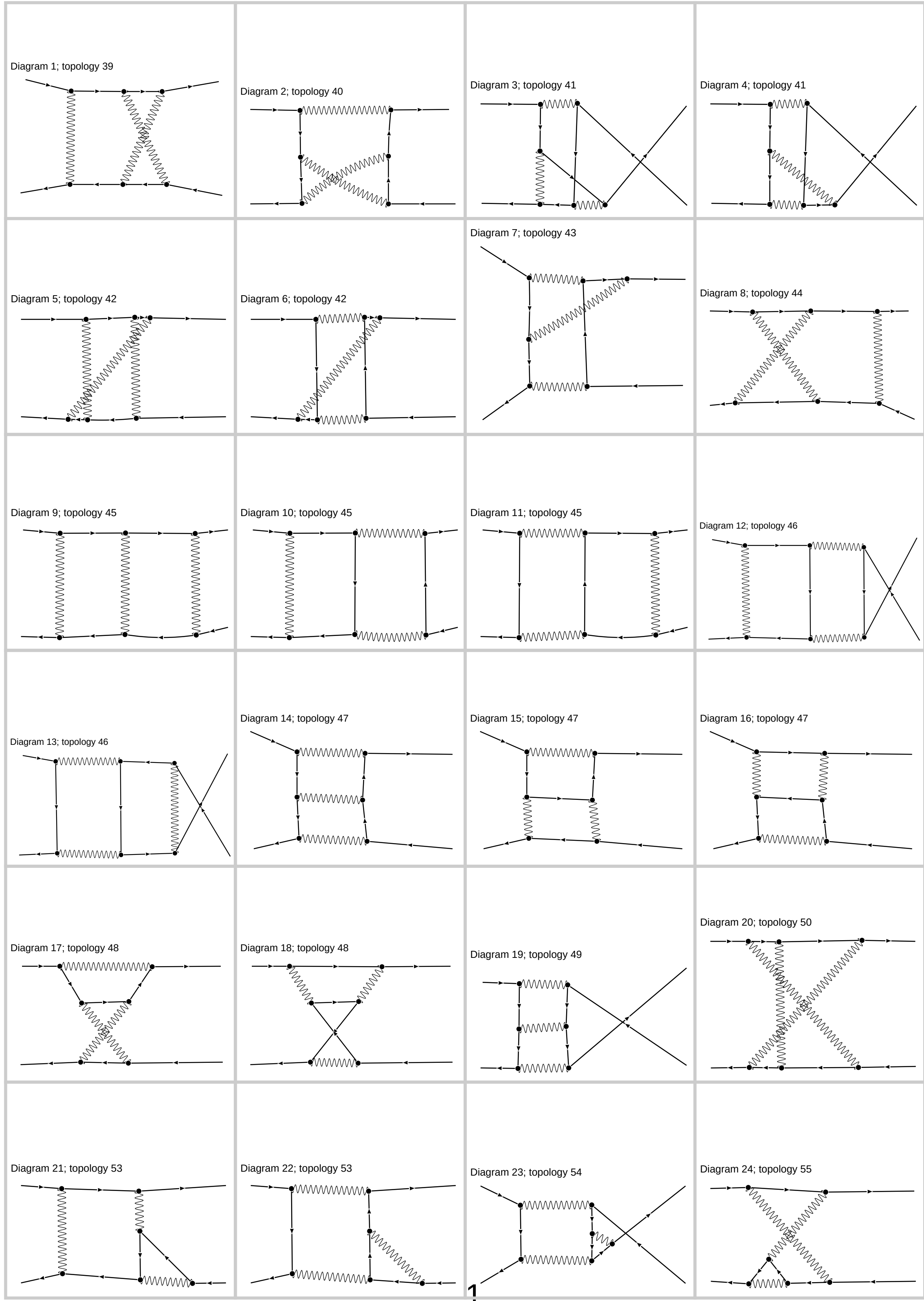
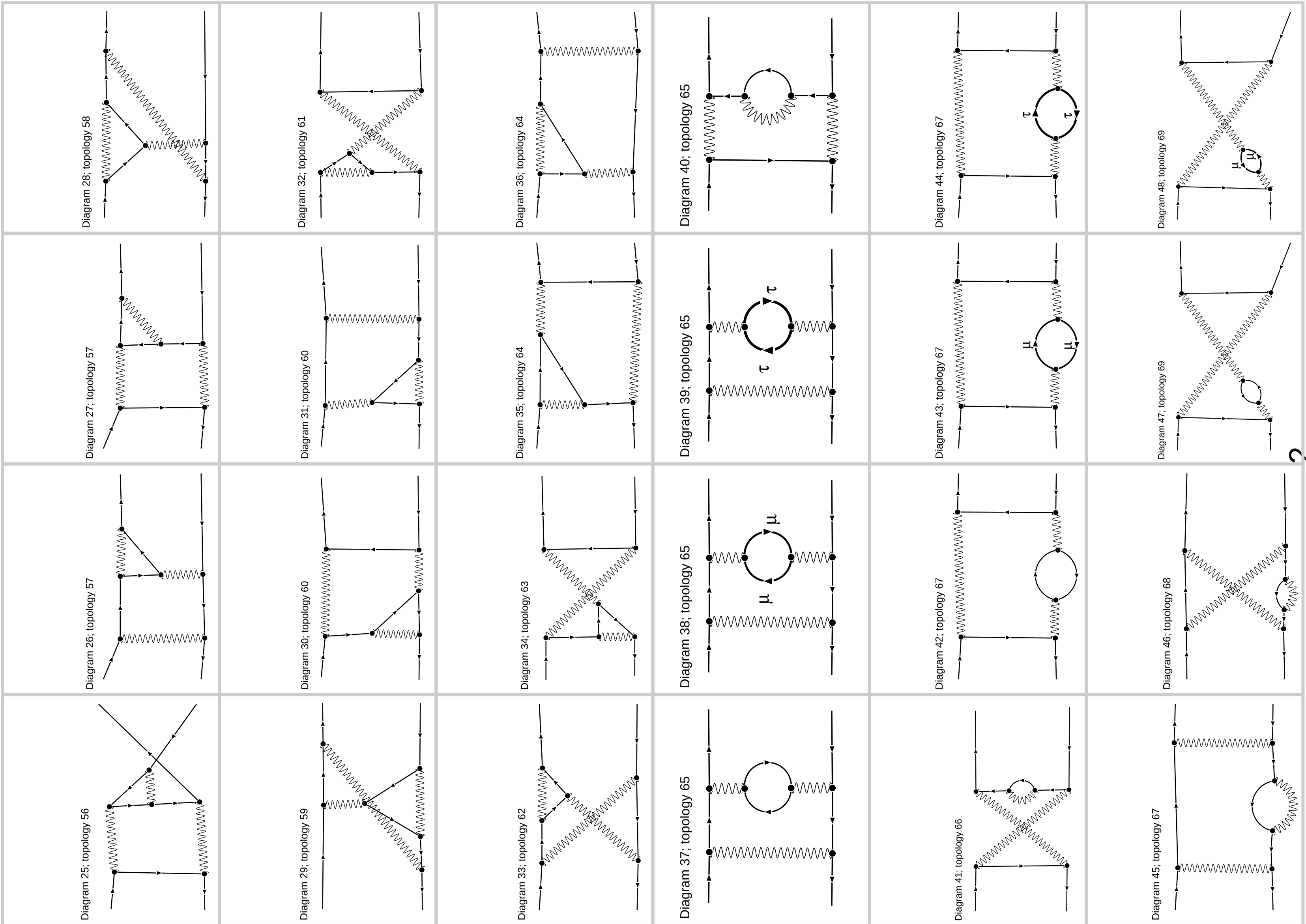
Abstract

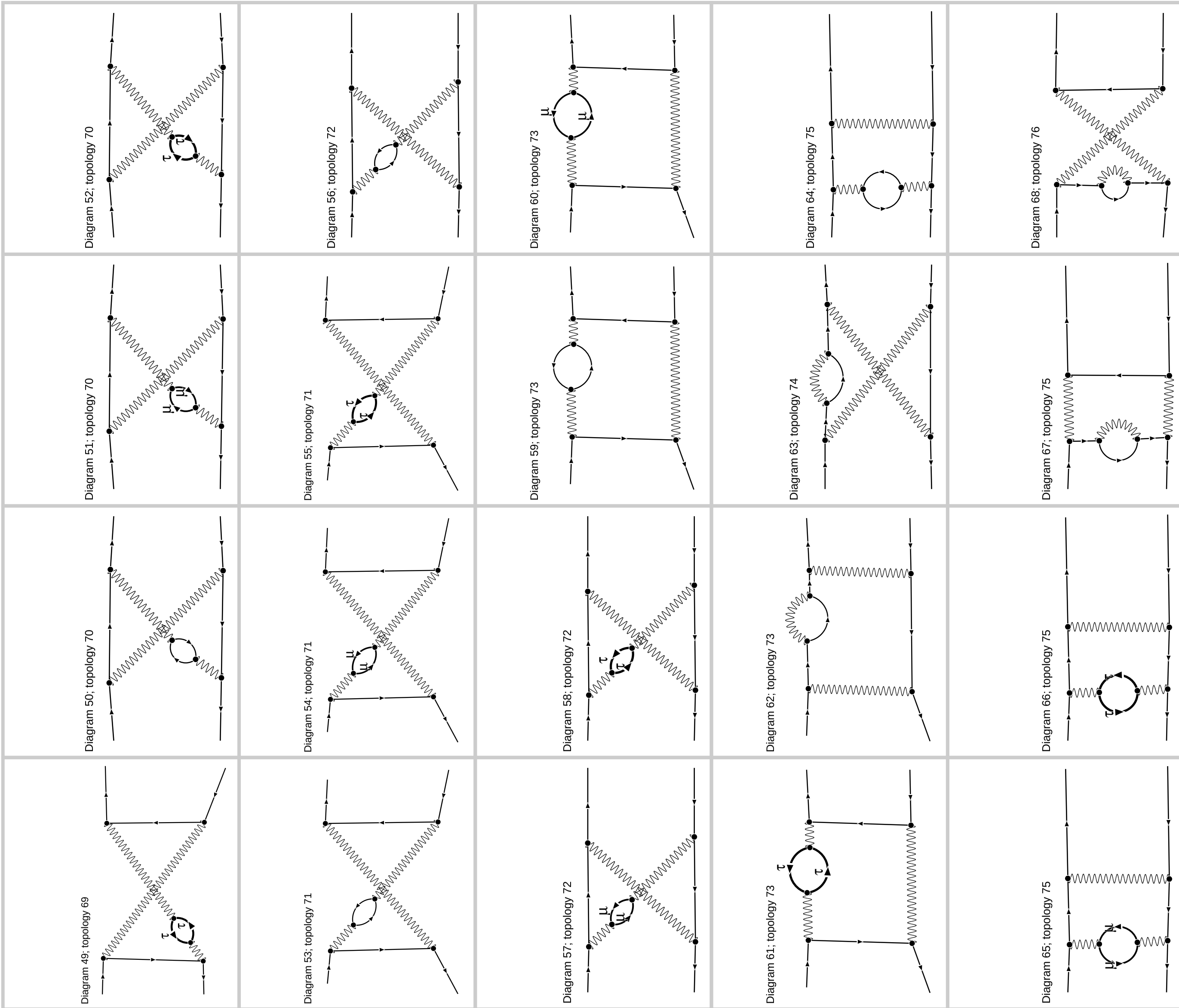
This memo is based on Monte Carlo simulations with the BHLUMI generator of Jadach and Was. It adresses the question how accurately electrons and positrons have to be reconstructed in the TESLA Lum-Cal in order to achieve a precision of 10^{-4} on the luminosity measurement.

Irreducible two-loop diagrams: 1/3



Irreducible two-loop diagrams: 2/3





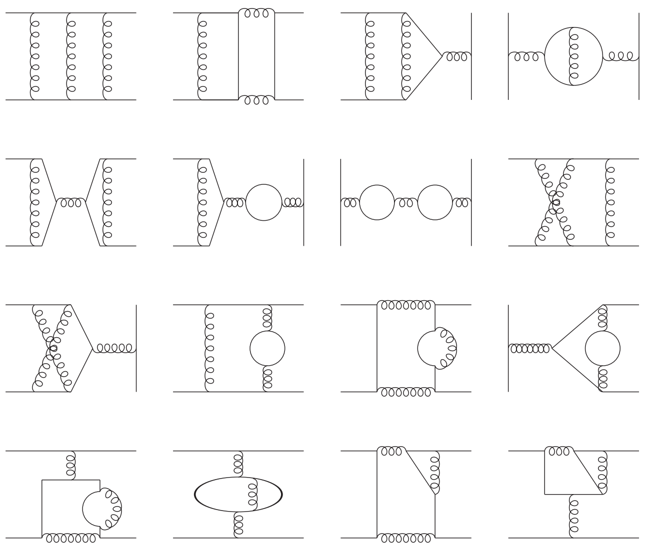
Irreducible two-loop diagrams: 3/3

$$m = 0$$

Two Loop Bhabha Scattering

To calculate Bhabha scattering it is best to first compute $e^+e^- \to \mu^+\mu^-$, since it's closely related but has less diagrams.

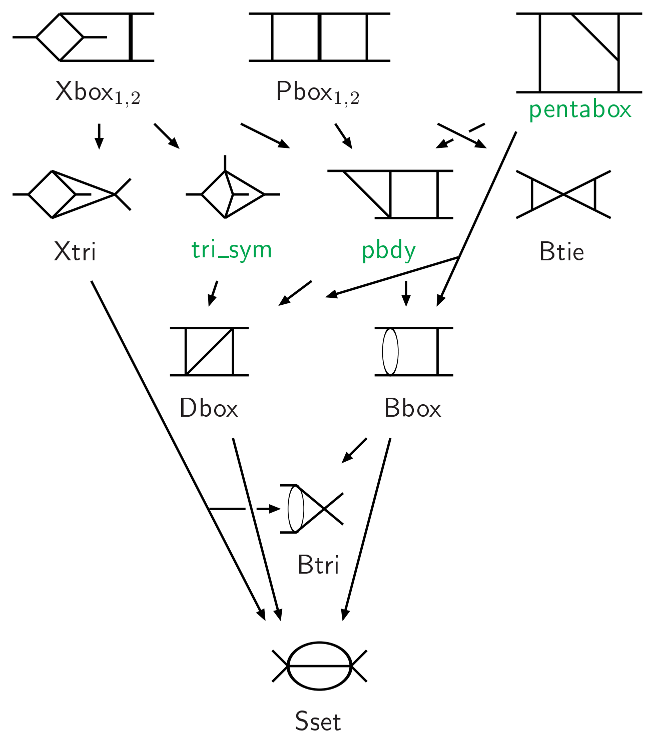
There are 47 QED diagrams contributing to $e^+e^- \to \mu^+\mu^-$.



In this calculation all particles massless.

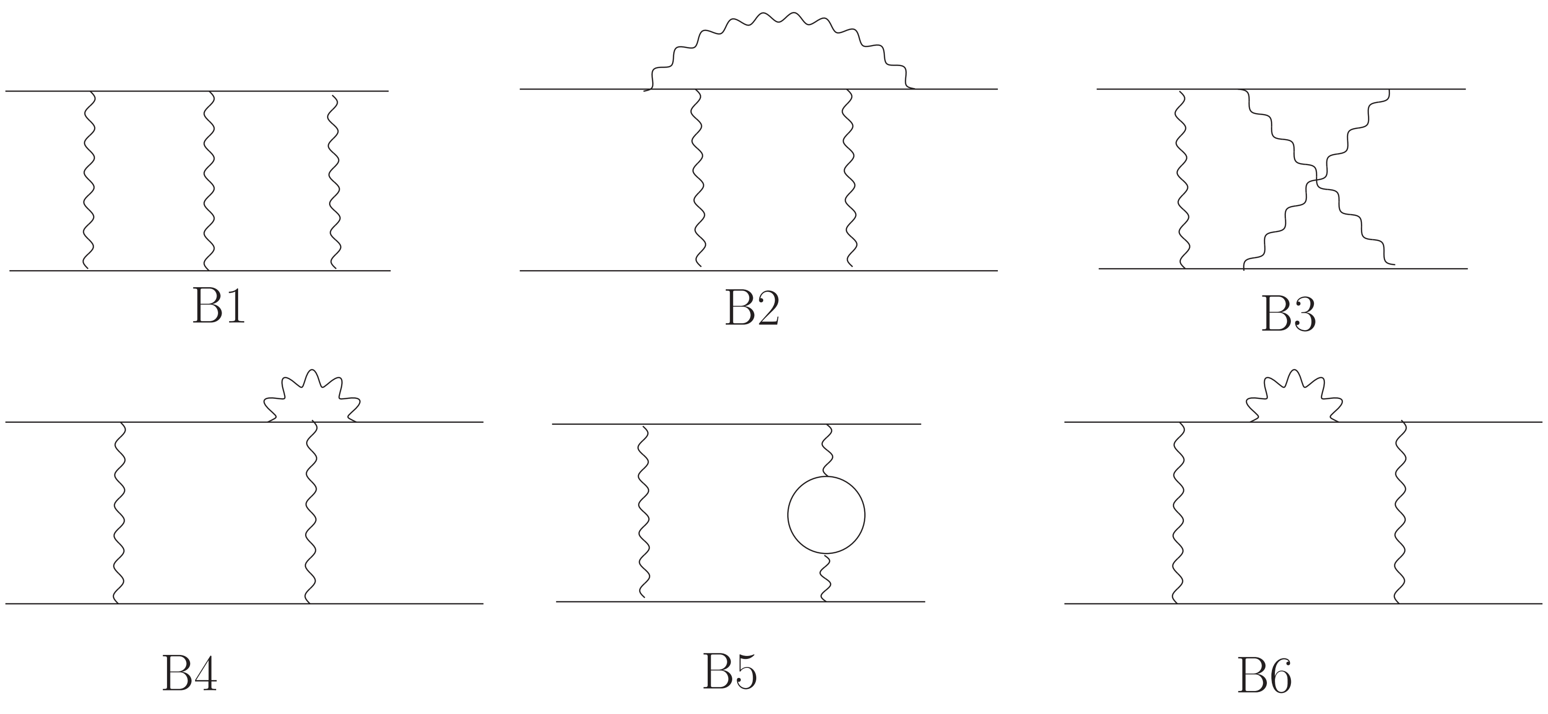
The Bhabha scattering amplitude can be obtained from $e^+e^- \to \mu^+\mu^-$ simply by summing it with the crossed amplitude (including fermi minus sign).

Two-loop integral inheritance chart



From Talk of Zvi Bern, LoopFest 2002

The two-loop box diagrams for massive Bhabha scattering



- B5: All masters known (2004)
  Bonciani, Ferroglia, Mastrolia, Remiddi, van der Bij: hep-ph/0405275, hep-ph/0411321
  Czakon, Gluza, Riemann: *http://www-zeuthen.desy.de/.../MastersBhabha.m*
- B1, B2, B3: two simplest masters for B1 known (Smirnov, Heinrich 2002,2004)
- B4, B6: they are no masters (see Czakon et al. 2004)

The basic planar 2-box master of B1, B7l4m, was a breakthrough

The two-loop Feynman integrals

One has to solve many, very complicated Feynman integrals with $L = 2$ loops and $N \leq 7$ internal lines:

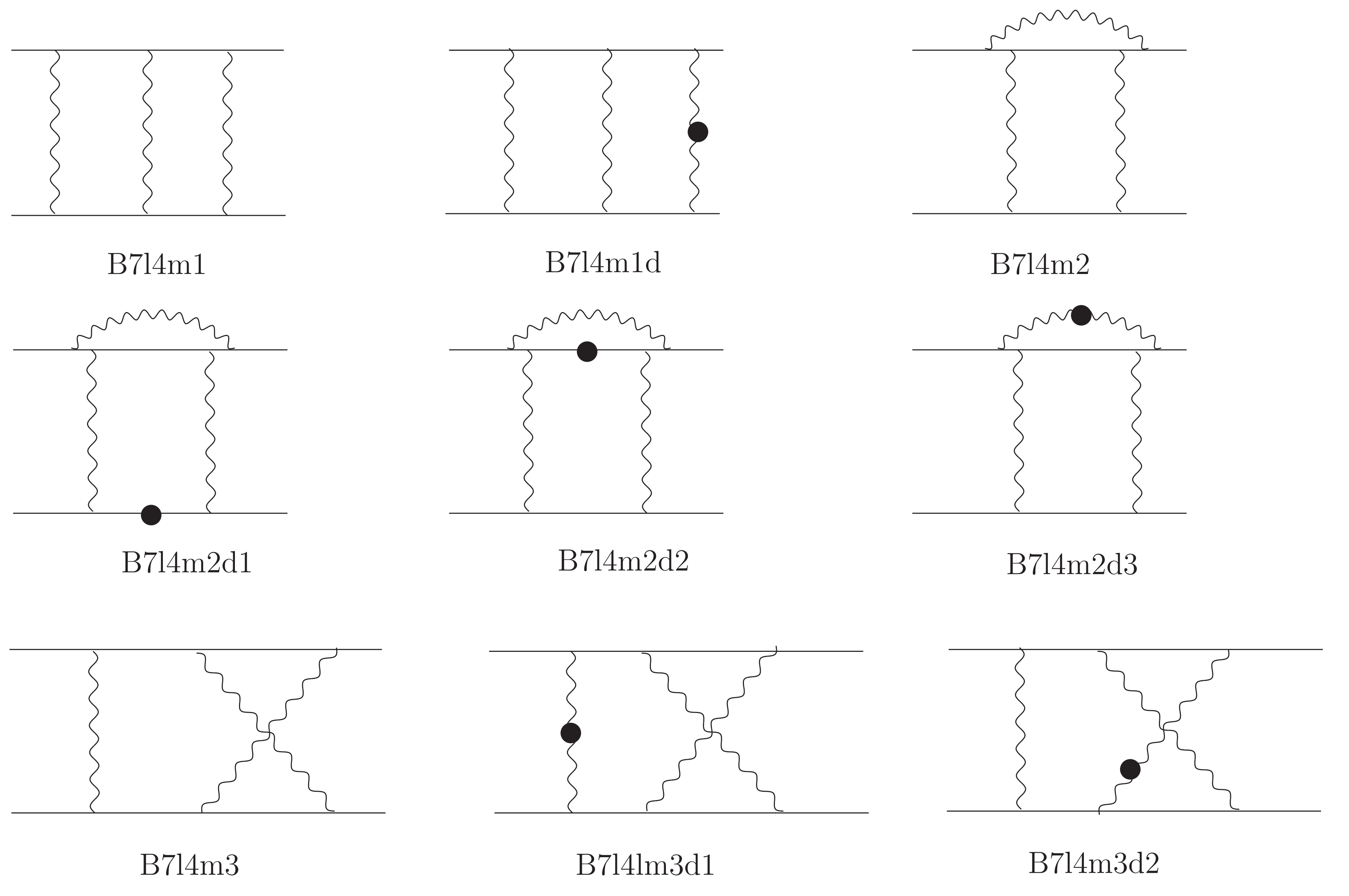
$$G(X) = \frac{1}{\left(i\pi^{d/2}\right)^2} \int \frac{d^D k_1 d^D k_2 \; X}{(q_1^2 - m_1^2)^{\nu_1} \ldots (q_j^2 - m_j^2)^{\nu_j} \ldots (q_N^2 - m_N^2)^{\nu_N}},$$

$$X = 1, (k_1 P), (k_1 k_2), (k_2 P), \cdots$$

where P is some external momentum: $p_1, ..p_4$

.... {X} contains IRREDUCIBLE NUMERATORS ....ν_i...index

We prefer to calculate the integrals analytically (where possible)
Derive a set of master integrals, and also the algebraic expressions (Substitutions) for all the other Feynman integrals in terms of them



The nine two-loop box MIs with seven internal lines.

.

So we need a

A table of master integrals ... LIST OF SUBSTITUTIONS

We use IdSolver, C++, Czakon 2004 with the Laporta/Remiddi algorithm:
Derive with integration-by-parts (and Lorentz-invariance) identities a system of *algebraic* equations for the Feynman integrals and solve the system.

- **2-loop self energies: 6 masters (all masters known)**
- **2-loop vertices: 19 masters (all masters known)**
- **2-loop boxes: 34 masters**

The calculation of the master integrals is mainly done with two methods:

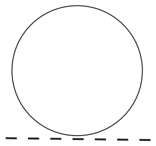
- **derive and solve (systems of) differential equations (with boundary conditions)**
- **derive and solve (up to 8-dimensional) Mellin-Barnes integral representations for single Feynman integrals**

From Czakon et al., PRD 71 (2005): 4-point 3 entering basic two-loop box diagrams. An asterisk denotes one-loop MI. MIs with a dagger: know singular parts only

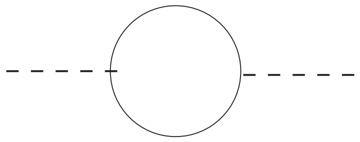
MI	B1	B2	B3	B4	B5	B6	ref.
B7l4m1	+	−	−	−	−	−	Smirnov:2001cm
B7l4m1N	+	−	−	−	−	−	Heinrich:2004iq
B7l4m2	−	+	−	−	−	−	Heinrich:2004iq†
B7l4m2[d1--d3]	−	+	−	−	−	−	
B7l4m3	−	−	+	−	−	−	Heinrich:2004iq†
B7l4m3[d1--d2]	−	−	+	−	−	−	
B6l3m1	+	−	+	−	−	−	
B6l3m1d	+	−	+	−	−	−	
B6l3m2	−	+	−	+	−	−	
B6l3m2d	−	+	−	+	−	−	
B6l3m3	−	−	+	−	−	−	
B6l3m3[d1--d5]	−	−	+	−	−	−	
B5l2m1	+	−	+	−	−	−	Czakon:2004tg
B5l2m2	−	+	−	+	−	+	Sec. IIIE1†
B5l2m2[d1--d2]	−	+	−	+	−	+	Sec. IIIE1†
B5l2m3	+	−	+	−	−	−	
B5l2m3[d1--d3]	+	−	+	−	−	−	Sec. IIIE1†
B5l3m	−	+	+	+	−	−	
B5l3m[d1--d3]	−	+	+	+	−	−	
B5l4m	−	+	+	+	+	−	Bonciani:2003cj
B5l4md	−	+	+	+	+	−	Sec. IIIE
B4l2m*	−	−	−	+	+	+	'tHooft:1972fi,Bonciani:2003cj
total = 33+1*	9	15	22	11+1*	2+1*	3+1*	

The simplest diagram is the tadpole:

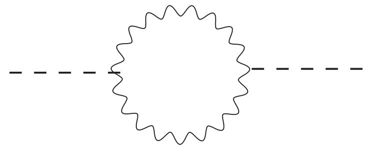
$$
\begin{aligned}
\texttt{T1l1m} &= \frac{e^{\epsilon\gamma_E}}{i\pi^{D/2}} \int \frac{d^D q}{q^2-1} \\
&= \frac{1}{\epsilon} + 1 + \left(1+\frac{\zeta_2}{2}\right)\epsilon + \left(1+\frac{\zeta_2}{2}-\frac{\zeta_3}{3}\right)\epsilon^2 + \ldots
\end{aligned}
$$



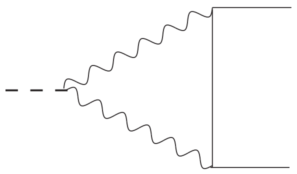
T1l1m



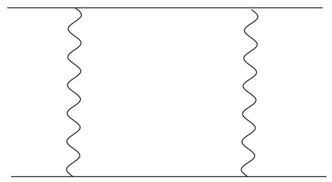
SE2l2m



SE2l0m



V3l1m



B4l2m

The IR-divergent QED vertex $C_0(s, m^2, m^2; 0, m, m)$ is no master:

$$
\texttt{V3l2m}(s) = \frac{(D-2)\texttt{T1l1m} - 2(D-3)\texttt{SE2l2m}(s)}{(D-4)(s-4)}
$$

The UV-divergencies cancel, the IR divergency comes from the denominator.

.

$fillPR0(1,1,1,1,1,1,0,0,-1) =$

$+PR1(1,1,0,0,0,0,0,0,0)*(((8*m^2+t-12)*ep^3+(-14*m^2-3*t+26)*ep^2+(4*m^2+3*t-16)*ep$

$+2*m^2-t+2)/((128*m^4+(24*t^2-128*t)*m^2)*ep^2+((16*t)*m^4+(-8*t^2+16*t)*m^2)*ep))$

$+PR11(1,1,1,1,0,0,0,0,0)*((((24*t-48)*m^2+3*t^2-24*t)*ep^2+((-2*t+16)*m^2-4*t^2+14*t)*ep$

$+(-2*t)*m^2+t^2-2*t)/((128*m^4+(24*t^2-128*t)*m^2)*ep+(16*t)*m^4+(-8*t^2+16*t)*m^2))$

$+PR11(1,1,1,2,0,0,0,0,0)*(((32*m^4+(-16*t)*m^2+2*t^2)*ep+16*m^4$

$+(-8*t)*m^2+t^2)/((64*m^4+(12*t^2-64*t)*m^2)*ep+(8*t)*m^4+(-4*t^2+8*t)*m^2))$

$+PR13(1,1,1,1,0,0,0,0,0)*((-ep+1)/(4*m^2))$

$+PR16(1,1,1,1,1,0,0,0,0)*(-1/4)$

$+PR2(1,1,0,0,0,0,0,0,0)*(((((12*t-24)*m^2-12*t^2+42*t)*ep^4+(8*m^4+(-39*t+44)*m^2$

$+31*t^2-98*t)*ep^3+(-18*m^4+(45*t-18)*m^2-27*t^2+72*t)*ep^2+(12*m^4+(-21*t)*m^2$

$+9*t^2-18*t)*ep-2*m^4+(3*t-2)*m^2-t^2+2*t)/((512*m^6+(96*t^2-1024*t)*m^4+(-96*t^3+512*t^2)*m^2)*ep^3$

$+((64*t-256)*m^6+(-144*t^2+576*t)*m^4+(80*t^3-320*t^2)*m^2)*ep^2+((-32*t)*m^6$

$+(48*t^2-32*t)*m^4+(-16*t^3+32*t^2)*m^2)*ep))$

$+PR4(1,1,1,0,0,0,0,0,0)*((((-36*t+72)*m^2+18*t)*ep^3+(-24*m^4+(45*t-84)*m^2-18*t)*ep^2+(22*m^4$

$+(-17*t+30)*m^2+4*t)*ep-4*m^4+(2*t-4)*m^2)/((256*m^4$

$+(48*t^2-256*t)*m^2)*ep^2+((32*t)*m^4+(-16*t^2+32*t)*m^2)*ep))$

$+PR4(1,1,2,0,0,0,0,0,0)*((((-12*t+24)*m^2+12*t-24)*ep^2+(-8*m^4+(7*t-8)*m^2-7*t+16)*ep+2*m^4$

$+(-t)*m^2+t-2)/((128*m^2+24*t^2-128*t)*ep^2$

$+((16*t)*m^2-8*t^2+16*t)*ep))$

$+PR6(1,1,1,0,0,0,0,0,0)*((((-36*t)*m^2+9*t^2)*ep^2+(24*m^4+(30*t)*m^2-9*t^2)*ep-16*m^4+(-4*t)*m^2$

$+2*t^2)/((256*m^6+(48*t^2-512*t)*m^4$

$+(-48*t^3+256*t^2)*m^2)*ep+(32*t)*m^6+(-48*t^2+32*t)*m^4+(16*t^3-32*t^2)*m^2))$

$+PR6(1,1,2,0,0,0,0,0,0)*((((36*t^2)*m^2-9*t^3)*ep^2+(-64*m^6+(80*t)*m^4+(-52*t^2)*m^2+9*t^3)*ep+16*m^6$

$+(-28*t)*m^4+(14*t^2)*m^2-2*t^3)/((512*m^6+(96*t^2-1024*t)*m^4$

$+(-96*t^3+512*t^2)*m^2)*ep^2+((64*t-256)*m^6+(-144*t^2+576*t)*m^4+(80*t^3-320*t^2)*m^2)*ep+(-32*t)*m^6$

$+(48*t^2-32*t)*m^4+(-16*t^3+32*t^2)*m^2))$

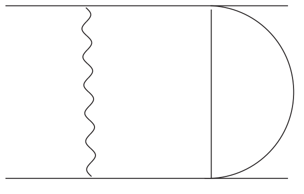
;

.

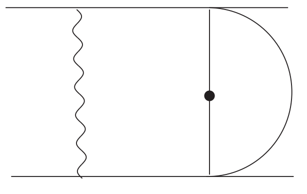
The 2-boxes with 5 lines

The B5l3m see later.

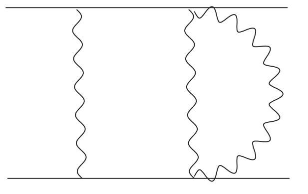
The completely known 2-boxes with 5 lines are B5l4m (Bonciani et al., Czakon et al. 2004) and B5l2m1 (Czakon et al. 2004) :



B5l4m1

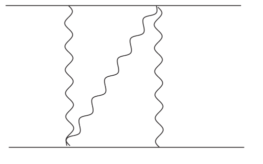


B5l4m1d1

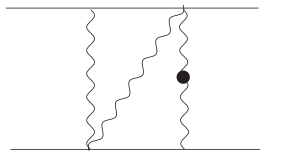


B5l2m1

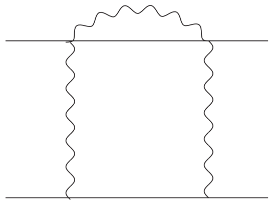
The divergent parts of the B5l2m2 and B5l2m3 type are known (Czakon et al. 2004):



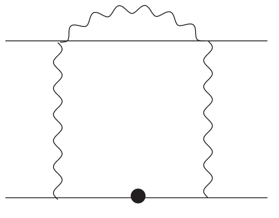
B5l2m3



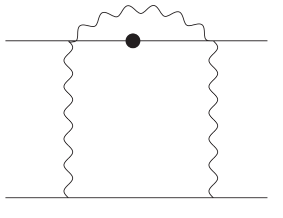
B5l2m3d1



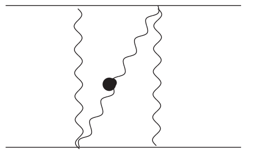
B5l2m2



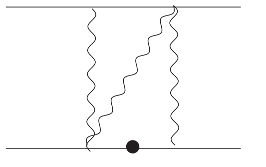
B5l2m2d1



B5l2m2d2



B5l2m3d2



B5l2m3d3

RADCOR 2005: The finite parts of B5l2m3d1-3, not of B5l2m3:

Derived with Differential Equations

$$
\begin{aligned}
\texttt{B5l2m3d1}[0,x,y] = {} & -\frac{1}{2}\frac{s}{t}\frac{1+x}{1-x}\frac{1}{(1+y)^2}\{(3-2y+3y^2)H[-1,0,x]-4yH[0,x](H[0,y]+2H[1,y]) \\
& -(1+y)^2(H[1,0,x]-H[0,x])\}+\frac{y}{8x(1-y)^2(1+y)^2}\{8(2y(1+x^2)+x(1-y)^2) \\
& (H[0,y]-2H[1,0,y]+2H[1,y]-4H[1,1,y])-4(1-2x+5x^2+2y+4xy+2x^2y+y^2 \\
& -2xy^2+5x^2y^2)H[0,0,x]-4(3x+4y-4xy+4x^2y+xy^2) \\
& (H[0,0,y]+2H[0,1,y]) \\
& -(15+4x+11x^2-10y+24xy \\
& +14x^2y+15y^2-28xy^2+11x^2y^2)\zeta_2\} \\
& -\frac{y(1+x^2)}{2x(1-y)^2} \\
& -\frac{(1-x)^2y^2}{x(1-y)(1+y)^3}\{2H[-1/y,-1,0,x] \\
& -H[-1/y,0,0,x]+H[-y,0,0,x] \\
& -2H[-y,-1,0,x]-H[-1,0,0,y] \\
& +5H[0,0,1,y]-2H[-1,0,1,y] \\
& +5/2H[0,0,0,y]+2H[0,1,0,y] \\
& +4H[0,1,1,y]+2H[1,0,0,y] \\
& +4H[1,0,1,y] \\
& +\zeta_3/2 \\
& -(4H[0,x]+4H[-1,y]-3H[-y,x] \\
& -5H[-1/y,x]-7/2H[0,y]-8H[1,y])\zeta_2 \\
& -(-H[-1/y,x]-H[-y,x]+H[0,x]) \\
& (H[0,0,y]+2H[0,1,y]) \\
& +(H[-1/y,0,x]-H[-y,0,x]) \\
& (H[0,y]+2H[1,y])\}.
\end{aligned}
$$

$$
\begin{aligned}
\texttt{B5l2m3d2}[0,x,y] = \frac{-4xy}{(1-x^2)(1-y)^2} \\
\Big\{ \frac{\zeta_2}{4}(20H[-1/y,x] - 12H[-y,x] - 10H[-1,x] \\
-3H[0,x] - 14H[0,y] + 2H[1,x] + 4H[1,y]) \\
+7/4\ \zeta_3 \\
-3H[-1,0,x](H[0,y] + 2H[1,y]) \\
+2H[0,x](H[1,0,y] + 2H[1,1,y]) \\
+(H[0,y] + 2H[1,y])(H[-1/y,0,x] \\
+H[-y,0,x] + H[1,0,x] + H[0,0,x]) \\
+H[0,0,y](H[-1/y,x] - H[-y,x] + H[0,x]) \\
+H[1,1,0,x] - H[0,0,0,y] - 2H[0,0,1,y] \\
+2H[0,1,y](H[-1/y,x] - H[-y,x] + 2H[0,x]) \\
+2H[-1/y,-1,0,x] - H[-1/y,0,0,x] \\
+2H[-y,-1,0,x] - H[-y,0,0,x] \\
+H[-1,0,0,x] - H[-1,1,0,x] - H[0,-1,0,x] \\
+H[0,1,0,x] - H[1,-1,0,x] + H[1,0,0,x] \\
-3H[-1,-1,0,x]\Big\}.
\end{aligned}
$$

$$
\begin{aligned}
\texttt{B5l2m3d3}[0,x,y] = \frac{y}{1-y^2}\{\zeta_2(4H[-1,y] \\
-7/2H[0,y] - 8H[1,y] - 5H[-1/y,x] \\
-3H[-y,x] + 4H[0,x]) \\
-\zeta_3/2 \\
+H[-1,0,0,y] + 2H[-1,0,1,y] \\
-5/2H[0,0,0,y] - 5H[0,0,1,y] - 2H[0,1,0,y] \\
-4H[0,1,1,y] + H[-1/y,0,0,x] \\
-4H[1,0,1,y] - 2H[-1/y,-1,0,x] \\
-2H[1,0,0,y] + 2H[-y,-1,0,x] - H[-y,0,0,x] \\
-(H[-1/y,0,x] - H[-y,0,x])(H[0,y] + 2H[1,y]) \\
-(H[-1/y,x] + H[-y,x] - H[0,x]) \\
(H[0,0,y] + 2H[0,1,y])\}.
\end{aligned}
$$

The singularities of the three dotted masters in $\epsilon = (4-d)/2$ have been determined recently using the method of differential equations, PRD 2005:

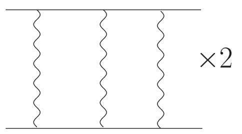
$$
\begin{aligned}
\texttt{B5l2m3d1} = &-\frac{1}{\epsilon^2}\frac{(1+x^2)y}{8x(1-y)^2} \\
&+\frac{1}{\epsilon}\Big\{\frac{y(1+x^2)}{4(1-y)^2} - \frac{2y^2(1-x)^2}{x(1-y)(1+y)^3}\zeta_2 \\
&-\frac{y}{2x(1-y)^2(1+y)^2}[2y(1+x^2)+x(1-y)^2] \\
&(H[0,y]+2H[1,y]) \\
&+\frac{y(1-x^2)}{4x(1-y)^2}H[0,x] \\
&-\frac{y^2(1-x)^2}{2x(1-y)(1+y)^3}\,(H[0,0,y]+2H[0,1,y])\Big\}
\end{aligned}
$$

$$
\begin{aligned}
\texttt{B5l2m3d2} = &-\frac{1}{\epsilon^2}\frac{xyH[0,x]}{(1-x^2)(1-y)^2} \\
&-\frac{1}{\epsilon}\frac{2xy}{(1-x^2)(1-y)^2}(H[1,0,x]-H[-1,0,x] \\
&+H[0,0,x]+\frac{\zeta_2}{2}+H[0,x](H[0,y] \\
&+2\,H[1,y]))
\end{aligned}
$$

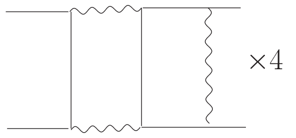
$$
\begin{aligned}
\texttt{B5l2m3d3} = &-\frac{1}{\epsilon}\frac{y}{2(1-y^2)}[4\zeta_2+H[0,0,y] \\
&+2H[0,1,y]]
\end{aligned}
$$

B5l2m3 is finite.

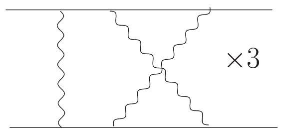
Ustron 2005: The divergences in $1/\epsilon$ of B5l3m



B7l4m1

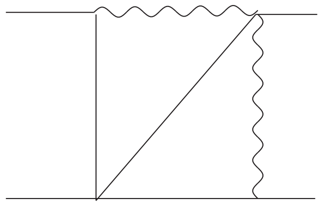


B7l4m2

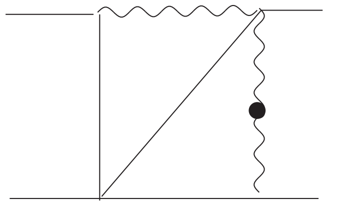


B7l4m3

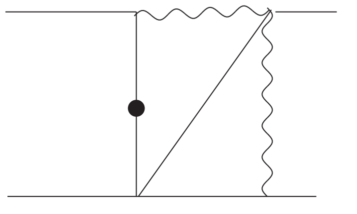
The B5l3m boxes, contribute to B2 (2nd planar 2-box) (shrink two lines...)



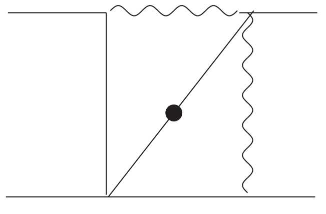
B5l3m



B5l3md1



B5l3md2



B5l3md3

The B5l3md2 topology appears twice as a master
but the B5l3md1 does not!

The B5l3m topology: Gross features

$$
\begin{aligned}
MB5l3m[x,y] &= Sum[B5l3m[k,x,y]*ep^k,k,0,1];\\
MB5l3md1[x,y] &= Sum[B5l3md1[k,x,y]*ep^k,k,-2,1];\\
MB5l3md2[x,y] &= Sum[B5l3md2[k,x,y]*ep^k,k,-2,1];\\
MB5l3md2a[x,y] &= Sum[B5l3md2a[k,x,y]*ep^k,k,-2,1];\\
MB5l3md3[x,y] &= Sum[B5l3md3[k,x,y]*ep^k,k,-1,1];
\end{aligned}
$$

Note:

- B5l3m – the basic master is finite
- B5l3md2 – use 4-dim. MB-Representation
- B5l3md2' – the same, but $(s \leftrightarrow t)$
- B5l3md1, B5l3md3 – system of 2 coupled differential eqns
  Only BLB5l3md1 has $1/\epsilon^2$ (so decouples), and last step is the two $1/\epsilon$ coefficients of B5l3md1 and B5l3md3.
  The first one is found by algebraic manipulations (see Czakon et al. LCWS Paris 2004), the second then fulfils a diff.eqn

Ustron 2005: The divergent parts of all B5l3m:

```
B5l3m[-2,x_,y_] = B5l3m[-1,x_,y_] = 0;

B5l3md1[-2,x_,y_] = ((-1 + x)^2*y*(-1 + y^2 + 2*y*H[0, y]))/
                    (8*x*(-1 + y)*(1 + y)^3);
B5l3md1[-1,x_,y_] = ((y*(6*(-1 + x - x^2 + x^3)*H[0, x]*
  (-1 + y^2 + 2*y*H[0, y]) - 6*(1 + x)*(-2 - 2*x^2 + 2*y^2 + 2*x^2*y^2 +
  y*z2 - 2*x*y*z2 + x^2*y*z2 + 2*(-2*x - y + 2*x*y - x^2*y - 2*x*y^2 +
  (-1 + x)^2*y*H[-1, -y] + 3*(-1 + x)^2*y*H[-1, y])*H[0, y] -
  6*(-1 + x)^2*y*H[0, -1, y] - 4*y*H[0, 0, y] + 8*x*y*H[0, 0, y] -
  4*x^2*y*H[0, 0, y] + 2*y*H[0, 1, y] - 4*x*y*H[0, 1, y] +
  2*x^2*y*H[0, 1, y])))/(24*x*(1 + x)*(-1 + y)*(1 + y)^3));

B5l3md2[-2,x_,y_] = -x/(1 - x^2)/4 H[0, x];
B5l3md2[-1,x_,y_] = ((x*(2*(1 + y^2)*H[0, x]*H[0, y] -
   (-1 + y^2)*(z2 + 6*H[-1, 0, x] - 4*H[0, 0, x] -
       2*H[1, 0, x])))/(4*(-1 + x^2)*(-1 + y^2)));

B5l3md2a[-a,x_,y_] = B5l3md2[-a,y,x],   a=-2,-1;

B5l3md3[-2,x_,y_] = 0;
B5l3md3[-1,x_,y_] = -((x*y*H[0, x]*H[0, y])/((-1 + x^2)*(-1 + y^2)));
```

.

Mellin-Barnes representations

Boos, Davydychev 1991, Smirnov 1999, Tausk 1999, Smirnov book 2004

$$\frac{1}{(A+B)^{\nu}} = \frac{B^{-\nu}}{(1-(-A/B))^{-\nu}} = \frac{B^{-\nu}}{2\pi i\Gamma(\nu)} \int\limits_{-i\infty}^{i\infty} d\sigma A^{\sigma} \; B^{-\sigma} \; \Gamma(-\sigma)\Gamma(\nu+\sigma)$$

Is special case of a well-known Mellin-Barnes integral for hypergeometric functions

$$\begin{aligned} \frac{1}{(1-z)^{\nu}} &= {}_2F_1(\nu, b, b', z)|_{b=b'} \\ &= \frac{1}{2\pi i\Gamma(\nu)} \frac{\Gamma(b')}{\Gamma(b)} \int_{-i\infty}^{+i\infty} d\sigma (-z)^{\sigma} \Gamma(\nu+\sigma)\Gamma(-\sigma) \frac{\Gamma(b+\sigma)}{\Gamma(b'+\sigma)} \end{aligned}$$

with $-z = A/B$.

How can this be made useful here?

.

Feynman Integrals after momentum integrations

$$G(X) \quad = \quad \frac{1}{(i\pi^{d/2})^L} \int \frac{d^D k_1 \ldots d^D k_L \;\; X}{(q_1^2 - m_1^2)^{\nu_1} \ldots (q_j^2 - m_j^2)^{\nu_j} \ldots (q_N^2 - m_N^2)^{\nu_N}}.$$

The denominator of G contains, after introduction of Feynman parameters x_i, the momentum dependent function

$$m^2 \quad = \quad \sum_{i=1}^{N} x_i (q_i^2 - m_i^2) \;=\; kMk - 2kQ + J.$$

$$U(x) \quad = \quad (\mathrm{det}\ M),$$

$$F(x) \quad = \quad (\mathrm{det}\ M)\mu^2 \;=\; -(\mathrm{det}\ M)\ (J + Q\ M^{-1}\ Q) \;=\; \sum A_{ij} x_i x_j.$$

The $\tilde{M}$ is defined with $M^{-1} = \tilde{M}/\mathrm{det}\ M$. $\qquad G(1) = \ldots$

$$G(k_{1\alpha}) \quad = \quad (-1)^{N_\nu} \frac{\Gamma\left(N_\nu - \frac{D}{2}L\right)}{\Gamma(\nu_1)\ldots\Gamma(\nu_N)} \int_0^1 \prod_{j=1}^{N} dx_j \; x_j^{\nu_j - 1} \delta\left(1 - \sum_{i=1}^{N} x_i\right) \frac{U(x)^{N_\nu - 1 - D(L+1)/2}}{F(x)^{N_\nu - DL/2}} \left[\sum_l \tilde{M}_{1l} Q_l\right]_\alpha ,$$

and

$$G(k_{1\alpha} k_{2\beta}) \quad = \quad (-1)^{N_\nu} \frac{\Gamma\left(N_\nu - \frac{D}{2}L\right)}{\Gamma(\nu_1)\ldots\Gamma(\nu_N)} \int_0^1 \prod_{j=1}^{N} dx_j \; x_j^{\nu_j - 1} \delta\left(1 - \sum_{i=1}^{N} x_i\right) \frac{U(x)^{N_\nu - 2 - D(L+1)/2}}{F(x)^{N_\nu - DL/2}}$$

$$\times \sum_l \left[[\tilde{M}_{1l} Q_l]_\alpha [\tilde{M}_{2l} Q_l]_\beta - \frac{\Gamma\left(N_\nu - \frac{D}{2}L - 1\right)}{\Gamma\left(N_\nu - \frac{D}{2}L\right)} \frac{g_{\alpha\beta}}{2} U(x) F(x) \frac{(V_{1l}^{-1})^{+}(V_{2l}^{-1})}{\alpha_l} \right].$$

For 2loops with N lines: $(9 - N)$ irreducible numerators $k_i^2, k_1 k_2, k_i p_e$

Feynman parameters II

In 2-loops, consider two subsequent sub-loops (the first: off-shell 1-loop, second on-shell 1-loop) and get e.g. for B7l4m2, the planar 2nd type 2-box:

$$K_{\text{1-loop Box,off}} = \frac{(-1)^{a_{4567}}\Gamma(a_{4567}-d/2)}{\Gamma(a_4)\Gamma(a_5)\Gamma(a_6)\Gamma(a_7)} \int_0^\infty \Pi_{j=4}^7 dx_j x_j^{a_j-1} \frac{\delta(1-x_4-x_5-x_6-x_7)}{F^{a_{4567}-d/2}}$$

where $a_{4567} = a_4 + a_5 + a_6 + a_7$ and the function F is characteristic of the diagram; here for the off-shell 1-box (2nd type):

$$\begin{aligned} F &= [-t]x_4x_7 + [-s]x_5x_6 + m^2(x_5+x_6)^2 \\ &\quad +(m^2-Q_1^2)x_7(x_4+2x_5+x_6) + (m^2-Q_2^2)x_7x_5 \end{aligned}$$

We want to apply now:

$$\int_0^1 \prod_i^4 dx_i\; x_i^{\alpha_i-1}\; \delta\left(1-x_1-x_2-x_3-x_4\right) = \frac{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)\Gamma(\alpha_4)}{\Gamma\left(\alpha_1+\alpha_2+\alpha_3++\alpha_4\right)}$$

with coefficients α_i dependent on a_i and on F

For this, we have to apply several MB-integrals here.
And repeat the procedure for the 2nd subloop.

For the 2nd planar 2-box, B7l4m2, one gets (Smirnov book 4.73):

$$B_{\mathrm{pl},2} = \frac{\text{const}}{(2\pi i)^6} \int_{-i\infty}^{+i\infty} \left[\frac{m^2}{-s}\right]^{z_5+z_6} \left[\frac{-t}{-s}\right]^{z_1} \prod_{j=1}^{6} [dz_j \Gamma(-z_j)] \frac{\prod_{k=7}^{18} \Gamma_k(\{z_i\}}{\prod_{l=19}^{24} \Gamma_l(\{z_i\}}$$

with $a = a_1 + ... + a_7$ and

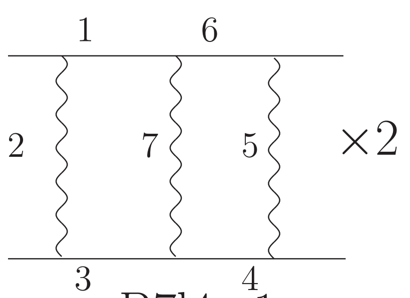
$$\begin{aligned} z_i &= \text{const} + i\, \Im m(z_i) \\ d &= 4 - 2\epsilon \\ \text{const} &= \frac{(i\pi^{d/2})^2(-1)^a(-s)^{d-a}}{\Gamma(a_2)\Gamma(a_4)\Gamma(a_5)\Gamma(a_6)\Gamma(a_7)\Gamma(d-a_{4567})} \end{aligned}$$

The integrand includes e.g.:

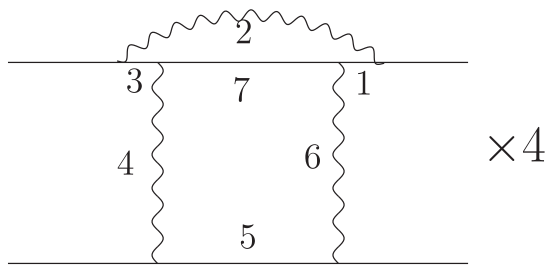
$$\begin{aligned} \Gamma_2 &= \Gamma(-z_2) \\ \Gamma_4 &= \Gamma(-z_4) \\ \Gamma_7 &= \Gamma(a_4 + z_2 + z_4) \\ \Gamma_8 &= \Gamma(D - a_{445667} - z_2 - z_3 - 2z_4) \\ &\ldots \end{aligned}$$

.

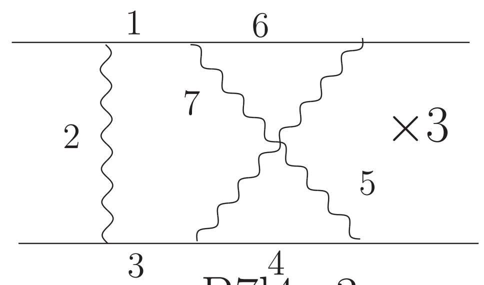
Shrinking of lines



B7l4m1



B7l4m2



B7l4m3

By shrinking selected lines, i.e. setting in the MB-integral representation

$$a_i = 0$$

one may derive from e.g. Smirnov's MB-integral for 7-line-diagrams simpler ones.

Done here e.g. for the topologies B6l3m1, B6l3m2.

For others, with irreducible numerators, one has to derive the MB-integral: here for B5l2m2, B5l2m3, B5l3m.

Further, sometimes a simpler representation may be obtained in a different way: here for B5l2m2, one integration less to be performed.

.

B5l2m2

$$\texttt{B5l2m2} = \frac{m^{4\epsilon}(-1)^{a_{12345}}e^{2\epsilon\gamma_E}}{\prod_{j=1}^{5}\Gamma[a_i]\Gamma[4-2\epsilon-a_{13}](2\pi i)^3}\int_{-i\infty}^{+i\infty}d\alpha\int_{-i\infty}^{+i\infty}d\beta\int_{-i\infty}^{+i\infty}d\gamma$$
$$(-s)^{2-\epsilon-a_{245}-\gamma-\alpha+\beta}(-t)^{\alpha}$$
$$\Gamma[-2+\epsilon+a_{13}+\beta]\Gamma[-\gamma]\Gamma[2-\epsilon-a_{245}-\gamma-\alpha]\Gamma[-\alpha]$$
$$\Gamma[a_2+\alpha]\Gamma[a_4+\alpha]\Gamma[4-2\epsilon-a_{113}-\beta]$$
$$\Gamma[-2+\epsilon+a_{245}+\gamma+\alpha-\beta]\Gamma[a_1+\beta]$$
$$\frac{\Gamma[4-2\epsilon-a_{2245}-2\alpha+\beta]\Gamma[2-\epsilon-a_{24}-\gamma-\alpha+\beta]}{\Gamma[4-2\epsilon-a_{245}+\beta]\Gamma[4-2\epsilon-a_{22445}-2\gamma-2\alpha+\beta]}$$

B5l3m$(p_e.k_1)$

$$\mathtt{B5l3m(p_e\cdot k_1)} = \frac{m^{4\epsilon}(-1)^{a12345}\ e^{2\epsilon\gamma_E}}{\prod_{j=1}^{5}\Gamma[a_i]\Gamma[5-2\epsilon-a_{123}](2\pi i)^4}\int_{-i\infty}^{+i\infty}d\alpha\int_{-i\infty}^{+i\infty}d\beta\int_{-i\infty}^{+i\infty}d\gamma\int_{-i\infty}^{+i\infty}d\delta$$

$$(-s)^{(4-2\ \epsilon)-a_{12345}-\alpha-\beta-\delta}\ (-t)^{\delta}$$

$$\frac{\Gamma[-4+2\ \epsilon+a_{12345}+\alpha+\beta+\delta]}{\Gamma[6-3\ \epsilon-a_{12345}-\alpha]}\frac{\Gamma[-\alpha]\ \Gamma[-\beta]}{\Gamma[7-3\ \epsilon-a_{12345}-\alpha]\ \Gamma[5-2\ \epsilon-a_{123}]}\frac{\Gamma[-\delta]}{\Gamma[4-2\ \epsilon-a_{1123}-2\ \alpha-\gamma]\ \Gamma[5-2\ \epsilon-a_{1123}-2\ \alpha-\gamma]}$$

$$\frac{\Gamma[2-\epsilon-a_{13}-\alpha-\gamma]}{\Gamma[8-4\ \epsilon-a_{112233445}-2\ \alpha-2\ \beta-2\ \delta-\gamma]}\frac{\Gamma[4-2\ \epsilon-a_{12345}-\alpha-\beta-\delta-\gamma]}{\Gamma[9-4\ \epsilon-a_{112233445}-2\ \alpha-2\ \beta-2\ \delta-\gamma]}\Big\{(p_e\cdot p_3)\ \Gamma[1+a_4+\delta]\ \Gamma[6-3\ \epsilon-a_{12345}-\alpha]$$

$$\Gamma[4-2\ \epsilon-a_{1234}-\alpha-\beta-\delta]\ \Gamma[3-\epsilon-a_{12}-\alpha]\ \Gamma[8-4\ \epsilon-a_{112233445}-2\ \alpha-2\ \delta-\gamma]\Gamma[9-4\ \epsilon-a_{112233445}-2\ \alpha-2\ \beta-2\ \delta-\gamma]$$

$$\Gamma[5-2\ \epsilon-a_{1123}-\gamma]\ \Gamma[4-2\ \epsilon-a_{1123}-2\ \alpha-\gamma]\ \Gamma[a_1+\gamma]\ \Gamma[-2+\epsilon+a_{123}+\alpha+\delta+\gamma]+\Gamma[a_4+\delta]\Big[-(p_e\cdot p_1)\ \Gamma[7-3\ \epsilon-a_{12345}-\alpha]$$

$$\Gamma[4-2\ \epsilon-a_{1234}-\alpha-\beta-\delta]\Gamma[8-4\ \epsilon-a_{112233445}-2\ \alpha-2\ \delta-\gamma]\Gamma[9-4\ \epsilon-a_{112233445}-2\ \alpha-2\ \beta-2\ \delta-\gamma]$$

$$\Big[\Gamma[3-\epsilon-a_{12}-\alpha]\ \Gamma[5-2\ \epsilon-a_{1123}-\gamma]\Gamma[4-2\ \epsilon-a_{1123}-2\ \alpha-\gamma]\ \Gamma[a_1+\gamma]+\Gamma[2-\epsilon-a_{12}-\alpha]\ \Gamma[4-2\ \epsilon-a_{1123}-\gamma]$$

$$\Gamma[5-2\ \epsilon-a_{1123}-2\ \alpha-\gamma]\ \Gamma[1+a_1+\gamma]\Big]\ \Gamma[-2+\epsilon+a_{123}+\alpha+\delta+\gamma]+\Gamma[6-3\ \epsilon-a_{12345}-\alpha]\ \Gamma[3-\epsilon-a_{12}-\alpha]$$

$$\Gamma[5-2\ \epsilon-a_{1123}-\gamma]\ \Gamma[4-2\ \epsilon-a_{1123}-2\ \alpha-\gamma]\Gamma[a_1+\gamma]\Big[((p_e\cdot(p_1+p_2))\ \Gamma[5-2\ \epsilon-a_{1234}-\alpha-\beta-\delta]\Gamma[9-4\ \epsilon-a_{112233445}-2\ \alpha-2\ \delta-\gamma]$$

$$\Gamma[8-4\ \epsilon-a_{112233445}-2\ \alpha-2\ \beta-2\ \delta-\gamma]\Gamma[-2+\epsilon+a_{123}+\alpha+\delta+\gamma]+(p_e\cdot p_1)\ \Gamma[4-2\ \epsilon-a_{1234}-\alpha-\beta-\delta]$$

$$\Gamma[8-4\ \epsilon-a_{112233445}-2\ \alpha-2\ \delta-\gamma]\Gamma[9-4\ \epsilon-a_{112233445}-2\ \alpha-2\ \beta-2\ \delta-\gamma]\Gamma[-1+\epsilon+a_{123}+\alpha+\delta+\gamma]\Big]\Big]\Big\}$$

Example:

derive from B7l4m2 the MB-integral for B5l3m by setting $a_1 = 0$ (trivial, gives B6l3m2) and then setting $a_4 = 0$.

The latter do with care because of

$$\frac{1}{\Gamma(a_4)} \rightarrow \frac{1}{\Gamma(0)} = 0$$

See by inspection that we will get factor $\Gamma(a_4)$ if $z_2, z_4 \rightarrow 0$.

$\rightarrow$ Start with the z_2, z_4 integrations by

taking the residues for closing the integration contours to the right:

$$\begin{aligned}
I_{2,4} &= \frac{(-1)^2}{(2\pi i)^2} \int dz_2 \Gamma(-z_2) \int dz_4 \frac{\Gamma(a_4 + z_2 + z_4)}{\Gamma(a_4)} \Gamma(-z_4) R(z_i) \\
&= \frac{1}{(2\pi i)} \int dz_2 \Gamma(-z_2) \sum_{n=0,1,..} \frac{-(-1)^n}{n!} \frac{\Gamma(a_4 + z_2 + n)}{\Gamma(a_4)} R(z_i) \\
&= \sum_{n,m=0,1,..} \frac{(-1)^{n+m}}{n!m!} \frac{\Gamma(a_4 + n + m)}{\Gamma(a_4)} R(z_i) \rightarrow_{a=0} 1
\end{aligned}$$

So, setting $a_1 = a_4 = 0$ and eliminating $\int dz_2 dz_4$ with setting $z_2 = z_4 = 0$

we got a 4-fold Mellin-Barnes integral for topology B5l3m (by "shrinking of lines")

with $24 - 3 = 21$ z_i-dependent Γ-functions which may yield residua within four-fold sums.

The MB-representation has to be calculated explicitely at fixed indices, e.g.

$$B5l3md2 = \frac{B_2}{\epsilon^2} + \frac{B_1}{\epsilon} + B_0$$

General Tasks:

- Find a region of definiteness of the n-fold MB-integral

  $$\Re(z_1) = -1/80, \Re(z_3) = -33/40, \Re(z_5) = -21/20, \Re(z_6) = -59/160, \Re(\epsilon) = -1/10!$$

- Then go to the physical region where $\epsilon << 1$ by distorting the integration path step by step (adding each crossed residuum – per residue this means one integral less!!!)
- Take integrals by sums over residua, i.e. introduce infinite sums
- Sum these infinite multiple series into some known functions of a given class, e.g. Nielsen polylogs, Harmonic polylogs or whatever is appropriate.

We derived an algorithmic solution for isolating the singularities in $1/\epsilon$
The automatization of that: MB.m (M. Czakon)

$$\begin{aligned} B5l3md2 \quad &\rightarrow \quad MB(\text{4-dim,fin}) + MB_3(\text{3-dim,fin}) \\ &+ \quad MB_{36}(\text{2-dim}, \epsilon^{-1}, fin) + MB_{365}(\text{1-dim}, \epsilon^{-2}, \epsilon^{-1,fin}) \\ &+ \quad MB_5(\text{3-dim,fin}) \end{aligned}$$

After these preparations e.g.:

$$\begin{aligned} MB_{365}(\text{1-dim}, \epsilon^{-2}) \quad &\sim \quad \frac{1}{\epsilon^2}\frac{1}{2\pi i}\int dz_6 \frac{(-s)^{(-z_6-1)}\Gamma(-z_6)^3\Gamma(1+z_6)}{8\Gamma(-2z_6)} \\ &= \quad \frac{1}{\epsilon^2}\sum_{n=0,\infty} -\frac{(-1)^n(-s)^n\Gamma(1+n)^3)}{8n!\Gamma(-2(-1-n))} \\ &= \quad -\frac{1}{\epsilon^2}\frac{ArcSin(\sqrt{s}/2)}{2\sqrt{4-s}\sqrt{s}} \\ &= \quad \frac{1}{\epsilon^2}\frac{-x}{4(1-x^2)}H[0,x] \end{aligned}$$

Here residua were taken at $z_6 = -n-1, n = 0, 1, ..,$ and $H[0,x] = \ln(x)$ and $x = \frac{\sqrt{-s+4}-\sqrt{-s}}{\sqrt{-s+4}+\sqrt{-s}}$.

.

Overview on exact results for massive 2-boxes

Exact solutions:

7 lines:

6 planar – 2x B7l4m1 (known) , 4x B7l4m2 (2nd planar, not much known)
(plus 3x B7l4m3, non-planar, nearly nothing known); Smirnov, Heinrich, CGR

6 lines:

4 planar – 4x B6l3m2 (plus 6x B6l3m3, non-planar)

5 lines:

all 15 masters are planar (all singularities are known)
B5l2m1, 3x B5l2m2 (all sing. kn.), 4x B5l2m3 (all sing. + 3 dotted kn.), 5x B5l3m (all sing. kn.), 2x B5l4m (known)

We have determined all the planar masters, in the limit of small electron mass.
To be published.

.

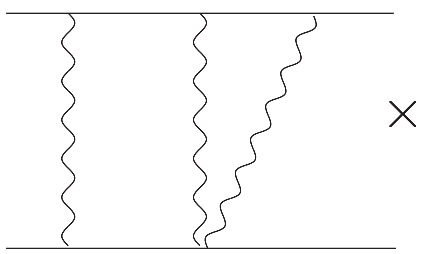
Solutions: The planar box masters for $m^2/s << 1, t/s = x$

$$
\begin{aligned}
\mathrm{L} &= \ln\left(-\frac{m^2}{s}\right) \\
x &= \frac{t}{s}
\end{aligned}
$$

The planar 7-line masters; examples

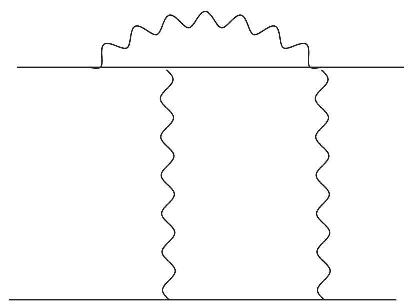
$$
\begin{aligned}
\texttt{B7l4m1} = \quad & + \frac{1}{\epsilon^2}\,\frac{2\mathrm{L}^2}{s^2 t} \\
& - \frac{1}{\epsilon}\,\frac{1}{3s^2 t}\left[-10\,\mathrm{L}^3 + 6\,\mathrm{L}\,\zeta_2 + 6\,\zeta_3 + 12\,\mathrm{L}^2\,\ln(x)\right] \\
& - \frac{1}{6s^2 t}\Big\{-12\,\mathrm{L}^4 + 28\,\mathrm{L}^3\,\ln(x) - 4\,\mathrm{L}^2\,(-30\,\zeta_2 + 3\,\ln^2(x)) \\
& - 4\,\mathrm{L}\,(9\,\zeta_3 + 30\,\zeta_2\,\ln(x) + \ln^3(x) - 18\,\zeta_2\,\ln(1+x) \\
& - 3\,\ln^2(x)\,\ln(1+x) - 6\,\ln(x)\,\mathrm{Li}_2(-x) + 6\,\mathrm{Li}_3(-x)) \\
& - 15\zeta_4 - 24\zeta_3\ln(x)\Big\}
\end{aligned}
$$

$$
\begin{aligned}
\mathtt{B714m2N}_3 = \quad & + \quad \frac{1}{\epsilon^2}\ x^2 \Big[\mathrm{L}^2 - \mathrm{L}\ \ln(x)\Big] \\
& + \quad \frac{1}{\epsilon}\ \Big[2\ \mathrm{L}^3\ x^2 + \mathrm{L}^2\ (-1/4 + x/2 - 3\ x^2\ \ln(x)) + \mathrm{L}\ (1 - x + x^2\ \ln^2(x)) \\
& - \quad 2\ \zeta_2 + 4\ x\ \zeta_2 + x^2\ \zeta_2\ \ln(x)\Big] \\
& + \quad \frac{1}{24}\Big\{25\ \mathrm{L}^4\ x^2 + \mathrm{L}^3\ (-8 + 16\ x + 84\ x^2 - 48\ x^2\ \ln(x)) \\
& + \quad \mathrm{L}^2\ (24 - 72\ x^2 - 456\ x^2\ \zeta_2 - 216\ x^2\ \ln(x)) \\
& + \quad \mathrm{L}\ (120 - 120\ x - 60\ \zeta_2 + 120\ x\ \zeta_2 + 576\ x^2\ \zeta_2 + 192\ x^2\ \zeta_3 \\
& + \quad 96\ x^2\ \ln(x) + 696\ x^2\ \zeta_2\ \ln(x) + 144\ x^2\ \ln^2(x) + 32\ x^2\ \ln^3(x) \\
& - \quad 288\ x^2\ \zeta_2\ \ln(1+x) - 48\ x^2\ \ln(x)^2\ \ln(1+x) \\
& - \quad 96\ x^2\ \ln(x)\ \mathrm{Li}_2(-x) + 96\ x^2\ \mathrm{Li}_3(-x))\Big\} \\
& + \quad \Big\{(3\ ((-6 + 22\ x + 60\ x^2)\ \zeta_2 + (-7 + 2\ (7 - 6\ x)\ x)\ \zeta_3 - 67\ x^2\ \zeta_4) \\
& + \quad 2\ x^2\ (-(\ln(x)\ (54\ \zeta_2 + 21\ \zeta_3 + \ln(x)\ (27\ \zeta_2 + \ln(x)\ (3 + \ln(x))))) \\
& + \quad (9 + 2\ \ln(x))\ (6\ \zeta_2 + \ln^2(x))\ \ln(1+x)))/6 \\
& - \quad 2\ x^2\ ((4\ \zeta_2 - 3\ \ln(x))\ \mathrm{Li}_2(-x) \\
& + \quad (3 - 2\ \ln(x))\ \mathrm{Li}_3(-x) + 4\ \mathrm{Li}_4(-x))\Big\}
\end{aligned}
$$



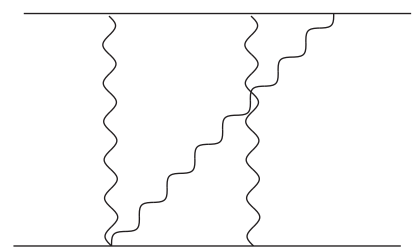
$\times 2$

B6l3m1



$\times 2$

B6l3m2



$\times 6$

B6l3m3

The two-loop box MIs with six internal lines. Czakon, Gluza, TR, Unpublished.

The planar 6-line masters; examples

$$
\begin{aligned}
\texttt{B6l3m1} = \quad & + \quad \frac{1}{\epsilon}\,\frac{1}{2st}\left[\mathrm{L}^3 - 2\mathrm{L}^2\ln(x) + \mathrm{L}[8\zeta_2 + \ln^2(x)]\right] \\
& + \quad \frac{1}{24st}\Big\{7\,\mathrm{L}^4 - 32\,\mathrm{L}^3\,\ln(x) + \mathrm{L}^2\,(24\,\zeta_2 + 42\,\ln^2(x)) \\
& + \quad \mathrm{L}\,(24\,\zeta_3 - 72\,\zeta_2\,\ln(x) - 16\,\ln^3(x) + 72\,\zeta_2\,\ln(1+x) \\
& + \quad 12\,\ln^2(x)\,\ln(1+x) + 24\,\ln(x)\,\mathrm{Li}_2(-x) - 24\,\mathrm{Li}_3(-x))\Big\} \\
& + \quad \frac{1}{24st}\Big\{-672\,\zeta_4 - 48\,\zeta_2\,\ln^2(x) - \ln^4(x) - 12\,(6\,\zeta_2 + \ln^2(x))\,\mathrm{Li}_2(-x) \\
& + \quad 48\,\ln(x)\,\mathrm{Li}_3(-x) - 72\,\mathrm{Li}_4(-x)\Big\} \\
\texttt{B6l3m1N} = \quad & - \quad \frac{1}{\epsilon}\,\frac{1}{2s}\left[-\mathrm{L}^2 - 4\mathrm{L} + 2\zeta_2\right] \\
& - \quad \frac{1}{6s}\Big\{\mathrm{L}^3 + 12\,\zeta_2 + 12\,\zeta_3 + \mathrm{L}^2\,(-6 - 12\,\ln(x)) \\
& + \quad \mathrm{L}\,(-24 + 24\,\zeta_2 + 6\,\ln^2(x))\Big\}
\end{aligned}
$$

The planar 5-line masters; examples

$$
\begin{aligned}
\texttt{B5l2m3} = \quad & + \quad \frac{1}{12u}\Big\{-6\,\mathrm{L}^2\,(6\,\zeta_2+\ln^2(x)) \\
& - \quad 6\,\mathrm{L}\,(-4\,\zeta_3+4\,\zeta_2\,\ln(x)-12\,\zeta_2\,\ln(1+x)-2\,\ln^2(x)\,\ln(1+x) \\
& - \quad 4\,\ln(x)\,\mathrm{Li}_2(-x)+4\,\mathrm{Li}_3(-x)) \\
& + \quad 312\,\zeta_4+72\,\zeta_3\,\ln(x)+36\,\zeta_2\,\ln^2(x)+\ln^4(x)-24\,\zeta_3\,\ln(1+x) \\
& + \quad 24\,\zeta_2\,\ln(x)\,\ln(1+x)-36\,\zeta_2\,\ln^2(1+x)-6\,\ln^2(x)\,\ln^2(1+x) \\
& - \quad 24\,\ln(x)\,\mathrm{S}_{1,2}(-x)+12\,(8\,\zeta_2+\ln^2(x)-2\,\ln(x)\,\ln(1+x))\,\mathrm{Li}_2(-x) \\
& - \quad 48\,\ln(x)\,\mathrm{Li}_3(-x)+24\,\ln(1+x)\,\mathrm{Li}_3(-x) \\
& + \quad 72\,\mathrm{Li}_4(-x)+24\,\mathrm{S}_{2,2}(-x)\Big\}
\end{aligned}
$$

$$
\begin{aligned}
\texttt{B5l2m3d2} = \quad & - \quad \frac{1}{\epsilon^2}\,\frac{1}{st}\mathrm{L} \\
& + \quad \frac{1}{\epsilon}\,\frac{1}{st}\Big[-3\,\mathrm{L}^2+2\,\mathrm{L}\,\ln(x)-\zeta_2\Big] \\
& + \quad \frac{1}{3st}\Big\{-10\,\mathrm{L}^3+12\,\mathrm{L}^2\,\ln(x)+51\,\mathrm{L}\,\zeta_2-21\,\zeta_3-42\,\zeta_2\,\ln(x)-2\,\ln^3(x)+36\,\zeta_2\,\ln(1+x) \\
& + \quad 6\,\ln^2(x)\,\ln(1+x)+12\,\ln(x)\,\mathrm{Li}_2(-x)-12\,\mathrm{Li}_3(-x)\Big\}
\end{aligned}
$$

Higher orders in ϵ may be determined:

$$
\begin{aligned}
& \texttt{B5l2m3d2} = \cdots \\
+ \quad & \epsilon \, \frac{1}{3st} \Bigg\{ \Bigg\{ -5\, \mathrm{L}^4 + 111\, \mathrm{L}^2\, \zeta_2 + 10\, \mathrm{L}^3\, \ln(x) + \mathrm{L}\, (104\, \zeta_3 - 126\, \zeta_2\, \ln(x) - 6\, \ln^3(x) + 108\, \zeta_2\, \ln(1+x) \\
+ \quad & 18\, \ln^2(x)\, \ln(1+x) + 36\, \ln(x)\, \mathrm{Li}_2(-x) - 36\, \mathrm{Li}_3(-x)) - 372\, \zeta_4 - 78\, \zeta_3\, \ln(x) + 30\, \zeta_2\, \ln^2(x) + 2\, \ln^4(x) + 12\, \zeta_3\, \ln(1+x) \\
- \quad & 60\, \zeta_2\, \ln(x)\, \ln(1+x) - 8\, \ln^3(x)\, \ln(1+x) + 18\, \zeta_2\, \ln^2(1+x) + 3\, \ln^2(x)\, \ln^2(1+x) + 12\, \ln(x)\, \mathrm{S}_{1,2}(-x) \\
- \quad & 6\, (4\, \zeta_2 + 3\, \ln^2(x) - 2\, \ln(x)\, \ln(1+x))\, \mathrm{Li}_2(-x) + 24\, \ln(x)\, \mathrm{Li}_3(-x) - 12\, \ln(1+x)\, \mathrm{Li}_3(-x) \\
- \quad & 12\, \mathrm{Li}_4(-x) - 12\, \mathrm{S}_{2,2}(-x) \Bigg\} \Bigg\}
\end{aligned}
\tag{1}
$$

$$
\begin{aligned}
\texttt{B5l2m3N1} \quad = \quad & \frac{1}{4}\left(\frac{s}{u}\right)^2 \Big\{ \mathrm{L}^2\ (6\ x\ \zeta_2 + 2\ x\ \ln(x) + 2\ x^2\ \ln(x) + x\ \ln^2(x)) \\
+ \quad & \mathrm{L}\ (16\ x\ \zeta_2 - 8\ x^2\ \zeta_2 - 4\ x\ \zeta_3 - 2\ \ln(x) + 2\ x^2\ \ln(x) \\
+ \quad & 4\ x\ \zeta_2\ \ln(x) + 2\ x\ \ln^2(x) - 2\ x^2\ \ln^2(x) - 12\ x\ \zeta_2\ \ln(1+x) \\
- \quad & 2\ x\ \ln^2(x)\ \ln(1+x) - 4\ x\ \ln(x)\ \mathrm{Li}_2(-x) + 4\ x\ \mathrm{Li}_3(-x)) \Big\} \\
+ \quad & \frac{1}{120}\left(\frac{s}{u}\right)^2 \Big\{ +120\ \zeta_2 + 360\ x\ \zeta_2 - 120\ x^2\ \zeta_2 - 1560\ x\ \zeta_4 - 480\ x\ \zeta_3 \\
- \quad & 240\ x^2\ \zeta_3 - 240\ x\ \zeta_2\ \ln(x) - 480\ x^2\ \zeta_2\ \ln(x) - 360\ x\ \zeta_3\ \ln(x) \\
+ \quad & 30\ \ln^2(x) + 60\ x\ \ln^2(x) - 30\ x^2\ \ln^2(x) - 180\ x\ \zeta_2\ \ln^2(x) \\
- \quad & 20\ x\ \ln^3(x) - 20\ x^2\ \ln^3(x) - 5\ x\ \ln^4(x) + 720\ x^2\ \zeta_2\ \ln(1+x) \\
+ \quad & 120\ x\ \zeta_3\ \ln(1+x) - 120\ x\ \zeta_2\ \ln(x)\ \ln(1+x) + 120\ x^2\ \ln^2(x)\ \ln(1+x) \\
+ \quad & 180\ x\ \zeta_2\ \ln^2(1+x) + 30\ x\ \ln^2(x)\ \ln^2(1+x) + 120\ x\ \ln(x)\ \mathrm{S}_{1,2}(-x) \\
+ \quad & 60\ x\ (-8\ \zeta_2 - \ln^2(x) + 2\ \ln(x)\ (2\ x + \ln(1+x)))\ \mathrm{Li}_2(-x) \\
- \quad & 240\ x^2\ \mathrm{Li}_3(-x) + 240\ x\ \ln(x)\ \mathrm{Li}_3(-x) - 120\ x\ \ln(1+x)\ \mathrm{Li}_3(-x) \\
- \quad & 360\ x\ \mathrm{Li}_4(-x) - 120\ x\ \mathrm{S}_{2,2}(-x) \Big\}
\end{aligned} \tag{2}
$$

The $N_f > 1$ contributions

Actis, Czakon, Gluza, TR, under study

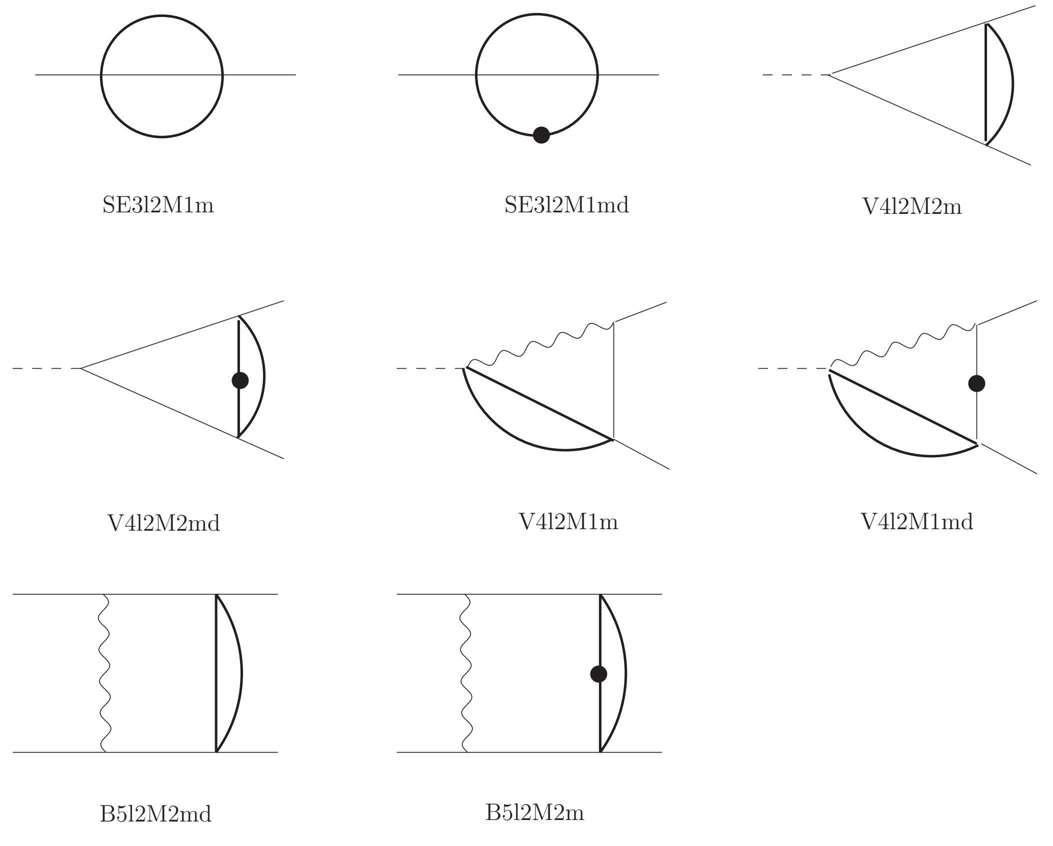


Figure 1: The eight additional master integrals with two different mass scales.

The 2-box-diagrams represent a three-scale problem: $s/m^2, t/m^2, M^2/m^2$

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Summary

Recent essential progress for the massive 2-box master integral determination:

- A complete List of Masters (2004)
- Huge files with algebraic Tables of Substitutions for all the reducible Feynman integrals needed for the interferences of boxes with Born
- Underway: Determination of all non-planar 2-box masters
  use Generalized Harmonic Polylogarithms, introduced by Remiddi, Vermaseren
  plus . . .)
- An unsolved problem is the systematic summation of the massive multiple sums after the MB-integral evaluation

Most important conclusion from the efforts of the last year:

It is possible to do the complete massive 2-loop calculation with present computers and human resources. An analytical solution in terms of known functions seems to be impossible.

Status by end of 2004

Established: 10^{-3} MC programs for LEP, ILC

Introduction to NLLBHA by Trentadue and to BHLUMI by Jadach in:
Proc. of Loops and Legs, Rheinsberg, Germany, 1996

Recent mini-review: Jadach, "Theoretical error of luminosity cross section at LEP", hep-ph/0306083 [1]

- BHLUMI v.4.04: Jadach, Placzek, Richter-Was, Was: CPC 1997
- see also: Jadach, Melles, Ward, Yost: PLB 1996, thesis Melles 1996 [2]
- NLLBHA: Arbuzov, Fadin, Kuraev, Lipatov, Merenkov, Trentadue: NPB 1997, CERN 96-01
- SAMBHA: Arbuzov, Haidt, Matteuzzi, Paganoni, Trentadue: hep-ph/0402211

See e.g.: Table 1 of [1] and Figure 3.1 of [2] $\rightarrow$ Conclude:
The nonlogarithmic $O(\alpha^2)$ terms, originating from pure QED radiative 1-loop and from 2-loop diagrams are not completely covered.

They have to be calculated and integrated into the MC programs.

Beware:

$$m_e, m_\gamma, (d-4), E_\gamma$$

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Status 2005

Know the constant term ($m_e = 0$) from 2-loop Bhabha scattering

A. Penin, Two-Loop Corrections to Bhabha Scattering, hep-ph/0501120 v.3, $\rightarrow$ PRL

Transform the massless 2-loop results of Bern, Dixon, Ghinculov (2002) with InfraRed (IR) regulation by $D = 4 - 2\epsilon$ into the on-mass-shell renormalization with $m_e \rightarrow 0$ and IR regulation by $\lambda = m_\gamma \neq 0$

Use IR-properties of amplitudes (see Penin):

[A] Exponentiation of the IR logarithms (Sudakov 1956,...)

[B] Factorization of the collinear logarithms into expernal legs (Frenkel, Taylor 1976)

[C] Non-renormalization of the IR exponents (YFS 1961, ...)

Isolate the closed fermion loop contribution (does not fulfil [C]) and add it separately (Burgers 1985, Bonciani et al. 2005, Penin)

If all this is correct, the constant term in m_e is known for the MCs (but the radiative one-loops with 5-point functions).