Feynman Integrals
Mellin-Barnes representations
Sums

Helmholtz International School

Calculations for Modern and Future Colliders

July 10 – 20, 2009, JINR, Dubna, Russia

Tord Riemann, DESY, Zeuthen

date of print: 2009-07-14 20:40

The slides of lecture are held at

http://www-zeuthen.desy.de/ riemann/

The slides of a quite similar lecture given at

CAPP 2009

– DESY School on Computer Algebra in Particle Physics

with exercises by J. Gluza are held at

https://indico.desy.de/conferenceDisplay.py?confId=1573

Plan of Lectures

- **Introduction + Motivation**
- **Mathematical Reminder on Γ-function, Residues, Cauchy-theorem**
- **Few simple Feynman integrals calculated conventionally**
- **The same Feynman integrals, their Mellin-Barnes [MB] representations and again their evaluation**
- **AMBRE – a tool for the derivation of MB-representations**
- **Expansions in a small parameter, e.g. $m^2/s << 1$**
- **If time left: More complicated Feynman integrals**

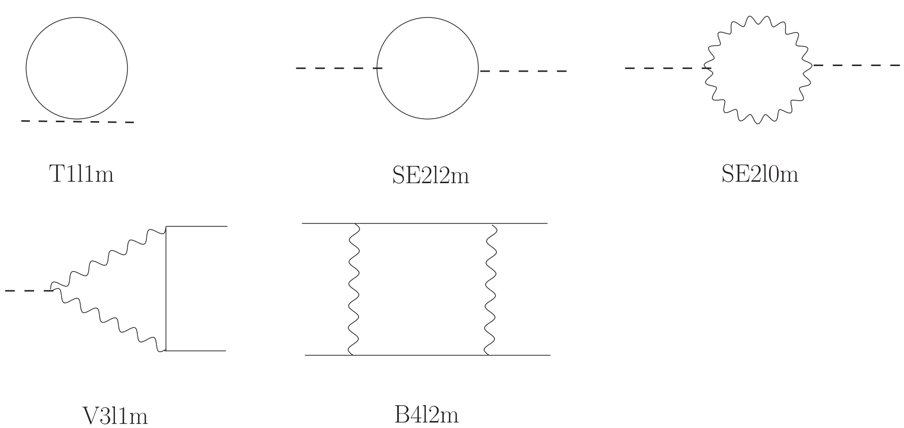
Introductory

For many problems of the past, a relatively simple approach to the evaluation of Feynman integrals was sufficient:

★ **Tensor reduction a la Passarino/Veltmann, 1979**

★ **Evaluate Feynman parameter integrals by direct integration**

Typically 1-loop (massless: 2-loop), typically $2 \longrightarrow 2$ scattering (plus bremsstrahlung)
Feynman parameters may be used and by direct integration over them one gets objects like: $\frac{23}{57}, \quad \zeta(3), \quad \ln(\frac{t}{s}), \quad \ln(\frac{t}{s}) \cdot \ln(\frac{s}{m^2}), \quad \mathrm{Li}_2(\frac{t}{s+i\epsilon})$ **etc.**
With more complexity of the reaction (more legs) and more perturbative accuracy (more loops), this approach appears to be not sufficiently sophisticated.



$$\texttt{T1l1m} = \frac{1}{\epsilon} + 1 + (1 + \frac{\zeta_2}{2})\epsilon + (1 + \frac{\zeta_2}{2} - \frac{\zeta_3}{3})\epsilon^2 +$$

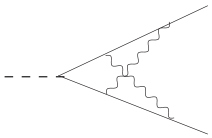
$$\texttt{B4l2m} = [-\frac{1}{\epsilon} + \ln(-s)]\frac{2y\ln(y)}{s(1-y^2)} + c_1\epsilon + \cdots$$

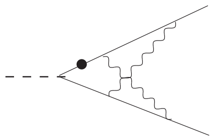
with $d = 4 - 2\epsilon$ **and** $m = 1$ **and**

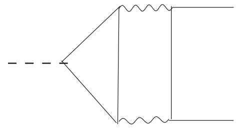
$$y = \frac{\sqrt{1-4/t}-1}{\sqrt{1-4/t}+1}$$

Figure shows so-called master integrals.

Some loops

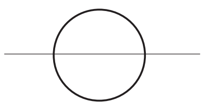

V6l4m1

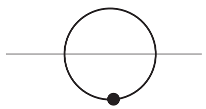

V6l4m1d

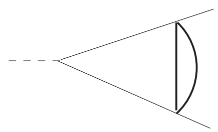

V6l4m2

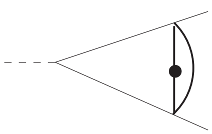
Two-loop vertex integrals with six internal lines

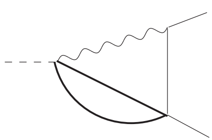
massless case: only fixed numbers and one scale factor

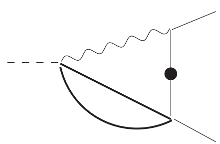

SE3l2M1m

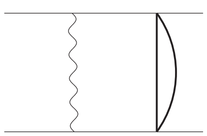

SE3l2M1md

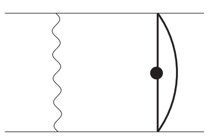

V4l2M2m


V4l2M2md

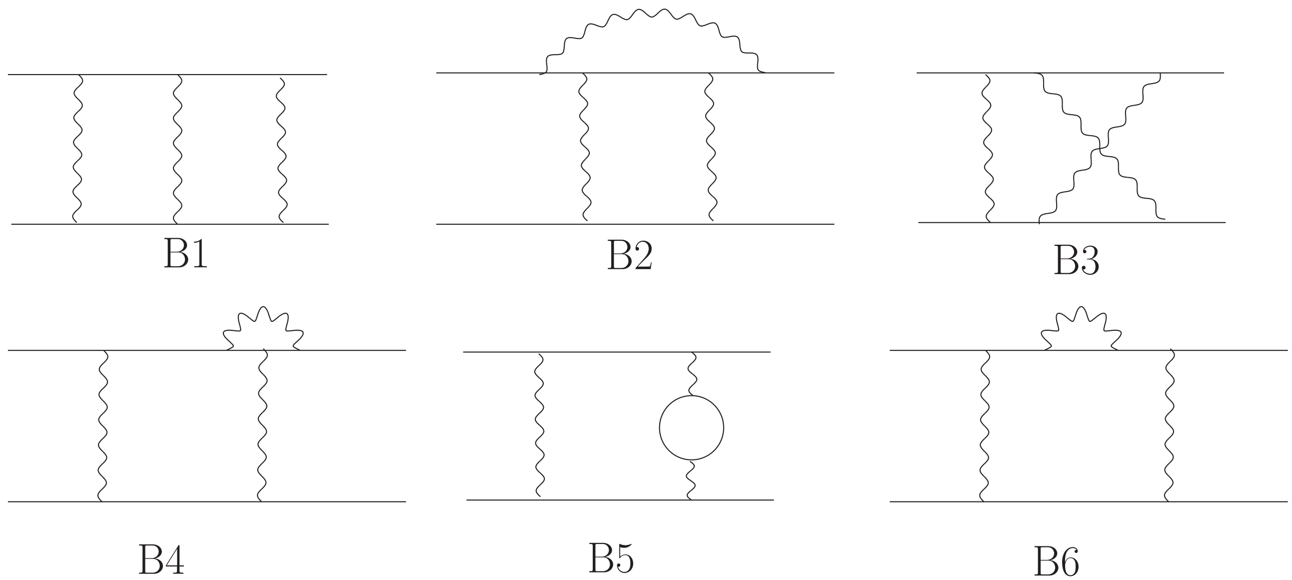

V4l2M1m


V4l2M1md


B5l2M2md


B5l2M2m

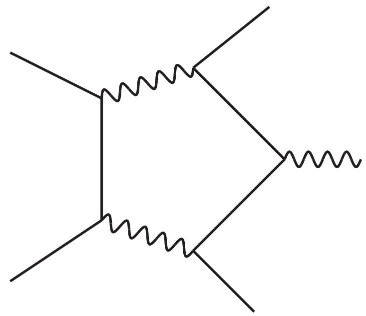
Integrals with two different mass scales m and M



Two-loop box diagrams for massive $2 \to 2$ scattering

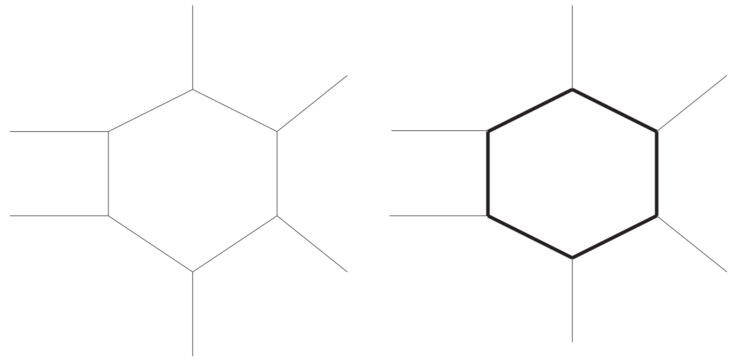
Variables for $2 \to 2$ scattering, i.e. box diagrams: s, t or s and $\cos\theta$

More legs



Massive pentagons: 5 kinematic variables + several masses

Variables for $2 \to 3$ scattering: 5 = 2 + 3 (three additional momenta of a particle)



Massless and massive hexagons: 8 kinematic variables + several masses

Variables for $2 \to 4$ scattering: 8 = 5 + 3 (another three additional)

→ See Lectures by Jochem Fleischer and Teo Diakonidis on 5- and 6-point functions

Some Mathematical Preparations

We will often use, for $d = 4 - 2\epsilon$:

$$a^{\epsilon} \; = \; e^{\epsilon \ln(a)} = 1 + \ln(a)\;\epsilon + \frac{1}{2}\ln^2(a)\;\epsilon^2 + \ldots$$

The Γ-function

The Γ-function may be defined by a difference equation:

$$z\Gamma(z) - \Gamma(z+1) = 0$$

$$\begin{aligned}
\Gamma(z) &= \int_0^\infty t^{z-1} e^{-t} dt \\
\Gamma(0) &= \infty \\
\Gamma(1) &= 1 \\
\Gamma(n) &= (n-1)!, \quad n = 2, 3, \cdots
\end{aligned}$$

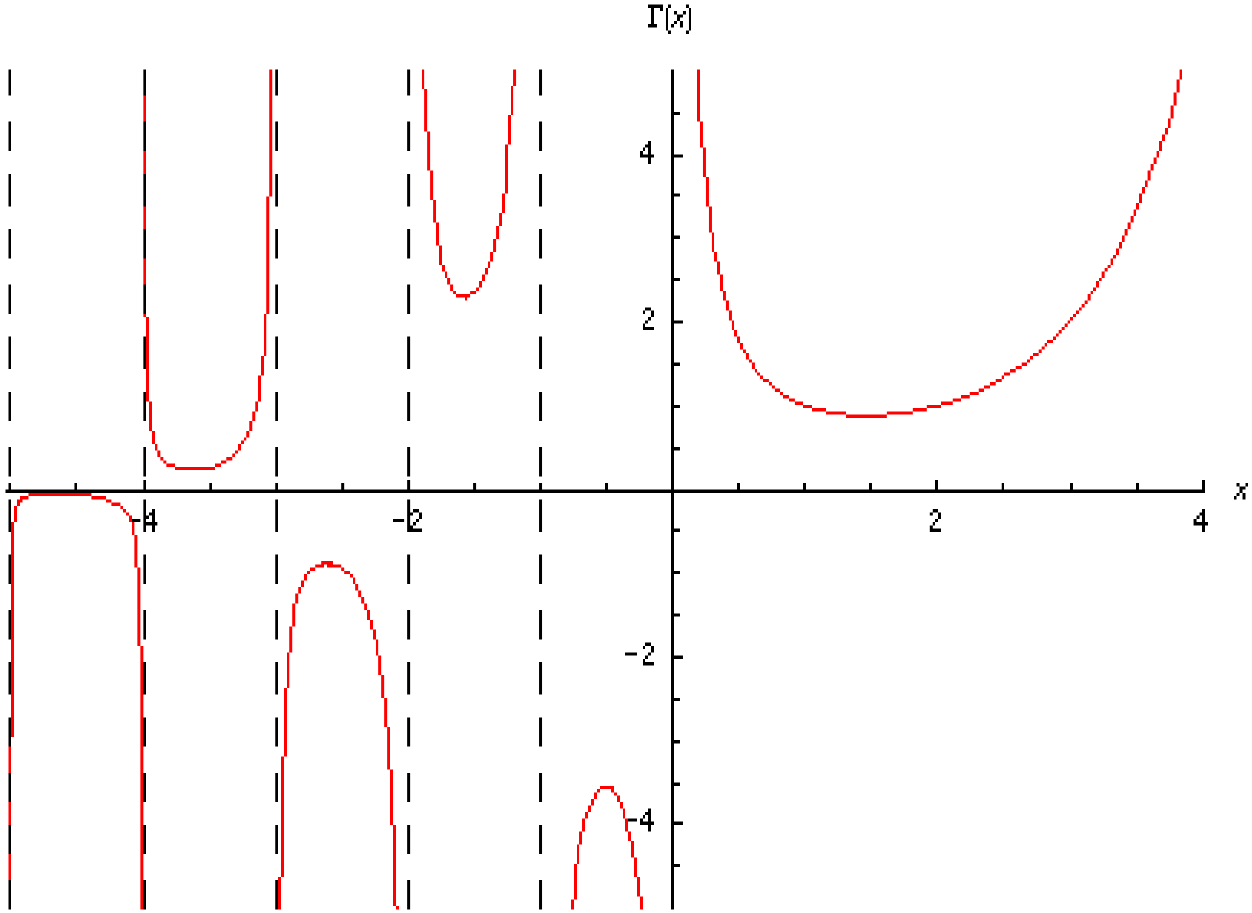
You remember that $\Gamma(z)$ has poles at $z = -n, n = 0, 1, 2, 3, \cdots$, and it is

$$\begin{aligned}
\Gamma[\epsilon] &= \frac{1}{\epsilon} - \gamma_E + \frac{1}{2}\left[{\gamma_E}^2 + \zeta(2)\right]\epsilon + \frac{1}{6}\left[-{\gamma_E}^3 - 3{\gamma_E}^2\zeta(2) - 2\zeta(3)\right]\epsilon^2 + \cdots \\
e^{\epsilon\gamma_E}\ \Gamma[\epsilon] &= \frac{1}{\epsilon} + \frac{1}{2}\zeta(2)\epsilon - \frac{1}{3}\zeta(3)\epsilon^2 + \cdots
\end{aligned}$$

For definitions of Riemann's zeta-numbers $\zeta(N)$ and the Euler constant γ_E see next slides.

Look at the singularities in the complex plane.

Figure shows the real part of Γ:



$$\text{Gamma}[-1 \pm 10i] = -4.9974\ 10^{-9} \pm 1.07847\ 10^{-8} i$$

$$\text{Gamma}[-1 \pm 100i] = 1.51438\ 10^{-71} \pm 1.27644\ 10^{-73} i$$

$$\text{Gamma}[\pm 100.1] \approx \pm 10^{\pm 157}$$

Just to remind:

$$\text{HarmonicNumber}[N,a] = \sum_{k=1}^{N} \frac{1}{k^a} = H_{N,a} = S_a(N)$$

$$\zeta(a) = \sum_{k=1}^{\infty} \frac{1}{k^a} = \text{HarmonicNumber}[\infty, a]$$

$$\gamma_E = \lim_{N\to\infty} \left[\sum_{k=1}^{N} \frac{1}{k^1} - \ln(N) \right] = 0.57721\cdots$$

$$\text{HarmonicNumber}[N] = \sum_{k=1}^{N} \frac{1}{k^1} = H_N = S_1(N)$$

We will also need derivatives of $\Gamma(z)$:

$$\text{PolyGamma [z]} \equiv \text{PolyGamma [0,z]} = \Psi(z) = \frac{1}{\Gamma(z)} \frac{d}{dz}\Gamma(z)$$

At integer values:

$$\Psi(N+1) = \text{PolyGamma [N+1]} = \sum_{k=1}^{N} \frac{1}{k} - \gamma_E = S_1(N) - \gamma_E$$

A well-known and useful expression for $\Gamma(n+\epsilon)$ using harmonic numbers $S_k(n-1)$:

$$\frac{\Gamma(n+\epsilon)}{\Gamma(n)} = \Gamma(1+\epsilon)\exp\left[-\sum_{k=1}^{\infty}\frac{(-\epsilon)^k}{k}S_k(n-1)\right].$$

The following properties hold:

$$\begin{aligned}\Psi(z+1) &= \Psi(z)+1/z\\ \Psi(1+\epsilon) &= -\gamma_E+\zeta_2\,\epsilon+\ldots\\ \Psi(1) &= -\gamma_E\\ \Psi(2) &= 1-\gamma_E\\ \Psi(3) &= 3/2-\gamma_E\end{aligned}$$

Finally:

$$\text{PolyGamma}[n,z]=\frac{d^n}{dz^n}\Psi(z)$$

It is e.g.

$$\text{PolyGamma}[2N,1]=-(2N)!\ \ \zeta(2N+1)$$

Cauchy Theorem and Residues

An integral over an anti-clockwise directed closed path C is:

$$\oint F(z)dz \quad = \quad 2\pi i \sum_{z=z_i} \text{Res}[F(z)]$$

where the residues $\text{Res}[F(z)]|_{z=z_i}$ are coefficients a^i_{-1} of the Laurent series of $F(z)$ around z_i:

$$F(z) = \sum_{n=-N}^{\infty} a^i_n (z-z_i)^n \quad = \quad \frac{a^i_{-N}}{(z-z_i)^N} + \cdots + \frac{a^i_{-1}}{(z-z_i)} + a^i_0 + \cdots$$

$$\text{Res}[F(z)]|_{z=z_i} \quad = \quad a^i_{-1}$$

If $G(z)$ has a Taylor expansion around z_0 , and $F(z)$ a Laurent expansion, then it is:

$$\text{Res}[G(z)\ F(z)]|_{z=z_i} = \sum_{n=1}^{N} \frac{a^i_{-n}}{k!} \frac{d^n}{dz^n} G(z)|_{z=z_i}$$

Due to this property, we need for applications not only $\Gamma(z)$, but also its derivatives.

Some residues with $\Gamma(z)$

$$\text{Residue}[\Gamma[z],\{z,-n\}] \quad = \quad \frac{(-1)^n}{n!}$$

$$\text{Residue}[F[z]\Gamma[z],\{z,-n\}] \quad = \quad \frac{(-1)^n}{n!}F[-n]$$

$$\text{Residue}[F[z]\Gamma[z]^2,\{z,-n\}] \quad = \quad \frac{2\text{PolyGamma}[n+1]F[-n]+F'[-n]}{(n!)^2}$$

where:

$$\Psi(z) \quad = \quad \text{PolyGamma}[z] = \text{PolyGamma}[0,z]$$

Some further examples
derived with Mathematica:

```
Series[Gamma[z]^2, {z, -3, -1}]
```

$$\frac{1}{36\,(z+3)^2}+\frac{\frac{11}{108}-\frac{\text{EulerGamma}}{18}}{z+3}+O[z+3]^0$$

```
In[8]:= Series[Gamma[z - 2] Gamma[z + 5]^2, {z, 2, -1}]
```

Out[8]= $$\frac{518\,400}{z-2}+O[z-2]^0$$

```
In[6]:= Series[Gamma[z + 2] Gamma[z - 1]^2, {z, -2, -1}]
```

Out[6]= $$\frac{1}{36\,(z+2)^3}+\frac{\frac{11}{108}-\frac{\text{EulerGamma}}{12}}{(z+2)^2}+\frac{97-132\,\text{EulerGamma}+54\,\text{EulerGamma}}{432\,(z+2)}$$

```
In[4]:= Series[Gamma[z + 2] Gamma[z - 1]^2, {z, 1, -1}]
```

Out[4]= $$\frac{2}{(z-1)^2}+\frac{3-6\,\text{EulerGamma}}{z-1}+O[z-1]^0$$

Integrals + sums of residues with Mathematica

Integral parallel to imaginary axis, residue sums when closing to the left and to the right:

$$\oint_{-1/3-9i}^{-1/3+9i} dz\Gamma[z] \quad = \quad (-i)\ 3.97173$$

$$2\pi i \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} = (2\pi i)\frac{1-e}{e} \quad = \quad (-i)\ 3.97173$$

while

$$(-1) * 2\pi i \sum_{n=0}^{0} \frac{(-1)^n}{n!} = (2\pi i) \quad \neq \quad (-i)\ 3.97173$$

More sophisticated sums of residues:

Sum[s^(n) Gamma[n + 1]^3/(n!Gamma[2 + 2n]), n, 0, Infinity] =

(4*ArcSin[Sqrt[s]/2])/(Sqrt[4 - s]*Sqrt[s])

Sum[s^(n) PolyGamma[0, n + 1], n, 0, Infinity] = (EulerGamma + Log[1 - s])/(-1 + s)

The above sums were done with Mathematica 5.2.

Mathematica versions 6 and 7 are more powerful.

L-loop n-point Feynman Integrals of tensor rank R with N internal lines

- **Internal loop momenta are** $k_l, \quad l = 1 \cdots L$
- **Propagators have mass** m_i **and momentum** $q_i, \quad i = 1 \cdots N$ **and indices** ν_i **– see** $G(X)$
- **External legs have momentum** $p_e, \quad e = 1 \cdots n$**, with** $p_e^2 = M_e^2$

Feynman integrals have the following general form:

$$G(X) = \frac{e^{\epsilon \gamma_E L}}{(i\pi^{d/2})^L} \int \frac{d^d k_1 \dots d^d k_L \quad X(k_{l_1}, \cdots, k_{l_R})}{D_1^{\nu_1} \dots D_i^{\nu_i} \dots D_N^{\nu_N}}.$$

The N propagators are:

$$D_i = q_i^2 - m_i^2 = [\sum_{l=1}^{L} c_i^l k_l + \sum_{e=1}^{n} d_i^e p_e]^2 - m_i^2$$

The numerator X may contain a tensor structure (see later for more on that):

$$X(k_{l_1}, \cdots, k_{l_R}) = (k_{l_1} P_{e_1}) \cdots (k_{l_R} P_{e_R}) = (P_{e_1}^{\alpha_1} \cdots P_{e_R}^{\alpha_R}) \, (k_{l_1}^{\alpha_1} \cdots k_{l_R}^{\alpha_R})$$

Tensor integrals

Tensor integrals appear naturally in Feynman diagrams, due to

- **fermion propagators**
- **non-abelian triple-boson vertices**
- **boson propagators in R_ξ gauges and unitary gauge**

Example: Fermionic vacuum polarization

$$\Pi^{\alpha\beta} \sim \frac{1}{(i\pi^{d/2})}\int d^dk \mathrm{Tr}\left[\frac{[\gamma k + m_1]}{D_1}\gamma^\beta \frac{[\gamma(k+p_1)+m_2]}{D_2}\gamma^\alpha\right]$$

$$\sim \frac{1}{(i\pi^{d/2})}\int \frac{d^dk}{D_1 D_2}\left[(m_1 m_2 - k^2 - kp_1)g^{\alpha\beta} + 2k^\alpha k^\beta + k^\alpha p_1^\beta + p_1^\alpha k^\beta\right]$$

So, one needs also efficient ways to evaluate tensor integrals

Tensor integrals

In general, the treatment of tensor integrals is a non-trivial task.

- One might think that all tensor numerators may be reduced to (even simpler) scalar integrands, as shown at the next slide.
  Important notice: For $Loop > 1$, this is not true, we have *irreducible numerators*.
- For many problems, it is preferrable to evaluate the tensors without knowing the scalar products.
  The reasons are different.

Simple tensor integrals

$$B^{\mu} \equiv p_1^{\mu} \, B_1 = \frac{1}{(i\pi^{d/2})} \int \frac{d^d k \quad k^{\mu}}{D_1 D_2}$$

$$B^{\mu\nu} \equiv p_1^{\mu} p_1^{\nu} B_{22} + g^{\mu\nu} B_{20} = \frac{1}{(i\pi^{d/2})} \int \frac{d^d k \quad k^{\mu} k^{\nu}}{D_1 D_2}$$

and B_1 and B_{22}, B_{20} have to be determined.

Reducible numerators

Some numerators are reducible – one may divide them out against the denominators:

$$\frac{2kp_1}{D_1\ [(k+p_1)^2-m_2^2]\ \ldots D_N} \equiv \frac{[(k+p_1)^2-m_2^2]-[k^2-m_1^2]+(-m_1^2+m_2^2-p_1^2)]}{D_1[(k+p_1)^2-m_2^2]\ldots D_N}$$
$$= \frac{1}{D_1D_3\ldots D_N}-\frac{1}{D_2D_3\ldots D_N}+\frac{-m_1^2+m_2^2-p_1^2}{D_1D_2D_3\ldots D_N}$$

This way one derives:

$$p_1^\mu B^\mu = p^2 B_1 = \frac{1}{(i\pi^{d/2})}\int d^dk\frac{p_1k}{D_1D_2}$$
$$= \frac{1}{(i\pi^{d/2})}\frac{1}{2}\int d^dk\left[\frac{1}{D_1}-\frac{1}{D_2}+\frac{-m_1^2+m_2^2-p_1^2}{D_1D_2}\right]$$

and finally:

$$B_1 = \frac{1}{2p_1^2}\left[A_0(m_1)-A_0(m_2)+(-m_1^2+m_2^2-p_1^2)B_0(m_1,m_2,p^2)\right]$$

Known: The Passarino-Veltman reduction scheme for 1-loop tensors worked out in *[Passarino:1979jh]*

until 4-point functions.

Irreducible numerators

For a two-loop QED box diagram, it is e.g. $L = 2$ loop momenta, $E = 4$ external lines, and we have as potential simplest numerators:
$k_1^2, k_2^2, k_1 k_2$ and $2(E-1)$ products $k_1 p_e, k_2 p_e$
compared to N internal lines, $N = 5, 6, 7$.
This gives Irreducible numerators, if $I > 0$,

$$I = L + L(L-1)/2 + L(E-1) - N$$

Here:

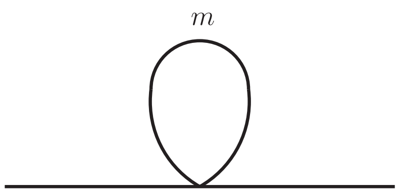
$$I(N) = 9 - N = 4, 3, 2$$

This observation is of practical importance:
Imagine you search for a list of potential master integrals. Then you may take into the list of masters $I(5) = 4$, or $I(6) = 3$, or $I(7) = 2$ such integrals.
Which momenta combinations are irreducible is dependent on the choice of momenta flows.
Message:
When evaluating Feynman integrals by Mellin-Barnes-integrals, one should also learn to handle numerator integrals
. . . and it is - in some cases - not too complicated compared to scalar ones

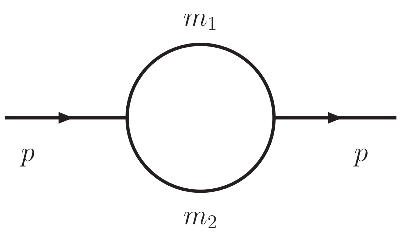
The one-loop case: $L = 1, E = N$, so

$$I(N) = 1 + (E-1) - N = 0 \text{ irreducible numerators}$$

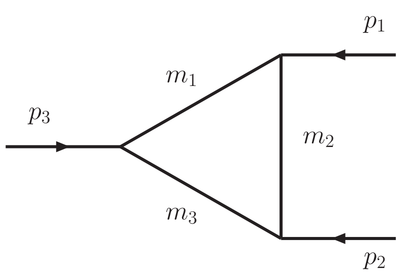
Simple examples of scalar integrals



$$A_0 = \frac{1}{(i\pi^{d/2})}\int \frac{d^dk}{D_1} \quad \rightarrow \mathrm{UV-divergent:} \quad \sim \frac{d^4k}{k^2}$$



$$B_0 = \frac{1}{(i\pi^{d/2})}\int \frac{d^dk}{D_1 D_2} \quad \rightarrow \mathrm{UV-divergent} \quad \sim \frac{d^4k}{k^4}$$



$$C_0 = \frac{1}{(i\pi^{d/2})}\int \frac{d^dk}{D_1 D_2 D_3} \quad \rightarrow \mathrm{UV-finite} \quad \sim \frac{d^4k}{k^6}$$

Dependent on conventions, where k starts to run in the loop, it is:

$$\begin{aligned} D_1 &= k^2 - m_1^2 \\ D_2 &= (k+p_1)^2 - m_2^2 \\ D_3 &= (k+p_1+p_2)^2 - m_3^2 \end{aligned}$$

Evaluate Feynman integrals

There are two strategies to solve a Feynman integral:

- Reduction

  Express the integral with the aid of recurrence relations by other, known integrals.
  These are then the Master Integrals. This approach will not be discussed here.
  → See Lectures by A. & V. Smirnov on tools for finding sets of master integrals

- Direct evaluation

Introduce Feynman parameters

$$\frac{1}{D_1^{\nu_1} D_2^{\nu_2} \ldots D_N^{\nu_N}} = \frac{\Gamma(\nu_1 + \ldots + \nu_N)}{\Gamma(\nu_1) \ldots \Gamma(\nu_N)} \int_0^1 dx_1 \ldots \int_0^1 dx_N \frac{x_1^{\nu_1 - 1} \ldots x_N^{\nu_N - 1} \delta(1 - x_1 \ldots - x_N)}{(x_1 D_1 + \ldots + x_N D_N)^{N_\nu}},$$

with $N_\nu = \nu_1 + \ldots \nu_N$.

The denominator of G contains, after introduction of Feynman parameters x_i, the momentum dependent function m^2 with index-exponent $N_\nu = (\nu_1 + \ldots + \nu_N)$:

$$(m^2)^{-(\nu_1 + \ldots + \nu_N)} = (x_1 D_1 + \ldots + x_N D_N)^{-N_\nu} = (k_i M_{ij} k_j - 2 Q_j k_j + J)^{-N_\nu}$$

Here M is an (LxL)-matrix, $Q = Q(x_i, p_e)$ an L-vector and $J = J(x_i x_j, m_i^2, p_{e_j} p_{e_l})$.
M, Q, J are linear in x_i. The momentum integration is now simple:
Shift the momenta k such that m^2 has no linear term in $\bar{k}$:

$$k = \bar{k} + (M^{-1}) Q,$$

$$m^2 = \bar{k} M \bar{k} - Q M^{-1} Q + J \equiv \bar{k} M \bar{k} + \mu^2(x)$$

Remember: $M_{1-\mathrm{loop}} = 1$, in general:

$$M^{-1} = \frac{1}{(\det M)} \tilde{M},$$

where $\tilde{M}$ is the transposed matrix to M. The shift leaves the integral unchanged.

The shift leaves the integral unchanged (rename $\bar{k} \to k$):

$$G(1) \quad = \quad \int \frac{Dk_1 \ldots Dk_L}{(kMk + J - QM^{-1}Q)^{N_\nu}}.$$

Go Euclidean: Rotate now the $k^0 \to iK_E^0$ with $k^2 \to -k_E^2$ (and again rename $k^E \to k$):

$$G(1) \to (i)^L \int \frac{Dk_1^E \ldots Dk_L^E}{(-k^E M k^E + J - QM^{-1}Q)^{N_\nu}} \;=\; (-1)^{N_\nu} (i)^L \int \frac{Dk_1 \ldots Dk_L}{[kMk - (J - QM^{-1}Q)]^{N_\nu}}.$$

Call

$$\mu^2(x) \quad = \quad -(J - QM^{-1}Q)$$

and get

$$G(1) \quad = \quad (-1)^{N_\nu} (i)^L \int \frac{Dk_1 \ldots Dk_L}{(kMk + \mu^2)^{N_\nu}}.$$

For 1-loop integrals it is $L = 1, M = 1$ - and we will use nearly only those - we are ready to do the k-integration.

Additional step for L-loop integrals

For L-loops go on and now diagonalize the matrix M by a rotation:

$$
\begin{aligned}
k \to k'(x) &= V(x)\; k, \\
kMk &= k' M_{diag} k' \\
&\to \sum \alpha_i(x) k_i^2(x), \\
M_{diag}(x) &= (V^{-1})^{+} M V^{-1} \;=\; (\alpha_1, \ldots, \alpha_L).
\end{aligned}
$$

This leaves both the integration measure and the integral invariant:

$$
G(1) = (-1)^{N_\nu} (i)^L \int \frac{Dk_1 \ldots Dk_L}{\left(\sum_i \alpha_i k_i^2 + \mu^2\right)^{N_\nu}}.
$$

Rescale now the k_i,

$$
\bar{k}_i = \sqrt{\alpha_i} k_i,
$$

with

$$
\begin{aligned}
d^d k_i &= (\alpha_i)^{-d/2} d^d \bar{k}_i, \\
\prod_{i=1}^{L} \alpha_i &= \text{det } M,
\end{aligned}
$$

and get the Euclidean integral to be calculated (and rename $\bar{k} \to k$):

$$G(1) \quad = \quad (-1)^{N_\nu} (i)^L (\text{det } M)^{-d/2} \int \frac{Dk_1 \ldots Dk_L}{(k_1^2 + \ldots + k_L^2 + \mu^2)^{N_\nu}}.$$

Use now (remembering that $Dk = dk/(i\pi^{d/2})$):

$$i^L \int \frac{Dk_1 \ldots Dk_L}{(k_1^2 + \ldots + k_L^2 + \mu^2)^{N_\nu}} \quad = \quad \frac{\Gamma\left(N_\nu - \frac{d}{2}L\right)}{\Gamma\left(N_\nu\right)} \frac{1}{(\mu^2)^{N_\nu - dL/2}},$$

$$i^L \int \frac{Dk_1 \ldots Dk_L \; k_1^2}{(k_1^2 + \ldots + k_L^2 + \mu^2)^{N_\nu}} \quad = \quad \frac{d}{2} \; \frac{\Gamma\left(N_\nu - \frac{d}{2}L - 1\right)}{\Gamma\left(N_\nu\right)} \frac{1}{(\mu^2)^{N_\nu - dL/2 - 1}}.$$

These formulae follow for $L = 1$ immediately from any textbook.

See 'Mathematical Interlude'.

For $L > 1$, get it iteratively, with setting $(k_1^2 + k_2^2 + m^2)^N = (k_1^2 + M^2)^N$, $M^2 = k_2^2 + m^2$, etc.

Mathematical interlude: d-dimensional integrals (I)

After the Wick rotation, the integrand of the momentum integration is positive definite.
Further it is independent of the angular variables.
The integral is understood as symmetric limit the infinity boundaries.

$$
\begin{aligned}
\int d^d k \; k_\mu \; F(k^2) &= 0 \\
\int d^d k \; F(k+C) &= \int d^d k \; F(k).
\end{aligned}
$$

Introduce d-dim. spherical coordinates. The vector k has d components:

$$
\begin{aligned}
k_d &= r \cos\theta_d \equiv \rho_d \cos\theta_d \\
k_{d-1} &= \rho_{d-1} \cos\theta_{d-1} \\
& \ldots \\
k_3 &= \rho_3 \cos\theta_3 \\
k_2 &= \rho_2 \sin\phi \\
k_1 &= \rho_2 \cos\phi \\
\rho_{d-1} &= \rho_d \sin\theta_d
\end{aligned}
$$

Mathematical interlude (II)

The above is the direct generalization of the 3- or 4-dimensional phase space parametrization.

With these variables, the integral over the complete d-dimensional phase space gets the following form:

$$\int_{-\infty}^{\infty} d^d k\ F(k) = \lim_{R\to\infty} \int_0^R dr r^{d-1} \int_0^\pi d\theta_{d-1} \sin^{d-2}\theta_{d-1} \int_0^\pi d\theta_{d-2} \sin^{d-3}\theta_{d-2} \ldots \int_0^{2\pi} d\theta_1 F(k)$$

The integrations met in the loop calculations may be performed using the following two integrals:

$$\int_0^\pi d\theta \sin^m\theta = \sqrt{\pi}\frac{\Gamma\left[\frac{1}{2}(m+1)\right]}{\Gamma\left[\frac{1}{2}(m+2)\right]},$$

$$\int_0^\infty dr \frac{r^\beta}{(r^2+M^2)^\alpha} = \frac{1}{2}\frac{\Gamma\left(\frac{\beta+1}{2}\right)\Gamma\left(\alpha-\frac{\beta+1}{2}\right)}{\Gamma(\alpha)}\frac{1}{(M^2)^{\alpha-(\beta+1)/2}}.$$

In general, the angular integrations are influenced by the integrand too. (Remember phase space integrals of bremsstrahlung!)

Mathematical interlude (III)

If $F(k) \to F(r),\ r = |k|$, the angular integrations yield the surface of the d-dimensional sphere with radius r:

$$\omega_d(r) = \frac{2\pi^{d/2}}{\Gamma\left[\frac{d}{2}\right]} r^{d-1}.$$

The remaining integration, over r, yields for $F(r) = 1$ the volume of the sphere with radius R:

$$V_d(R) = \frac{\pi^{d/2}}{\Gamma\left[1+\frac{d}{2}\right]} R^d,$$

$$G(1) = \int d^dk \frac{1}{(k^2+M^2)^{N_\nu}}.$$

$$= \int_0^\infty dr \frac{\omega_d(r)}{(r^2+M^2)^{N_\nu}}$$

and we get immediately, with $M^2 \equiv M^2(x_1, x_2, \ldots)$:

$$G(1) = \left[\frac{i\pi^{d/2}\Gamma\left(N_\nu - d/2\right)}{\Gamma(N_\nu)} \frac{1}{(M^2)^{N_\nu - d/2}} \right].$$

Finally, one gets for **Scalar integrals:**

$$G(1) = (-1)^{N_\nu} \frac{\Gamma\left(N_\nu - \frac{d}{2}L\right)}{\Gamma(\nu_1)\ldots\Gamma(\nu_N)} \int_0^1 \prod_{j=1}^{N} dx_j \; x_j^{\nu_j-1} \delta\left(1-\sum_{i=1}^{N} x_i\right) \frac{(\text{det } M)^{-d/2}}{(\mu^2)^{N_\nu - dL/2}},$$

or

$$G(1) = (-1)^{N_\nu} \frac{\Gamma\left(N_\nu - \frac{d}{2}L\right)}{\Gamma(\nu_1)\ldots\Gamma(\nu_N)} \int_0^1 \prod_{j=1}^{N} dx_j \; x_j^{\nu_j-1} \delta\left(1-\sum_{i=1}^{N} x_i\right) \frac{U(x)^{N_\nu - d(L+1)/2}}{F(x)^{N_\nu - dL/2}}$$

with

$$U(x) = (\text{det } M) \; (\rightarrow 1 \text{ for } L=1)$$

$$F(x) = (\text{det } M)\; \mu^2 = -(\text{det } M)\; J + Q\; \tilde{M}\; Q \; (\rightarrow -J + Q^2 \text{ for } L=1)$$

Trick for one-loop functions:

$U = \text{det } M = 1 = \sum x_i$ and so U 'disappears' and the construct $F_1(x)$ is bilinear in $x_i x_j$:

$$F_1(x) = -J(\sum x_i) + Q^2 = \sum A_{ij} x_i x_j .$$

Examples for one-loop F-polynomials

One-loop vertex:

$$F(t,m^2) = m^2(x_1+x_2)^2 + [-t]x_1x_2$$

one-loop box:

$$F(s,t,m^2) = m^2(x_1+x_2)^2 + [-t]x_1x_2 + [-s]x_3x_4$$

one-loop pentagon:

$$F(s,t,t',v_1,v_2,m^2) = m^2(x_1+x_3+x_4)^2 + [-t]x_1x_3 + [-t']x_1x_4 + [-s]x_2x_5 + [-v_1]x_3x_5 + [-v_2]x_2x_4$$

2-loop example: B7l4m2, has a box-type sub-loop with 2 off-shell legs:

$$F^{-(a_{4567}-d/2)} = \Big\{[-t]x_4x_7 + [-s]x_5x_6 + m^2(x_5+x_6)^2 + (m^2-Q_1^2)x_7(x_4+2x_5+x_6) + (m^2-Q_2^2)x_7x_5\Big\}^{-(a_{4567}-d/2)}$$

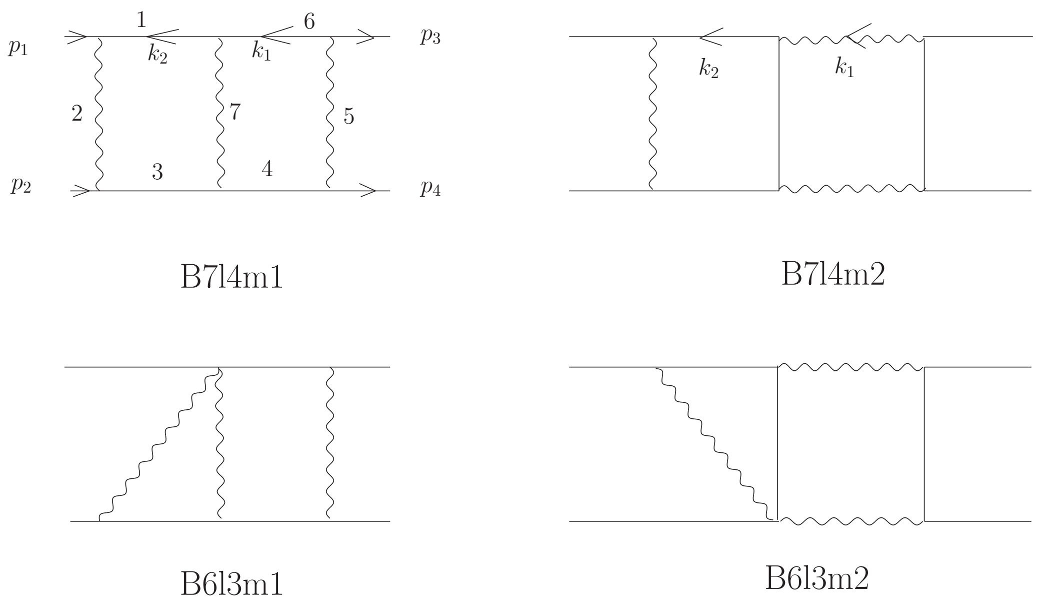


Figure 1: The planar 6- and 7-line topologies.

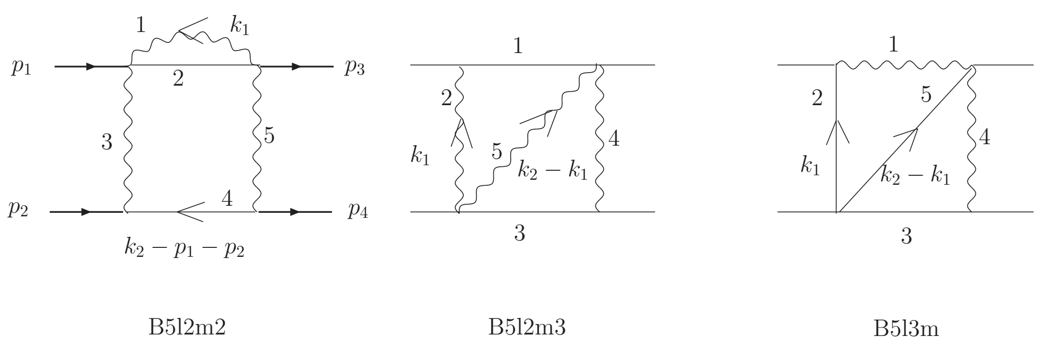
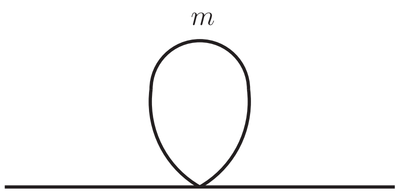


Figure 2: The 5-line topologies. B7l4m2: shrink line 1 get B6l3m2, then line 4 get B5l3m

The Tadpole $A_0(m)$



$$T1l1m[a] = A_0 = \frac{e^{\epsilon\gamma_E}}{(i\pi^{d/2})} \int \frac{d^dk}{(k^2-m^2)^a} \quad \to \mathrm{UV-divergent}$$

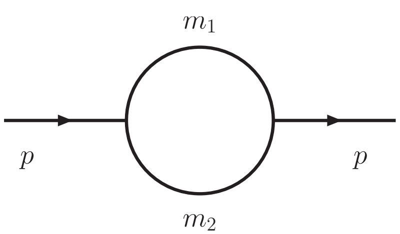
With our general formulae we get, in the 1-dimensional Feynman parameter integral, for the numerator

$$\begin{aligned} N &= (k^2-m^2)x_1 \equiv k^2 + J \\ F &= m^2 x_1 \equiv m^2 x_1^2 \end{aligned}$$

and thus

$$\begin{aligned} T1l1m[a] &= (-1)^a e^{\epsilon\gamma_E} \frac{\Gamma[a-d/2]}{\Gamma[a]} \int_0^1 dx x^{a-1} \delta[1-x] \frac{1}{F^{a-d/2}} \\ &= (-1)^a e^{\epsilon\gamma_E} (m^2)^{2-a-\epsilon} \frac{\Gamma[a-2+\epsilon]}{\Gamma[a]} \\ &\to -e^{\epsilon\gamma_E}\Gamma[-1+\epsilon] \quad \mathrm{for\ a=1,\ m=1} \\ &= \frac{1}{\epsilon} + 1 + \left(1+\frac{\zeta_2}{2}\right)\epsilon + \left(1+\frac{\zeta_2}{2}-\frac{\zeta_3}{3}\right)\epsilon^2 + \cdots \end{aligned}$$

The Self-energy $B_0(s, m_1, m_2)$



$$SE2l = B_0[s, m_1, m_2] = (2\sqrt{\pi}\mu)^{4-d} \frac{e^{\epsilon\gamma_E}}{(i\pi^{d/2})} \int \frac{d^d k}{[k^2 - m^2][(k+p)^2 - m_2^2]}$$

The $SE2l$ is UV-divergent and the corresponding F-function is:

$$F[s, m_1, m_2] = m_1^2 x_1^2 + m_2^2 x_2^2 - [s - m_1^2 - m_2^2] x_1 x_2$$

and for special cases:

$$F[s, m_1, 0] = m_1^2 x_1^2 - [s - m_1^2] x_1 x_2$$

$$F[s, m_1, m_1] = m_1^2 (x_1 + x_2)^2 - [s] x_1 x_2$$

$$F[s, 0, 0] = -[s] x_1 x_2$$

The 'conventional' Feynman parameter integral is 1-dimensional because $x_2 \equiv 1 - x_1$:

$$F(x) = -sx(1-x) + m_2^2(1-x) + m_1^2 x \equiv -s(x - x_a)(x - x_b)$$

The result is of logarithmic type for the constant term in ϵ:

$$B_0[s, m_1, m_2] = (4\pi\mu^2)^\epsilon \; e^{\epsilon\gamma_E} \; \frac{\Gamma(1+\epsilon)}{\epsilon} \int_0^1 \frac{dx}{F(x)^\epsilon}$$

$$= \frac{1}{\epsilon} - \int_0^1 dx \ln\left(\frac{F(x)}{4\pi\mu^2}\right)$$

$$+ \epsilon \; \left\{ \frac{\zeta_2}{2} + \frac{1}{2}\int_0^1 dx \ln^2\left(\frac{F(x)}{4\pi\mu^2}\right)\right\} + \mathcal{O}(\epsilon^2).$$

Here we used the expansion:

$$e^{\epsilon\gamma_E}\Gamma(1+\epsilon) = 1 + \frac{\zeta_2}{2}\epsilon^2 - \frac{\zeta_3}{3}\epsilon^3 \cdots$$

When using `LoopTools`, the corresponding call returns exactly the constant term of B_0 in ϵ (with use of $e^{\epsilon\gamma_E} = 1 + \epsilon\gamma_E + \cdots \to 1$):

$$B_0^{(0)}(s, m_1^2, m_2^2) = \mathtt{b0}(\mathtt{s}, \mathtt{am12}, \mathtt{am22})$$

For $4\pi\mu^2 \to 1$ B_0 looks quite compact:

$$B_0(s, m_1, m_2) = \frac{1}{\epsilon} - \int_0^1 dx \; \ln[F(x)] + \frac{\epsilon}{2}\left[\zeta_2 + \int_0^1 dx \ln^2[F(x)]\right] + \cdots$$

Explicitly, one has to integrate

$$\begin{aligned} \ln[F(x)] &= \ln[-s(x-x_a)(x-x_b)] \\ \ln^2[F(x)] &= \ln^2[-s(x-x_a)(x-x_b)] \end{aligned}$$

So we will need the integrals:

$$\int_0^1 dx \left\{\ln(x-x_a),\ ln(x-x_a)ln(x-x_b)\right\}$$

which is trivial, together with some complex algebra rules how to handle complex arguments of logarithms with

$$s \to s + i\epsilon$$

wherever needed.

For the case $m_1 = m_2 = 1$, one gets for the first terms in ϵ:

$$B_0[s,1,1] = \frac{1}{\epsilon} + 2 + \frac{1+y}{1-y} H(0,y),$$
$$H(0,y) = \ln(y).$$

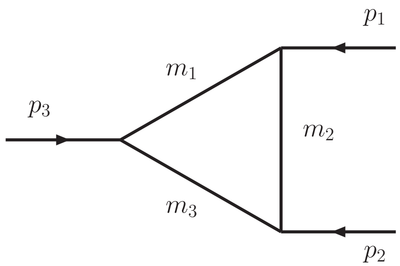
The $H(0,y)$ is a harmonic polylogarithmic function, and

$$y = \frac{\sqrt{-s+4} - \sqrt{-s}}{\sqrt{-s+4} + \sqrt{-s}}$$
$$s = -\frac{(1-y)^2}{y}$$

The other case treated later again is $m_1 = 0, m_2 = m$:

$$B_0[s,m^2,0] = \frac{1}{\epsilon} + 2 + \frac{1-s/m^2}{s/m^2} \ \ln(1-s/m^2)$$

The massive one-loop vertex $C_0(s, m_1, m_2)$



$$C_0 = \frac{e^{\epsilon\gamma_E}}{(i\pi^{d/2})} \int \frac{d^d k}{[(k+p_1)^2 - m^2][k^2][(k-p_2)^2 - m^2]} \sim |_{k\to\infty} \frac{d^4 k}{k^6} \longrightarrow \mathrm{UV-fi}$$

The massive vertex (all $m_1, m_2, m_3 \neq 0$) is a finite quantity.

We assume immediately $m_1 = m_3 = 0$.

A problem now is IR-divergence.

Appears when a massive internal line is between two external on-shell lines.

Incoming $p_1^2 = m^2$ and $p_2^2 = m^2$, look at $k \to 0$:

$$\begin{aligned} & d^4k \; \frac{1}{(k-p_2)^2 - m^2} \; \frac{1}{(k)^2} \; \frac{1}{(k+p_1)^2 - m^2} \\ = \; & d^4k \; \frac{1}{k^2 - 2kp_2} \; \frac{1}{(k)^2} \; \frac{1}{k^2 + 2kp_1} \\ \to \; & \frac{d^4k}{k^{1+2+1}} \sim \frac{k^3 dk}{k^4} \sim \frac{dk}{k}|_{k\to 0} \longrightarrow div \end{aligned}$$

An IR-regularization is needed, must take $d > 4$.

Both UV-div (with $d < 4$) and IR-div together: must allow for a complex $d = 4 - 2\epsilon$, and take limit at the end.

First we have a look, for later use, at the F-function:

$$\begin{aligned} N &= D_1 x + D_2 y + D_3 z \\ &= k^2 x + (k^2 + 2kp_1) y + (k^2 - 2kp_2) z \\ &= k^2 (x + y + z) + 2k(p_1 y - p_2 z) \\ &= (k+Q)^2 - Q^2 \end{aligned}$$

We used $1 = x + y + z$ here. And the F-function is $F = Q^2 - J = Q^2$ (there is no constant term in N here), as was shown before:

$$F = m^2 (y+z)^2 + [-s] yz$$

This F-function does not factorize in y and z. But now back to the direct Feynman parameter integration.

Start with change $y \to y' = (1-x)y$, then $y' \to y$:

$$\begin{aligned}
\frac{1}{D_1 D_2 D_3} &= \int_0^1 dx dy dz \frac{\delta(1-x-y-z)}{(D_2 x + D_1 y + D_3 z)^3} \\
&= \int_0^1 dx \int_0^{1-x} \frac{dy}{(D_2 x + D_1 y + D_3 z)^3} \\
&= \int_0^1 dx \int_0^1 \frac{x dy}{(D_2 x + D_1 y + D_3 z)^3}
\end{aligned}$$

After this change of variables, the integrand factorizes in x and y:

$$\begin{aligned}
N &= (k + x p_y)^2 - x^2 p_y^2 \\
&= (k+Q)^2 - Q^2
\end{aligned}$$

resulting into

$$\begin{aligned}
F = Q^2 &= x^2 \ p_y^2 \\
p_y^2 &= -sy(1-y) + m^2
\end{aligned}$$

Such a factorization is the result, if systematically done, of:

⟶ Sector Decomposition

For C_0 we obtain (with $N_\nu = 3$ and $N_\nu - d/2 = 1 + \epsilon$):

$$C_0[s, m, m, 0] = (-1) e^{\epsilon \gamma_E} \Gamma[1+\epsilon] \int_0^1 \frac{dx}{x^{1+2\epsilon}} \int_0^1 \frac{dy}{(p_y^2)^{1-\epsilon}}$$

The problem is at $x = 0$.

The C_0 is integrable for $\epsilon < 0$, or $d > 4$, or more general: $d \not< 4$.

The x-integral made simple here:

$$\int_0^1 \frac{dx}{x^{1+2\epsilon}} = \frac{x^{-2\epsilon}|_0^1}{-2\epsilon} = -\frac{1^{-2\epsilon} - 0^{-2\epsilon}}{2\epsilon} = -\frac{1-0}{2\epsilon}$$
$$= -\frac{1}{2\epsilon}$$

We see that the IR-singularity is an end-point-singularity in Feynman parameter space.

Further:

$$-\frac{1}{2\epsilon} \frac{dy}{(p_y^2)^{1-\epsilon}} = -\frac{1}{2\epsilon} \frac{dy}{(p_y^2)} \; (p_y^2)^{\epsilon}$$
$$= -\frac{1}{2\epsilon} \frac{dy}{(p_y^2)} \; e^{\epsilon \ln(p_y^2)}$$
$$= -\frac{1}{2\epsilon} \frac{dy}{(p_y^2)} \left[1 + \epsilon \ln(p_y^2) + \epsilon^2 \ln^2(p_y^2) + \cdots \right]$$

Here I stop this study.

We see that the further integrations proceed quite similar as for the 2-point function, in fact the $p_y^2 = -sy(1-y) + m^2$ is the same building block.
The integrals to be solved now are more general, they include also denominators $1/p_y^2$:

Some integrals

$$
\begin{aligned}
\int dy \ln(y - y_0) &= (y-y_0)\ln(y-y_0) - y + C \\
\int dy \frac{1}{y-y_0} &= \ln(y-y_0) + C \\
\int dy \frac{\ln(y-y_0)}{y-y_0} &= \frac{1}{2}\ln^2(y-y_0) + C
\end{aligned}
$$

Here, often y is real and y_0 is complex. Then no special care about phases is necessary.

$$
\int_0^1 \frac{dx}{x-x_0}\left[\ln(x-x_A) - \ln(x_0-x_A)\right] = Li_2\left(\frac{x_0}{x_0-x_A}\right) - Li_2\left(\frac{x_0-1}{x_0-x_A}\right).
$$

This formula is valid if x_0 is real.

C_0 with a small photon mass λ

In

[Berends:1976zp,'tHooft:1979xw]

, the C_0-integral is treated with a finite photon mass λ:

$$\int \frac{d^4k}{(k^2-\lambda^2)(k^2+2kp_1)(k^2-2kp_2)}$$

$$= \quad -i\pi^2 \int_0^1 dy dx \frac{y}{x^2 p_y^2 + (1-x)\lambda^2}$$

$$= \quad i\pi^2 \int_0^1 dy \left[\frac{1}{2p_y^2} \ln \frac{\lambda^2}{p_y^2} + \mathcal{O}\left(\lambda/\sqrt{p_y^2}\right)\right],$$

It is easy to see from the term $1/(2p_y^2)\ \ln(\lambda^2)$ the correspondence of $(d-4)$ and λ^2, which is a universal relation in all 1-loop cases.

Now using Mellin-Barnes Representations

Perform the x-integrations

Find an as-general-as-possible general formula

Make it ready for algorithmic analytical and/or numerical evaluation

Computercodes:

- Ambre.m - Derive Mellin-Barnes representations for Feynman integrals *[Gluza:2007rt]*
- MB.m - Find an ϵ-expansion and evaluate numerically in Euclidean region *[Czakon:2005rk]*

Integrating the Feynman parameters – get MB-Integrals

We derived:

$$SE2l1m = B_0(s,m,0) = e^{\epsilon\gamma_E}\,\Gamma(\epsilon)\int_0^1 dx_1 dx_2 \delta(1-x_1-x_2)\frac{\delta(1-x_1-x_2)}{F(x)^{\epsilon}}$$

$$V3l2m = C_0(s,m,m,0) = e^{\epsilon\gamma_E}\,\Gamma(1+\epsilon)\int_0^1 dx_1 dx_2 dx_3 \frac{\delta(1-x_1-x_2-x_3)}{F(x)^{1+\epsilon}}$$

and

$$F_{SE2l1m} = m^2 x_1^2 - (s-m^2)x_1 x_2$$

$$F_{V3l2m} = m^2(x_1+x_2)^2 - (s)x_1 x_2$$

We want to apply now:

$$\int_0^1 \prod_{j=1}^{N} dx_j\; x_j^{\alpha_j-1}\,\delta\left(1-\sum x_i\right) = \frac{\Gamma(\alpha_1)\Gamma(\alpha_2)\cdots\Gamma(\alpha_7 N)}{\Gamma\left(\alpha_1+\alpha_2+\cdots+\alpha_N\right)}$$

with coefficients α_i dependent on ν_i and on the structure of the F

See in a minute:
For this, we have to apply one or several MB-integrals here.

$$\int_0^1 \prod_{j=1}^{N} dx_j \; x_j^{\alpha_j-1} \; \delta\left(1-\sum_{i=1}^{N} x_i\right) \quad = \quad \frac{\prod_{i=1}^{N}\Gamma(\alpha_i)}{\Gamma\left(\sum_{i=1}^{N}\alpha_i\right)}$$

Simplest cases:

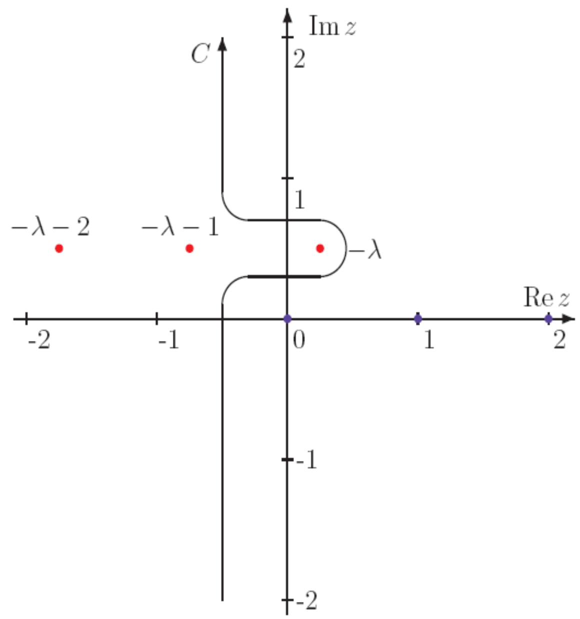
$$\int_0^1 dx_1 \; x_1^{\alpha_1-1} \; \delta\left(1-x_1\right) \quad = \quad 1$$

$$\int_0^1 \prod_{j=1}^{2} dx_j \; x_j^{\alpha_j-1} \; \delta\left(1-\sum_{i=1}^{N} x_i\right) = \int_0^1 dx_1 x_1^{\alpha_1-1}(1-x_1)^{\alpha_2-1} \quad = \quad B(\alpha_1,\alpha_2)$$

$$= \quad \frac{\Gamma(\alpha_1)\Gamma(\alpha_2)}{\Gamma\left(\alpha_1+\alpha_2\right)}$$

Here we want to go:

$$\frac{1}{(A+B)^{\lambda}} = \frac{1}{\Gamma(\lambda)} \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} dz \Gamma(\lambda+z)\Gamma(-z) \frac{B^z}{A^{\lambda+z}}$$

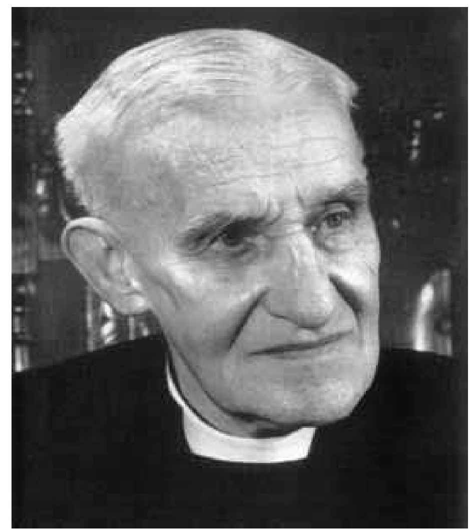


The integration path separates poles of $\Gamma[\lambda+z]$ and $\Gamma[-z]$.

The formula looks a bit unusual to loop people, but for persons with a mathematical background it is common knowledge.

One might well assume that these two gentlemen did not dream of so heavy use of their results in basic research ···

Mellin, Robert, Hjalmar, 1854-1933
Barnes, Ernest, William, 1874-1953

.

Barnes' contour integrals for the hypergeometric function

Exact proof and further reading: Whittaker & Watson (CUP 1965) 14.5 - 14.52, pp. 286-290

Consider

$$F(z) \quad = \quad \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} d\sigma (-z)^{\sigma}\frac{\Gamma(a+\sigma)\Gamma(b+\sigma)\Gamma(-\sigma)}{\Gamma(c+\sigma)}$$

where $|\mathrm{arg(-z)}| < \pi$ (i.e. $(-z)$ is not on the neg. real axis) and the path is such that it separates the poles of $\Gamma(a+\sigma)\Gamma(b+\sigma)$ from the poles of $\Gamma(-\sigma)$.
$1/\Gamma(c+\sigma)$ has no pole.
Assume $a \neq -n$ and $b \neq -n, n = 0, 1, 2, \cdots$ so that the contour can be drawn.

The poles of $\Gamma(\sigma)$ are at $\sigma = -n, n = 1, 2, \cdots$, and it is:

```
Residue[ F[s] Gamma[-s] , {s,n} ]  =  (-1)^n/n!  F(n)
```

Closing the path to the right gives then, by Cauchy's theorem, for $|z| < 1$ the

hypergeometric function $_2F_1(a,b,c,z)$ (for proof see textbook):

$$\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} d\sigma (-z)^\sigma \frac{\Gamma(a+\sigma)\Gamma(b+\sigma)\Gamma(-\sigma)}{\Gamma(c+\sigma)} = \sum_{n=0}^{N\to\infty} \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)} \frac{z^n}{n!}$$

$$= \frac{\Gamma(a)\Gamma(b)}{\Gamma(c)} \quad {}_2F_1(a,b,c,z)$$

The continuation of the hypergeometric series for $|z| > 1$ is made using the intermediate formula

$$F(z) = \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(1-c+a+n)\sin[(c-a-n)\pi]}{\Gamma(1+n)\Gamma(1-a+b+n)\cos(n\pi)\sin[(b-a-n)\pi]}(-z)^{-a-n}$$

$$+\sum_{n=0}^{\infty} \frac{\Gamma(b+n)\Gamma(1-c+b+n)\sin[(c-b-n)\pi]}{\Gamma(1+n)\Gamma(1-a+b+n)\cos(n\pi)\sin[(a-b-n)\pi]}(-z)^{-b-n}$$

and yields

$$\frac{\Gamma(a)\Gamma(b)}{\Gamma(c)} \quad {}_2F_1(a,b,c,z) = \frac{\Gamma(a)\Gamma(a-b)}{\Gamma(a-c)}(-z)^{-a} \quad {}_2F_1(a,1-c+a,1-b+ac,z^{-1})$$

$$+\frac{\Gamma(b)\Gamma(b-a)}{\Gamma(b-c)}(-z)^{-b} \quad {}_2F_1(b,1-c+b,1-a+b,z^{-1})$$

Corollary I

Putting $b = c$, we see that

$$
{}_2F_1(a,b,b,z) = \sum_{n=0}^{\infty} \frac{\Gamma(a+n)}{\Gamma(a)} \frac{z^n}{n!}
$$

$$
= \frac{1}{(1-z)^a} = \frac{1}{2\pi i\ \Gamma(a)} \int_{-i\infty}^{+i\infty} d\sigma\ (-z)^{\sigma}\ \Gamma(a+\sigma)\Gamma(-\sigma)
$$

This allows to replace sum by product:

$$
\frac{1}{(A+B)^a} = \frac{1}{B^a[1-(-A/B)]^a} = \frac{1}{2\pi i \Gamma(a)} \int_{-i\infty}^{i\infty} d\sigma A^{\sigma}\ B^{-\sigma-a}\ \Gamma(a+\sigma)\Gamma(-\sigma)
$$

Barnes' lemma

If the path of integration is curved so that the poles of $\Gamma(c-\sigma)\Gamma(d-\sigma)$ lie on the right of the path and the poles of $\Gamma(a+\sigma)\Gamma(b+\sigma)$ lie on the left, then

$$\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} d\sigma \Gamma(a+\sigma)\Gamma(b+\sigma)\Gamma(c-\sigma)\Gamma(d-\sigma) \quad = \quad \frac{\Gamma(a+c)\Gamma(a+d)\Gamma(b+c)\Gamma(b+d)}{\Gamma(a+b+c+d)}$$

It is supposed that a,b,c,d are such that no pole of the first set coincides with any pole of the second set.

Scetch of proof: Close contour by semicircle C to the right of imaginary axis. The integral exists and $\int_C$ vanishes when $\Re(a+b+c+d-1)<0$. Take sum of residues of the integrand at poles of $\Gamma(c-\sigma)\Gamma(d-\sigma)$. The double sum leads to two hypergeometric functions, expressible by ratios of Γ-functions, this in turn by combinations of sin, may be simplifies finally to the r.h.s.

Analytical continuation: The relation is proved when $\Re(a+b+c+d-1)<0$. Both sides are analytical functions of e.g. a. So the relation remains true for all values of a,b,c,d for which none of the poles of $\Gamma(a+\sigma)\Gamma(b+\sigma)$, as a function of σ, coincide with any of the poles of $\Gamma(c-\sigma)\Gamma(d-\sigma)$.

Corollary II Any real shift k: $\sigma+k, a-k, b-k, c+k, d+k$ together with $\int_{-k-i\infty}^{-k+i\infty}$ leaves the result true.

How can the Mellin-Barnes formula be made useful in the context of Feynman integrals?

- Apply corollary I to propagators and get:

$$\frac{1}{(p^2 - m^2)^a} = \frac{1}{2\pi i \ \Gamma(a)} \int_{-i\infty}^{i\infty} d\sigma \frac{(-m^2)^\sigma}{(p^2)^{a+\sigma}} \ \Gamma(a+\sigma)\Gamma(-\sigma)$$

  which transforms a massive propagator to a massless one (with index a of the line changed to $(a+\sigma)$).

- Apply corollary I after introduction of Feynman parameters and after the momentum integration to the resulting F- and U-forms, in order to get a single monomial in the x_i, which allows the integration over the x_i:

$$\frac{1}{[A(s)x_1^{a_1} + B(s)x_1^{b_1}x_2^{b_2}]^a} = \frac{1}{2\pi i \ \Gamma(a)} \int_{-i\infty}^{i\infty} d\sigma [A(s)x_1^{a_1}]^\sigma [B(s)x_1^{b_1}x_2^{b_2}]^{a+\sigma} \ \Gamma(a+\sigma)\Gamma(-\sigma)$$

Both methods leave Mellin-Barnes (MB-) integrals to be performed afterwards.

A short remark on history

- N. Usyukina, 1975: "ON A REPRESENTATION FOR THREE POINT FUNCTION", Teor. Mat. Fiz. 22;
  a finite massless off-shell 3-point 1-loop function represented by 2-dimensional MB-integral
- E. Boos, A. Davydychev, 1990: "A Method of evaluating massive Feynman integrals", Theor. Math. Phys. 89 (1991);
  N-point 1-loop functions represented by n-dimensional MB-integral
- V. Smirnov, 1999: "Analytical result for dimensionally regularized massless on-shell double box", Phys. Lett. B460 (1999);
  treat UV and IR divergencies by analytical continuation: shifting contours and taking residues 'in an appropriate way'
- B. Tausk, 1999: "Non-planar massless two-loop Feynman diagrams with four on-shell legs", Phys. Lett. B469 (1999);
  nice algorithmic approach to that, starting from search for some unphysical space-time dimension d for which the MB-integral is finite and well-defined
- M. Czakon, 2005 (with experience from common work with J. Gluza and TR): "Automatized analytic continuation of Mellin-Barnes integrals", Comput. Phys. Commun. (2006);
  Tausk's approach realized in Mathematica program MB.m, published and available for use

A self-energy: SE2l1m

This is a nice example, being simple but showing a lot of essentials in a nutshell.

We get for this $F(x) = m^2 x_1^2 - (s - m^2) x_1 x_2$ the following representation:

$$SE2l1m = \frac{e^{\epsilon\gamma_E}}{2\pi i} \frac{(m^2)^{-\epsilon}}{\Gamma[2-2\epsilon]} \int_{\Re z = -1/8} dz \left[\frac{-s+m^2}{m^2}\right]^{-\epsilon - z} \Gamma_1[1-\epsilon-z]\Gamma_2[-z]\Gamma_3[1-\epsilon+z]\Gamma_4[\epsilon+z]$$

Tausk approach:

Seek a configuration where all arguments of Γ-functions have positive real part. Then the $SE2l1m$ is well-defined and finite.

For small ϵ this is - here - evidently impossible; set $\epsilon \to 0$ and look at Γ_2 Γ_4:

$$\Gamma_1[1-z]\Gamma_2[-z]\Gamma_3[1+z]\Gamma_4[+z]$$

What to do ????

Tausk: Set ϵ such that all arguments of Γ-functions get positive real parts, e.g. with the choice:

$$\epsilon = 3/8$$

To make physics we have now to deform the integrand or the path such that $\epsilon \to 0$; when crossing a residue, take it and add it up.

Varying $\epsilon \to 0$ from 3/8 makes crossing in $\Gamma_4[\epsilon+z]$ a pole at $\epsilon=-z=+1/8$; there is $\epsilon+z=0$:

$$\textbf{Residue}[\textbf{SE2l1m},\{z,-\epsilon\}] = e^{\epsilon\gamma_E}\frac{(m^2)^{-\epsilon}}{\Gamma[2-2\epsilon]}\Gamma_1[1]\ \Gamma_2[\epsilon]\ \Gamma_3[1-2\epsilon]$$

Here we 'loose' one integration (easier term!) and catch the IR-singularity in $\Gamma_2[\epsilon]\sim 1/\epsilon$!
The function becomes now, for small ϵ:

$$\begin{aligned}\textbf{SE2l1m} &= \frac{e^{\epsilon\gamma_E}}{2\pi i}\frac{(m^2)^{-\epsilon}}{\Gamma[2-2\epsilon]}\int_{\Re z=-1/8} dz \left[\frac{-s+m^2}{m^2}\right]^{-\epsilon-z}\Gamma_1[1-\epsilon-z]\ \Gamma_2[-z]\ \Gamma_3[1-\epsilon+z]\ \Gamma_4[\epsilon+z] \\ &+ e^{\epsilon\gamma_E}\frac{(m^2)^{-\epsilon}}{\Gamma[2-2\epsilon]}\Gamma_2[\epsilon]\ \Gamma_3[1-2\epsilon]\end{aligned}$$

Now we may take the limit of small ϵ because the integral will stay finite and well-defined:

$$\textbf{SE2l1m} = \frac{e^{\epsilon\gamma_E}}{2\pi i}\int_{\Re z=-1/8} dz \left[\frac{-s+m^2}{m^2}\right]^{-z}\Gamma_1[1-z]\ \Gamma_2[-z]\ \Gamma_3[1+z]\ \Gamma_4[z] + e^{\epsilon\gamma_E}\left(2+\frac{1}{\epsilon}-\ln[m^2]\right) + O(\epsilon$$

Now we close the integration path to the left, catch all residues from $\Gamma_3\ \Gamma_4$ for $\Re z < -1/8$, i.e. at $z=-n,\ n=1,2,\ldots$:

$$\textbf{Res}\left\{\left[\frac{-s+m^2}{m^2}\right]^{-z}\Gamma_1[1-z]\Gamma_2[-z]\Gamma_3[1+z]\Gamma_4[z],\{z,-n\}\right\} = (-s+m^2)^n\ln(-s+m^2)$$

The sum to be done is trivial (in this trivial case!!):

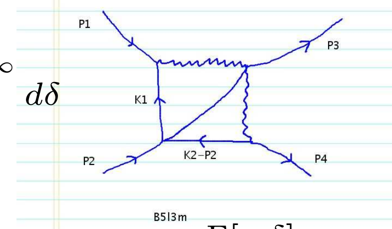
$$\sum_{n=1}^{\infty} \left[\frac{-s+m^2}{m^2}\right]^n = \frac{1}{1-\frac{-s+m^2}{m^2}} - 1$$

and we end up with:

$$\textbf{SE2l1m} \quad = \quad \frac{1}{\epsilon} + 2 + \left[\frac{1-s/m^2}{s/m^2}\ln(1-s/m^2)\right]$$

This is what we had also from the direct Feynman parameter integration above

. A nice box with numerator, B5l3m($p_e.k_1$)



$$\mathtt{B5l3m}(p_e \cdot k_1) \;=\; \frac{m^{4\epsilon}(-1)^{a_{12345}}\; e^{2\epsilon\gamma_E}}{\prod_{j=1}^{5}\Gamma[a_i]\Gamma[5-2\epsilon-a_{123}](2\pi i)^4}\int_{-i\infty}^{+i\infty}d\alpha\int_{-i\infty}^{+i\infty}d\beta\int_{-i\infty}^{+i\infty}d\gamma\int_{-i\infty}^{+i\infty}d\delta$$

$$(-s)^{(4-2\,\epsilon)-a_{12345}-\alpha-\beta-\delta} \quad \times \quad (-t)^{\delta}$$

$$\frac{\Gamma[-4+2\,\epsilon+a_{12345}+\alpha+\beta+\delta]}{\Gamma[6-3\,\epsilon-a_{12345}-\alpha]}\;\frac{\Gamma[-\alpha]\;\Gamma[-\beta]}{\Gamma[7-3\,\epsilon-a_{12345}-\alpha]\;\Gamma[5-2\,\epsilon-a_{123}]}\;\frac{\Gamma[-\delta]}{\Gamma[4-2\,\epsilon-a_{1123}-2\,\alpha-\gamma]\;\Gamma[5-2\,\epsilon-a_{1123}-}$$

$$\frac{\Gamma[2-\epsilon-a_{13}-\alpha-\gamma]}{\Gamma[8-4\,\epsilon-a_{112233445}-2\,\alpha-2\,\beta-2\,\delta-\gamma]}\;\frac{\Gamma[4-2\,\epsilon-a_{12345}-\alpha-\beta-\delta-\gamma]}{\Gamma[9-4\,\epsilon-a_{112233445}-2\,\alpha-2\,\beta-2\,\delta-\gamma]}\Big\{(p_e\cdot p_3)\;\Gamma[1+a_4+\delta]\;\Gamma[6-3$$

$$\Gamma[4-2\,\epsilon-a_{1234}-\alpha-\beta-\delta]\;\Gamma[3-\epsilon-a_{12}-\alpha]\;\Gamma[8-4\,\epsilon-a_{112233445}-2\,\alpha-2\,\delta-\gamma]\Gamma[9-4\,\epsilon-a_{112233445}-2\,\alpha-2\,\beta$$

$$\Gamma[5-2\,\epsilon-a_{1123}-\gamma]\;\Gamma[4-2\,\epsilon-a_{1123}-2\,\alpha-\gamma]\;\Gamma[a_1+\gamma]\;\Gamma[-2+\epsilon+a_{123}+\alpha+\delta+\gamma]+\Gamma[a_4+\delta]\Big[-(p_e\cdot p_1)\;\Gamma[7-3\,\epsilon$$

$$\Gamma[4-2\,\epsilon-a_{1234}-\alpha-\beta-\delta]\Gamma[8-4\,\epsilon-a_{112233445}-2\,\alpha-2\,\delta-\gamma]\Gamma[9-4\,\epsilon-a_{112233445}-2\,\alpha-2\,\beta-2\,\delta-\gamma]$$

$$\Big[\Gamma[3-\epsilon-a_{12}-\alpha]\;\Gamma[5-2\,\epsilon-a_{1123}-\gamma]\Gamma[4-2\,\epsilon-a_{1123}-2\,\alpha-\gamma]\;\Gamma[a_1+\gamma]+\Gamma[2-\epsilon-a_{12}-\alpha]\;\Gamma[4-2\,\epsilon-a_{1123}-\gamma$$

$$\Gamma[5-2\,\epsilon-a_{1123}-2\,\alpha-\gamma]\;\Gamma[1+a_1+\gamma]\Big]\;\Gamma[-2+\epsilon+a_{123}+\alpha+\delta+\gamma]+\Gamma[6-3\,\epsilon-a_{12345}-\alpha]\;\Gamma[3-\epsilon-a_{12}-\alpha]$$

$$\Gamma[5-2\,\epsilon-a_{1123}-\gamma]\;\Gamma[4-2\,\epsilon-a_{1123}-2\,\alpha-\gamma]\Gamma[a_1+\gamma]\Big[((p_e\cdot(p_1+p_2))\;\Gamma[5-2\,\epsilon-a_{1234}-\alpha-\beta-\delta]\Gamma[9-4\,\epsilon-a_{11}$$

$$\Gamma[8-4\,\epsilon-a_{112233445}-2\,\alpha-2\,\beta-2\,\delta-\gamma]\Gamma[-2+\epsilon+a_{123}+\alpha+\delta+\gamma]+(p_e\cdot p_1)\;\Gamma[4-2\,\epsilon-a_{1234}-\alpha-\beta-\delta]$$

$$\Gamma[8-4\,\epsilon-a_{112233445}-2\,\alpha-2\,\delta-\gamma]\Gamma[9-4\,\epsilon-a_{112233445}-2\,\alpha-2\,\beta-2\,\delta-\gamma]\Gamma[-1+\epsilon+a_{123}+\alpha+\delta+\gamma]\Big]\Big\}$$

DESY 07-037
HEPTOOLS 07-009
SFB/CPP-07-14

AMBRE – a Mathematica package for the construction of Mellin-Barnes representations for Feynman integrals

J. Gluza, K. Kajda, T. Riemann

AMBRE v.1.0 Abstract

The Mathematica toolkit AMBRE derives Mellin-Barnes (MB) representations for Feynman integrals in $d = 4 - 2\varepsilon$ dimensions. It may be applied for tadpoles as well as for multi-leg multi-loop scalar and tensor integrals. AMBRE uses a loop-by-loop approach and aims at lowest dimensions of the final MB representations. The present version of AMBRE works fine for planar Feynman diagrams. The output may be further processed by the package MB for the determination of its singularity structure in ε. The AMBRE package contains various sample applications for Feynman integrals with up to six external particles and up to four loops.

A AMBRE functions list

The basic functions of AMBRE are:

- **Fullintegral[{numerator},{propagators},{internal momenta}]** – is the basic function for input Feynman integrals
- **invariants** – is a list of invariants, e.g. **invariants = {p1*p1 → s}**
- **IntPart[iteration]** – prepares a subintegral for a given internal momentum by collecting the related numerator, propagators, integration momentum
- **Subloop[integral]** – determines for the selected subintegral the U and F polynomials and an MB-representation
- **ARint[result,i_]** – displays the MB-representation number i for Feynman integrals with numerators
- **Fauto[0]** – allows user specified modifications of the F polynomial **fupc**
- **BarnesLemma[repr,1,Shifts->True]** – function tries to apply Barnes' first lemma to a given MB-representation; when **Shifts->True** is set, AMBRE will try a simplifying shift of variables
  BarnesLemma[repr,2,Shifts->True] – function tries to apply Barnes' second lemma

.

AMBRE – **Automatic Mellin-Barnes Representations** for Feynman diagrams

For the Mathematica package AMBRE, many examples, and the program description, see:

http://prac.us.edu.pl/∼gluza/ambre/
http://www-zeuthen.desy.de/theory/research/CAS.html

Authors: J. Gluza, K. Kajda, T. Riemann

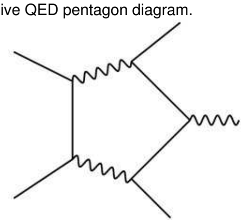
See also here:
http://www-zeuthen.desy.de/∼riemann/Talks/capp07/

with additional material presented at the CAPP – School on Computer Algebra in Particle Physics, DESY, Zeuthen, March 2007

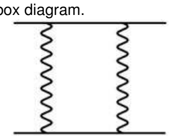
AMBRE - Automatic Mellin-Barnes REpresentation (arXiv:0704.2423)

To download 'right click' and 'save target as'.

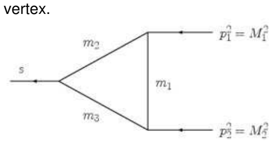
- The package AMBRE.m
- Kinematics generator for 4- 5- and 6- point functions with any external legs KinematicsGen.m
- Tarball with examples given below examples.tar.gz
  - example1.nb, example2.nb - Massive QED pentagon diagram.



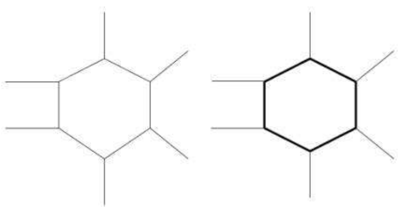
  - example3.nb - Massive QED one-loop box diagram.



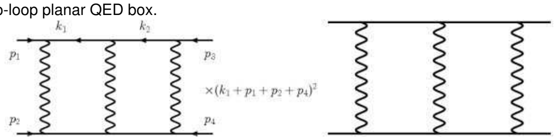
  - example4.nb - General one-loop vertex.



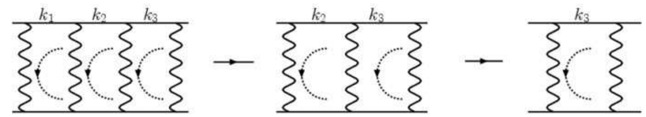
  - example5.nb - Six-point scalar functions;
    left: massless case,
    right: massive case.



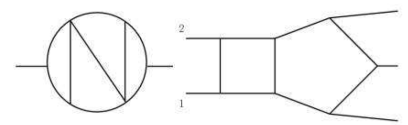
  - example6.nb - **left**, example7.nb - **right**
    Massive two-loop planar QED box.



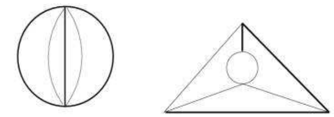
  - example8.nb - The loop-by-loop iterative procedure.



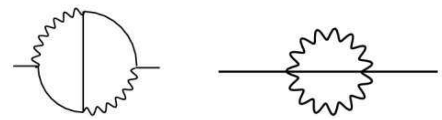
  - $ 2)& $ (3 & /
    ")
    !"



  - $ ((&) $ (* & !"
    4 "



  - $ (−)& $ (& 5 /
    . − *)
    !" . − (



General Tasks after deriving the MB-integral

The first three steps are automated by `MB.m [M. Czakon]`:

- **Find a region of definiteness of the n-fold MB-integral**

  $$\Re(z_1) = -1/80, \Re(z_3) = -33/40, \Re(z_5) = -21/20, \Re(z_6) = -59/160, \Re(\epsilon) = -1/10!$$

- **Analytical continuation to the physical region where $\epsilon << 1$ by distorting the integration path step by step (adding each crossed residuum – per crossed residue, this means one integral less!!!)**

- **ϵ-expansion, get a sequence of multi-dimensional finite MB-integrals**

- **Perform numerical integration, – or –**

- **Take integrals by sums over residua, i.e. introduce infinite sums**

- **Sum these infinite multiple series into some known functions of a given class, e.g. Nielsen polylogs, Harmonic polylogs or whatever is appropriate.**

A vertex: V3l2m

The Feynman integral V3l2m is the QED one-loop vertex function, which is no master.
It is infrared-divergent (see this by counting of powers of loop integration momentum k or know it from: massless line between two external on-shell lines)

$$F = m^2(x_1+x_2)^2 + [-s]x_1x_2$$

We will also use the variable

$$y = \frac{\sqrt{-s+4}-\sqrt{-s}}{\sqrt{-s+4}+\sqrt{-s}}$$

$$\begin{aligned} \texttt{V3l2m} &= \frac{e^{\epsilon\gamma_E}\Gamma(-2\epsilon)}{2\pi i}\int_{-i\infty-1/2}^{-i\infty-1/2} dz(-s)^{-\epsilon-1-z}\frac{\Gamma^2(-\epsilon-z)\Gamma(-z)\Gamma(1+\epsilon+z)}{\Gamma(1-2\epsilon)\Gamma(-2\epsilon-2z)} \\ &= \frac{\texttt{V3l2m}[-1]}{\epsilon} + \texttt{V3l2m}[0] + \epsilon\ \texttt{V3l2m}[1] + \cdots . \end{aligned}$$

One may slightly shift the contour by $(-\epsilon)$ and then close the path to the left and get residua from (and only from) $\Gamma(1+z)$:

[Gluza:2007bd]

:

$$V(s) = \frac{1}{2s\epsilon}\frac{e^{\epsilon\gamma_E}}{2\pi i}\int\limits_{-i\infty-1/2}^{-i\infty-1/2} dz \quad (-s)^{-z}\frac{\Gamma^2(-z)\Gamma(-z+\epsilon)\Gamma(1+z)}{\Gamma(-2z)}$$

$$= -\frac{e^{\epsilon\gamma_E}}{2\epsilon}\sum_{n=0}^{\infty}\frac{s^n}{\binom{2n}{n}(2n+1)}\frac{\Gamma(n+1+\epsilon)}{\Gamma(n+1)}.$$

This series may be summed directly with Mathematica![a], and the vertex becomes:

$$V(s) = -\frac{e^{\epsilon\gamma_E}}{2\epsilon}\Gamma(1+\epsilon) \quad {}_2F_1\left[1, 1+\epsilon; 3/2; s/4\right].$$

Alternatively, one may derive the ϵ-expansion by exploiting the well-known relation with harmonic numbers $S_k(n) = \sum_{i=1}^{n} 1/i^k$:

$$\frac{\Gamma(n+a\epsilon)}{\Gamma(n)} = \Gamma(1+a\epsilon)\exp\left[-\sum_{k=1}^{\infty}\frac{(-a\epsilon)^k}{k}S_k(n-1)\right].$$

[a] The expression for $V(s)$ was also derived in

[Huber:2007dx]

; see additionally

[Davydychev:2000na]

.

The product $\exp(\epsilon\gamma_E)\Gamma(1+\epsilon) = 1 + \frac{1}{2}\zeta[2]\epsilon^2 + O(\epsilon^3)$ yields expressions with zeta numbers $\zeta[n]$, and, taking all terms together, one gets a collection of inverse binomial sums[b]; the first of them is the IR divergent part:

$$V(s) = \frac{V_{-1}(s)}{\epsilon} + V_0(s) + \cdots$$

$$V_{-1}(s) = \frac{1}{2}\sum_{n=0}^{\infty}\frac{s^n}{\binom{2n}{n}(2n+1)} = \frac{1}{2}\frac{4\arcsin(\sqrt{s}/2)}{\sqrt{4-s}\sqrt{s}} = \frac{y}{y^2-1}\ln(y).$$

[b]For the first four terms of the ϵ-expansion in terms of inverse binomial sums or of polylogarithmic functions, see

[Gluza:2007bd]

.

The constant term:

$$\texttt{V312m}[0] \quad = \quad \frac{1}{2\pi i}\int_{-i\infty+u}^{+i\infty+u} dr(-s)^{-1-r}\ \frac{\Gamma^3[-r]\Gamma[1+r]}{\Gamma[-2r]}$$
$$\frac{1}{2}\left[\gamma_E - \ln(-s) + 2\Psi[-2r] - 2\Psi[-r] + \Psi[1+r]\right]$$

$$= \quad \frac{1}{2}\sum_{n=0}^{\infty}\frac{s^n}{\binom{2n}{n}(2n+1)}S_1(n),$$

There is also the opportunity to evaluate the MB-integrals numerically by following with e.g. a Fortran routine the straight contour.
This applies after the ϵ-expansion.

$\int_{-5i+\Re z}^{+5i+\Re z}$ is usually sufficient.

But: This works fast and stable for Euclidean kinematics where $-s > 0$.

and the ϵ-term:

$$
\begin{aligned}
\texttt{V312m}[1] &= \frac{1/4}{2\pi i}\int_{-i\infty+u}^{+i\infty+u} dr(-s)^{-1-r}\ \frac{\Gamma^3[-r]\Gamma[1+r]}{\Gamma[-2r)]} \\
&\Big[\gamma_E^2 + Log[-s]^2 + Log[-s](-2\gamma_E - 4\Psi[-2z] + 4\Psi[-z] - 2\Psi[1+z]) \\
&+\gamma_E(4\Psi[-2z] - 4\Psi[-z] + 2\Psi[1+z]) \\
&-4\Psi[1,-2z] + 2\Psi[1,-z] + \Psi[1,1+z] \\
&+4(\Psi[-2z]^2 - 2\Psi[-2z]\Psi[-z] + \Psi[-z]^2 + \Psi[-2z]\Psi[1+z] \\
&-\Psi[-z]\Psi[1+z]) + \Psi[1+z]^2\Big] \\
&= [const=1?]\times\frac{1}{4}\sum_{n=0}^{\infty}\frac{(t)^n}{\binom{2n}{n}(2n+1)}\left[S_1(n)^2+\zeta_2-S_2(n)\right].
\end{aligned}
$$

Here, $\Psi[r] = \ldots$ and $\Psi[1,r] = \ldots$, and the harmonic numbers $S_k(n)$ are

$$
S_k(n) = \sum_{i=1}^{n}\frac{1}{i^k},
$$

The sums appearing above may be obtained from sums listed in Table 1 of Appendix D in [Gluza:2007bd,Davydychev:2003mv]

:

$$\sum_{n=0}^{\infty} \frac{s^n}{\binom{2n}{n}(2n+1)} = \frac{y}{y^2-1} 2\ln(y),$$

$$\sum_{n=0}^{\infty} \frac{s^n}{\binom{2n}{n}(2n+1)} S_1(n) = \frac{y}{y^2-1}\left[-4\mathrm{Li}_2(-y) - 4\ln(y)\ln(1+y) + \ln^2(y) - 2\zeta_2\right],$$

$$\sum_{n=0}^{\infty} \frac{s^n}{\binom{2n}{n}(2n+1)} S_1(n)^2 = \frac{y}{y^2-1}\Bigg[16 S_{1,2}(-y) - 8\mathrm{Li}_3(-y) + 16\mathrm{Li}_2(-y)\ln(1+y)$$
$$+8\ln^2(1+y)\ln(y) - 4\ln(1+y)\ln^2(y) + \frac{1}{3}\ln^3(y) + 8\zeta_2\ln(1+y)$$
$$-4\zeta_2\ln(y) - 8\zeta_3\Bigg],$$

$$\sum_{n=0}^{\infty} \frac{s^n}{\binom{2n}{n}(2n+1)} S_2(n) = -\frac{y}{3(y^2-1)} \ln^3(y),$$

Expansion in a small parameter: vertex V3l2m for m^2/s

Use as an example for determining the small mass expansion:

$$\text{V3coefm1} = \text{Coefficient}[\text{V3l2m}[[1,1]],\epsilon,-1]$$
$$= -\frac{1}{2s}\frac{1}{2\pi i}\int_{-i\infty-1/2}^{+i\infty-1/2} dz \left(-\frac{m^2}{s}\right)^z \frac{\Gamma_1[-z]^3\Gamma_2[1+z]}{\Gamma_3[-2z]}$$

If $|m^2/s| << 1$, then the smallest [positive] power of it gives the biggest contribution: its exponent has to be positive and small.

So, close the contour to the right (positive $\Re z$), and leading terms come from the residua expansion of $\Gamma_1[-z]^3/\Gamma_3[-2z]$ at $z=+1,+2,\cdots$. The residues are terms of a binomial sum:

$$Residue = -\frac{1}{s}\left(\frac{m^2}{s}\right)^n \frac{(2n)!}{(n!)^2}\left[2HarmonicNumber[n] - 2HarmonicNumber[2n] - \ln\left(-\frac{m^2}{s}\right)\right]$$

with first terms equal to (-1)*Residua:

$$V3l2m = \frac{1}{s}\ln\left(-\frac{m^2}{s}\right) + O(m^4/s^2)$$

End of 2x 60 minutes lecture at CALC, Dubna, 2009