9. Radiative transfer in moving media

9.1 Radiative transfer for spectral lines

In the preceding chapters we have separately derived equations and techniques to describe the dynamical evolution of matter and the transport of radiation. In practise, the two are coupled problems, because the radiating medium may move with different velocities or in different directions in different parts of the emission region. The effect will be most prominent for spectral lines. In the following we will discuss the (easier) case of pure emission and absorption and neglect scattering.

Lines have a natural minimal width with a profile called a Lorentzian that can be understood using classical physics. Emission corresponds to an energy loss for the emitter or to a damping of its state. Consider a damped harmonic oscillator, whose energy decrease with a time constant γ. Then the amplitude is

$$x(t) = x_0 \, \exp(\imath\omega_0 t) \, \exp(-\frac{\gamma}{2}t) \qquad\qquad t \geq 0 \tag{9.1}$$

We derive the line profile by a Fourier-Transformation

$$\tilde{x}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x(t) \, \exp(-\imath\omega t) \, dt = \frac{x_0}{\sqrt{2\pi}} \frac{1}{\frac{\gamma}{2} + \imath(\omega - \omega_0)} \tag{9.2}$$

and the spectral power as

$$P(\omega) \propto |\tilde{x}|^2 = \frac{1}{2\pi} \frac{x_0^2}{\left(\frac{\gamma}{2}\right)^2 + (\omega - \omega_0)^2} \tag{9.3}$$

The line may be further broadened by the Doppler effect on account of the isotropic thermal motion of the emitters, described by a Gaussian, or on account of bulk motion. In some cases we may disentangle the effect of thermal motion and bulk motion.

Suppose an observer, located at $\vec{r}_0$, sees line radiation from gas at $\vec{r}$. It is useful to introduce a path length coordinate along the line-of-sight.

$$\vec{r}_0 = \vec{r} + \vec{e}_k \, s \tag{9.4}$$

At each location a line is emitted (and absorbed) with a width, that is determined by the thermal velocity of the emitters, $v_{\rm th}$, and a Doppler shift on account of the bulk flow with velocity $\vec{V}$.

$$\nu = \nu_0 \left(1 + \frac{\vec{e}_k \cdot \vec{V}}{c}\right) \qquad \text{for } |\vec{V}| \ll c \tag{9.5}$$

If the thermal velocity is of the same order as or larger than the bulk velocity or change thereof, the bulk velocity will only moderately change the line profile. If, on the other hand, the change of bulk velocity is fast in comparison with the thermal velocity, i.e.

$$l_k = \frac{v_{\rm th}}{\vec{e}_k \cdot \vec{\nabla}(\vec{e}_k \cdot \vec{V})} \ll L \tag{9.6}$$

where L is any other physical scale length of the system, then the total line profile will be dominated by the Doppler shift (9.5) and we may neglect the thermal line width.
We will discuss two methods to treat radiative transfer in moving media, the first one good for unresolved sources if absorption can be neglected, and the second more complicated, but valid for arbitrary optical depth of the line.

9.2 Pure emission lines

Let us first discuss a case in which absorption plays no role. A nice example are nuclear de-excitation gamma-ray lines that are emitted by the radioactive nuclei produced in a supernova explosion. Let us suppose the explosion was isotropic and the nuclei were blasted away with a flat probability distribution for the non-relativistic radial velocity, v,

$$P(v) = \frac{2\,v}{v_{\rm max}^2}\,\Theta(v_{\rm max} - v) \tag{9.7}$$

where Θ is a step function. The measured gamma-ray energy for each decay is given by the Doppler shift formula.

$$E_\gamma = E_0\left(1 + \frac{\vec{e}_k \cdot \vec{V}}{c}\right) \tag{9.8}$$

The observed line profile is a distribution in E_γ, $P(E_\gamma)$. To derive the line shape we have to integrate over the velocity distribution and over the angular distribution of emitted nuclei, which we assume to be isotropic here, $P(\Omega) = \frac{1}{4\pi}$. The distributions are differential quantities and thus care must be exercized. We can set the integrals equal, though.

$$P(E_\gamma)\;dE_\gamma = P(v)\,dv\,P(\Omega)\,d\Omega = P(v)\,dv\,P(\Omega)\,d\phi\,d\cos\theta \tag{9.9}$$

If we can transform one the differentials on the RHS to dE_γ, then it can be cancelled from Eq.9.9, and we derive $P(E_\gamma)$ after all other integrations on the RHS have been worked out. Using 9.8 and choosing polar angles θ with respect to the line-of-sight, we get the Jacobian and the limits on E_γ that arise from the integral limits on $\cos\theta$.

$$E_\gamma = E_0\left(1 + \frac{v\,\cos\theta}{c}\right) \qquad \Rightarrow \qquad \frac{dE_\gamma}{d\cos\theta} = E_0\,\frac{v}{c} \qquad\qquad v \geq c\left|\frac{E_\gamma}{E_0} - 1\right| \tag{9.10}$$

Then

$$P(E_\gamma) = \int P(v)\,dv \int P(\Omega)\,d\phi\,\frac{c}{v\,E_0}\,\Theta\left(v - c\left|\frac{E_\gamma}{E_0} - 1\right|\right)$$

$$= \frac{c}{E_0\,v_{\max}^2}\int_{c\left|\frac{E_\gamma}{E_0}-1\right|}^{v_{\max}} dv\;\Theta\left(v_{\max} - c\left|\frac{E_\gamma}{E_0} - 1\right|\right)$$

$$\Rightarrow \qquad P(E_\gamma) = \frac{c}{E_0\,v_{\max}}\left(1 - \left|\frac{E_\gamma - E_0}{E_c - E_0}\right|\right) \qquad E_c = E_0\left(1 \pm \frac{v_{\max}}{c}\right) \tag{9.11}$$

where E_γ is within the bounds given by E_c. The line would have the shape of a triangle.
Let us now consider the optically thin line profile resulting from material that is expelled in a supernova explosion. In contrast to the time-independent treatment, we now want to know the temporal evolution of the line profile, $P(E_{\text{obs}}, t_{\text{obs}})$. We assume all particles have the same velocity $P(v) = \delta(v - v_0)$, have been expelled isotropically at the same location and time ($t = 0$). Further, we posit that the particles decay exponentially after instantaneous injection as would befit a radioactive particle ($P(t) = \exp(-t/t_0)\,\Theta(t)$).
The problem is similar to that earlier except that now we would wish to calculate rates, i.e. also temporal probabilities. The ansatz would be

$$P(E_\gamma, t_{\text{obs}})\;dE_\gamma\,dt_{\text{obs}} = P(v)\,dv\,P(\Omega)\,d\phi\,d\cos\theta\,P(t)\,dt$$

where we can use the Doppler shift formula

$$E_\gamma = E_0\left(1 + \frac{\vec{e}_k\cdot\vec{V}}{c}\right) = E_0\left(1 + \frac{v\,\cos\theta}{c}\right)$$

The relation between the observed time t_{obs} and the standard time coordinate t is given by the changes in light travel time. Radiation from particles that are located on the far side of the supernova needs more time to arrive at the observer, and vice versa. After time t the particle has a separation $x = vt$ from the supernova. If the supernova is located at a distance d to the observer, the particle will be at a distance

$$D = \sqrt{d^2 + x^2 - 2\,x\,d\,\cos\theta} = \sqrt{d^2 + v^2\,t^2 - 2\,v\,t\,d\,\cos\theta}\;.$$

The observed time therefore is (we can add any constant)

$$t_{\text{obs}} = t + \frac{D}{c} = t + \frac{1}{c}\sqrt{d^2 + v^2\,t^2 - 2\,v\,t\,d\,\cos\theta} = t + \frac{d}{c}\sqrt{1 + \frac{v^2\,t^2}{d^2} - 2\,\frac{v\,t}{d}\,\cos\theta}$$

We are supposed to assume $x = v\,t \ll d$. Taylor expansion to first order yields

$$t_{\text{obs}} \simeq t + \frac{d}{c}\left(1 - \frac{v\,t}{d}\,\cos\theta\right) = t + t_c - \frac{v\,t}{c}\,\cos\theta = t_c + t\left(1 - \frac{E_\gamma - E_0}{E_0}\right) = t_c + t\left(2 - \frac{E_\gamma}{E_0}\right)\;.$$

after inserting the Doppler shift formula.
We again need to transform the differentials in the probability equation above.

$$\frac{dE_\gamma}{d\cos\theta} = E_0\,\frac{v}{c} \qquad\qquad v \geq c\left|\frac{E_\gamma}{E_0} - 1\right|$$

$$dt_{\rm obs} \simeq dt\,\left(2 - \frac{E_\gamma}{E_0}\right)$$

Then with $t_c = 0$ for simplicity

$$P(E_\gamma, t_{\rm obs}) \simeq \int P(v)\,dv \int P(\Omega)\,d\phi\,\frac{c}{v\,E_0}\,\Theta\left(v - c\left|\frac{E_\gamma}{E_0} - 1\right|\right)\,P(t)\,\frac{1}{2 - \frac{E_\gamma}{E_0}}$$

$$= \int\int \frac{\delta(v - v_0)\,dv}{4\pi}\,\frac{d\phi\,c}{v\,(2\,E_0 - E_\gamma)}\,\Theta\left(\frac{v}{c} - \left|\frac{E_\gamma}{E_0} - 1\right|\right)\,\exp\left(-\frac{t_{\rm obs}}{t_0}\,\frac{1}{2 - \frac{E_\gamma}{E_0}}\right)\,\Theta\left(\frac{t_{\rm obs}}{2 - \frac{E_\gamma}{E_0}}\right)$$

$$\Rightarrow\;\; P(E_\gamma, t_{\rm obs}) \simeq \frac{c}{2\,v_0\,(2\,E_0 - E_\gamma)}\,\Theta\left(\frac{v_0}{c} - \left|\frac{E_\gamma}{E_0} - 1\right|\right)\,\exp\left(-\frac{t_{\rm obs}}{t_0}\,\frac{1}{2 - \frac{E_\gamma}{E_0}}\right)\,\Theta\left(t_{\rm obs}\right)$$

Initially the line profile is skewed to the blue side ($E_\gamma > E_0$), after the decay sets in the exponential factor suppresses the blue wing relative to the red wing. At any point of time the observer sees the particles on the far side in a younger state, i.e. less severely decayed. This corresponds to a relative enhancement of the red wing of the line.