6. Moment equations for radiation

6.1 Conservation equations

Instead of solving the full radiation transport equation one may be content with knowing the moments of the radiation field, much like hydrodynamics is often preferred over kinetic theory. We already know the zeroth moment of the specific intensity, the mean intensity of differential energy density.

$$\oint I_\nu \; d\Omega = c\, u_\nu = 4\pi \, J_\nu \tag{6.1}$$

Rather than velovity moments we would now be interested in angular moments such as the net flux of radiation. For that purpose we must recall that Eq.5.1, though written as a scalar expression, tacitly involves the direction of the radiation, $\vec{e}_k$. The intensity I_ν refers to the energy transfer through an area element with unit normal $\vec{e}_k$, i.e. perpendicular to the direction of radiation. As observers, however, we might be interested in the energy transport rate through a fixed area element with unit normal $\vec{e}_n$. The net flux through the area element $d\vec{A}_n$ would then be

$$F_{\nu,\vec{e}_n} = \oint d\Omega \; I_\nu \, \vec{e}_n \cdot \vec{e}_k = \int_0^{2\pi} d\phi \int_{-1}^{1} d\cos\theta_k \;\; \cos\theta_k \, I_\nu(\phi, \theta_k) \tag{6.2}$$

using $\vec{e}_n$ as the polar axis of the coordinate system. The unit normal $\vec{e}_n$ can be pulled out of the integral, and hence what we have written is only the projection of a general flux vector

$$\vec{F}_\nu = \oint d\Omega \; I_\nu \, \vec{e}_k \tag{6.3}$$

In the same line of thought the second moment would be the monochromatic pressure tensor

$$c\, P_{\nu,n,m} = c \oint d\Omega \; I_\nu \, (\vec{e}_n \cdot \vec{e}_k) \, (\vec{e}_m \cdot \vec{e}_k) \qquad\qquad \mathbf{P}_\nu = \oint d\Omega \; I_\nu \, \vec{e}_k \, \vec{e}_k \tag{6.4}$$

Let us now calculate the zeroth angular moment of the radiation transport equation

$$\frac{1}{c}\frac{\partial I_\nu}{\partial t} + \vec{e}_k \cdot \vec{\nabla}\, I_\nu = j_\nu - \alpha_\nu \, I_\nu - I_\nu \oint d\Omega' \; \sigma(\Omega, \Omega') + \oint d\Omega' \; I_\nu(\Omega') \, \sigma(\Omega', \Omega)$$

Using that $\vec{e}_k$ commutes with $\vec{\nabla}$ we obtain

$$\frac{\partial u_\nu}{\partial t} + \vec{\nabla} \cdot \oint d\Omega \; I_\nu \, \vec{e}_k =$$

$$\oint d\Omega \; j_\nu - \oint d\Omega \; \alpha_\nu \, I_\nu - \oint d\Omega \; I_\nu(\Omega) \oint d\Omega' \; \sigma(\Omega, \Omega') + \oint d\Omega' \; I_\nu(\Omega') \oint d\Omega \; \sigma(\Omega', \Omega)$$

$$\Leftrightarrow \qquad \frac{\partial u_\nu}{\partial t} + \vec{\nabla} \cdot \vec{F}_\nu = \oint d\Omega \; j_\nu - \oint d\Omega \; \alpha_\nu \, I_\nu \quad \overset{\text{isotropy}}{=} \quad 4\pi \, j_\nu - \alpha_\nu \, c\, u_\nu \tag{6.5}$$

We have thus derived an equation similar to the mass conservation equation in hydrodynamics. For isotropic radiation coefficients the first-order equation would read

$$\frac{1}{c}\frac{\partial \vec{F}_\nu}{\partial t} + c\,\vec{\nabla}\cdot\mathbf{P}_\nu = -\kappa_\nu\,\vec{F}_\nu = -(\alpha_\nu+\sigma)\,\vec{F}_\nu = -\left(\alpha_\nu + \oint d\Omega'\;\sigma(\Omega,\Omega')\right)\,\vec{F}_\nu \tag{6.6}$$

As in hydrodynamics we have the problem that with each new equation we get a number of new variables, so the problem is underdetermined and we need closure relations. These closure relations are very difficult to derive. Let us now study an approximation that is applicable to high-opacity situations.

6.2 The radiation conduction approximation

Inside a star the photon mean free path

$$l_\nu = \frac{1}{\kappa_\nu} \approx 1\ \mathrm{cm} \ll R_* \tag{6.7}$$

Therefore the optical depth to the surface of a star is extremely high, so thermodynamic equilibrium should be established and the radiation field must be isotropic and a Planckian with the temperature of the ambient matter. Let us calculate the differential energy density, energy flux, and pressure tensor for these conditions.

$$I_\nu \simeq B_\nu(T) \qquad \Rightarrow \qquad u_\nu \simeq \frac{4\pi}{c}\,B_\nu(T) \qquad \vec{F}_\nu \simeq 0 \qquad \mathbf{P}_\nu \simeq \frac{4\pi}{3\,c}\,B_\nu(T)\,(\delta_{ij}) \tag{6.8}$$

$$\Rightarrow \qquad \mathbf{P}_\nu \simeq P_\nu\,(\delta_{ij}) \qquad \text{with} \qquad P_\nu = \frac{1}{3}\,u_\nu \tag{6.9}$$

let us now compare with the moment equations for isotropic radiation coefficients, Eqs. 6.5 and 6.6, where we note that in 6.5 we can probably neglect the explicit time derivative because

$$\frac{\partial u_\nu}{\partial t} \approx \frac{u_\nu}{t_\odot} \approx \frac{u_\nu}{t_\odot}\,\frac{c}{c}\,l_\nu\,\alpha_\nu = \frac{l_\nu}{c\,t_\odot}\,\alpha_\nu\,c\,u_\nu \ll \alpha_\nu\,c\,u_\nu \tag{6.10}$$

where $t_\odot$ is the evolution time of the sun. Similarily we may neglect $\frac{1}{c}\frac{\partial \vec{F}_\nu}{\partial t}$ in Eq.6.6, which then writes

$$\text{Eq.6.6} \;\Rightarrow \qquad c\,\vec{\nabla}\cdot\mathbf{P}_\nu \simeq -\kappa_\nu\,\vec{F}_\nu \tag{6.11}$$

Using Eq.6.9 we then obtain

$$\vec{F}_\nu \simeq -\frac{4\pi}{3\,\kappa_\nu}\,\frac{\partial B_\nu(T)}{\partial T}\,\vec{\nabla}T \tag{6.12}$$

The total, frequency-integrated radiation flux can then be written

$$\vec{F}_{\mathrm{rad}} \simeq -\frac{4\pi}{3}\,(\vec{\nabla}T)\int_0^\infty d\nu\;\frac{1}{\kappa_\nu}\,\frac{\partial B_\nu(T)}{\partial T} = -\frac{4\pi}{3\,\kappa_R}\int_0^\infty d\nu\;\frac{\partial B_\nu(T)}{\partial T} \tag{6.13}$$

where we have defined the Rosseland mean opacity

$$\kappa_R \equiv \frac{\int_0^\infty d\nu\ \frac{\partial B_\nu(T)}{\partial T}}{\int_0^\infty d\nu\ \frac{1}{\kappa_\nu}\ \frac{\partial B_\nu(T)}{\partial T}} \tag{6.14}$$

Since we know

$$\int_0^\infty d\nu\ \frac{\partial B_\nu(T)}{\partial T} = \frac{d}{dT}\int_0^\infty d\nu\ B_\nu(T) = \frac{c}{4\pi}\,\frac{du(T)}{dT}$$

$$\Rightarrow \qquad \vec{F}_{\rm rad} \simeq -\frac{c}{3\,\kappa_R}\,\vec{\nabla}u(T) = -\frac{c}{3\,\kappa_R}\,\vec{\nabla}(a\,T^4) \tag{6.15}$$

Eq.6.15 has the general form of a diffusion equation with diffusion coefficient D, it follows Fick's law

$$\text{(diffusive flux)} = -D\vec{\nabla}\text{(density of diffusing medium)}$$

Here the radiative diffusion coefficient would be

$$D = \frac{c}{3\,\kappa_R} = \frac{c\,l_R}{3} \tag{6.16}$$

This diffusive random walk implies that the energy transport by radiation in stars is slow, for the timescale of diffusive transport to the stellar surface is much larger than the timescale for free escape, $\tau_{\rm free}$.

$$\tau_{\rm dif} \simeq \frac{R_*^2}{D} = 3\,\frac{R_*}{l_R}\,\frac{R_*}{c} = 3\,\frac{R_*}{l_R}\,\tau_{\rm free} \gg \tau_{\rm free} \tag{6.17}$$

The absorption of radiation with a net flux corresponds to a momentum transfer to the ambient medium and hence to a radiative force. Using 6.12 we obtain for the radiative force per unit volume

$$\vec{f}_{\rm rad} = \frac{1}{c}\int_0^\infty d\nu\ \kappa_\nu\,\vec{F}_\nu \simeq -\frac{4\pi}{3\,c}\,(\vec{\nabla}T)\int_0^\infty d\nu\ \frac{\partial B_\nu(T)}{\partial T} = -\frac{1}{3}\,\vec{\nabla}u(T) = -\vec{\nabla}P_{\rm rad} \tag{6.18}$$