5. Radiation transport

Reading: Shu, Vol.I, Ch.1 and Ch.3

5.1 The radiation transport equation

Being equipped with techniques to describe non-relativistic matter we now turn our attention to radiation. We know that in many cirumstances we can use classical optics, i.e. describe radiation as freely propagating rays and waves or photons when interacting with matter. We also know that when interactions with individual photons are concerned, such as the emission or absorption of photons by atoms, a quantummechanical treatment is needed.

Let us first define a few quantities. Assume a area element dA perpendicular to incoming radiation. All rays through dA, whose direction is within the solid angle element $d\Omega$, transport the energy dE through dA in the time interval dt and frequency interval $d\nu$. We define

$$I_\nu = \frac{dE}{dA\,dt\,d\nu\,d\Omega} \qquad \text{specific intensity} \tag{5.1}$$

Averaging over solid angle yields

$$J_\nu = \frac{1}{4\pi}\oint I_\nu\,d\Omega \qquad \text{mean intensity} \tag{5.2}$$

The energy density spectrum per solid angle element then is

$$u_\nu(\Omega) = \frac{dE}{dV\,d\nu\,d\Omega} = \frac{dE}{c\,dt\,dA\,d\nu\,d\Omega} = \frac{I_\nu}{c} \tag{5.3}$$

And the total energy density spectrum

$$u_\nu = \oint u_\nu(\Omega)\,d\Omega = \oint \frac{I_\nu}{c}\,d\Omega = \frac{4\pi}{c}J_\nu \tag{5.4}$$

How do these quantities compare with the notion of a photon distribution function for spin state i? Using $p = h\nu/c$ we obtain

$$\sum_i f_i(\vec{x},\vec{p},t) = \sum_i \frac{dN_i}{d^3x\,d^3p} = u_\nu(\Omega)\,\frac{d\nu}{E\,p^2\,dp} = \frac{I_\nu}{c}\,\frac{c^3}{h\nu\,(h\nu)^2\,h}$$

$$\Rightarrow \qquad I_\nu = \frac{h^4\,\nu^3}{c^2}\,\sum_i f_i(\vec{x},\vec{p},t) \tag{5.5}$$

In thermodynamic equilibrium the radiation field should be a black-body or Planck spectrum.

$$\text{Planck} \qquad I_\nu = 2\,\frac{h\,\nu^3}{c^2}\,\frac{1}{\exp\left(\frac{h\nu}{kT}\right)-1}$$

or because the blackbody emission is unpolarized ($\sum \to$ factor 2 for two polarization directions).

$$f_i(\vec{x},\vec{p},t) = \frac{1}{h^3}\,\frac{1}{\exp\left(\frac{h\nu}{kT}\right)-1} = \frac{1}{h^3}\,n_i(\vec{x},\vec{p},t) \tag{5.6}$$

where n_i is called the photon occupation number.

If radiation passes through matter, its specific intensity may change. Energy may be added by emission or taken from the radiation field by absorption processes. Using $dV = dA\,ds$ with pathlength element ds let us define the spontaneous emission coefficient as

$$j_\nu = \frac{dE}{dV\,dt\,d\nu\,d\Omega} = \frac{dE}{ds\,dA\,dt\,d\nu\,d\Omega} = \frac{dI_\nu}{ds} \tag{5.7}$$

For the absorption let us visualize an ensemble of particles with density n, each of which blocks the radiation over a area σ. The total absorbing area in a test volume then is $dA_a = n\,\sigma\,dA\,ds$. The absorbed energy is

$$dE = I_\nu\,dA_a\,d\Omega dt d\nu = -dI_\nu\,dA\,d\Omega dt d\nu \qquad \Rightarrow \qquad dI_\nu = -n\sigma\,I_\nu\,ds = -\alpha_\nu\,I_\nu\,ds = -I_\nu\,d\tau \tag{5.8}$$

where we define the absorption coefficient α_ν and the optical depth τ as

$$\alpha_\nu = n\sigma \qquad\qquad \tau = \int_{s_0}^{s} \alpha_\nu\;ds \qquad d\tau = \alpha_\nu\,ds \tag{5.9}$$

In total we have derived the radiation transport equation without scattering

$$\frac{dI_\nu}{ds} = j_\nu - \alpha_\nu\,I_\nu \qquad \text{or} \qquad \frac{dI_\nu}{d\tau} = S_\nu - I_\nu \qquad \text{mit } S_\nu = \frac{j_\nu}{\alpha_\nu} \tag{5.10}$$

In practice j_ν and α_ν are the sums of the respective coefficients for all radiation processes. The quantity S_ν is called the source function. The radiation transport equation has the formal solution

$$I_\nu(\tau) = I_\nu(0)\exp(-\tau) + \int_0^\tau d\tau'\;S_\nu(\tau')\exp(-\tau+\tau') \tag{5.11}$$

If the source function is a constant, then

$$I_\nu(\tau) = I_\nu(0)\exp(-\tau) + S_\nu\;[1-\exp(-\tau)]$$

$$\Rightarrow \quad I_\nu \simeq \begin{cases} I_\nu(0) + j_\nu\,s & \text{for } \tau \ll 1 \quad \text{optically thin} \\ S_\nu & \text{for } \tau \gg 1 \quad \text{optically thick} \end{cases} \tag{5.12}$$

In an optically thick situation we obviously have $I_\nu = S_\nu$, whereas in optically thin conditions the intensity is determined by the emission coefficient, so we expect the spectrum to change at the frequency where the system transitions from being optically thick to optically thin.

Example: let us consider two regions on the line-of-sight with different properties τ_i and $S_{\nu,i}$, where I will use index 1 for the background region and index 2 for the foreground. Let us ignore irradiation from behind the background system, that is we set $I_\nu(0) = 0$. The intensity at the front of the background system, $I_\nu(\tau_1)$, is then given by (5.12),

$$I_\nu(\tau_1) = S_{\nu,1}\ [1 - \exp(-\tau_1)] \tag{5.12a}$$

To reach the observer, this intensity still has to pass through cloud 2, where it will be subject to absorption with optical depth τ_2, so the observer see from cloud 1

$$I_\nu(\text{cloud 1}) = S_{\nu,1}\ \exp(-\tau_2)\ [1 - \exp(-\tau_1)] \tag{5.12b}$$

In addition to this, the observer will measure the emission of cloud 2, which we can simply add to that of cloud 1, because the radiation transport equation is linear. The observed emission of cloud 2 is also given by (5.12) as

$$I_\nu(\text{cloud 2}) = S_{\nu,2}\ [1 - \exp(-\tau_2)] \tag{5.12c}$$

so the total intensity is the sum of (5.12b) and (5.12c).

$$I_\nu(\tau_1 + \tau_2) = S_{\nu,2}\ [1 - \exp(-\tau_2)] + S_{\nu,1}\ \exp(-\tau_2)\ [1 - \exp(-\tau_1)] \tag{5.12d}$$

It is instructive to ask yourself what the asymptotic cases would be. If $\tau_2 >> 1$, then emmision from cloud 1 would be fully absorbed and it would probably beinvisible, so we only see cloud 2 under optically thick conditions, and $I_\nu(\tau_1 + \tau_2) \simeq S_{\nu,2}$.
If $\tau_2 << 1$, then we have two cases, $\tau_1 << 1$ and $\tau_1 >> 1$. If $\tau_1 << 1$, then the entire system is optically thin, absorption plays no role, and

$$I_\nu(\tau_1 + \tau_2) \simeq S_{\nu,2}\,\tau_2 + S_{\nu,1}\,\tau_1 \qquad \text{for } \tau_1 << 1 \quad \text{and } \tau_2 << 1$$

If $\tau_1 >> 1$, then the background system is optically thick and

$$I_\nu(\tau_1 + \tau_2) \simeq S_{\nu,2}\,\tau_2 + S_{\nu,1} \qquad \text{for } \tau_1 >> 1 \quad \text{and } \tau_2 << 1$$

Considering that generally both the source functions, S_ν, and the optical depths, τ, are frequency-dependent, we must expect spectral changes at characteristic frequencies, for which the individual clouds transition from being optically thin to optically thick.
Example 2: to calculate the intensity distribution of an extended source in the sky we need to solve the radiation transport equation for each line-of-sight through the source and, to derive

the integrated radiation flux from an unresolved source, integrate the intensity distribution over solid angle, $F_\nu = \int_{\rm source} d\Omega\; I_\nu$.

For an extended source let us assume a sphere of radius R filled with emitting material of constant emission and absorption coefficient. If our line-of-sight intersects the sphere at a projected distance from the center r_0 (or impact parameter), the length of the line-of-sight is $l = 2\sqrt{R^2 - r_0^2}$ and the optical depth is $\tau = \alpha_\nu\, l = 2\,\alpha_\nu\,\sqrt{R^2 - r_0^2}$. According to 5.12 the intensity as a function of r_0 then is

$$I_\nu(r_0) = S_\nu\; [1 - \exp(-\tau)]$$

which gives the brightness distribution across the source on the sky. If as usual the absorption coefficient is a function of frequency, Eq.5.12 also describes the variation of the spectrum across the source.

To calculate the radiation flux, we need to integrate over all directions or, for a source at a distance $D \gg R$ equivalently over the apparent surface area, A, because $d\Omega \simeq \frac{1}{D^2}\,dA$. In our case

$$F_\nu = \frac{1}{D^2}\int dA\; I_\nu = \frac{2\pi}{D^2}\int_0^R dr_0\; r_0\, S_\nu\; [1 - \exp(-\tau)]$$

Using

$$\tau = \alpha_\nu\, 2\sqrt{R^2 - r_0^2} \qquad \rightarrow\; \tau\, d\tau = -4\,\alpha_\nu^2\, r_0\, dr_0$$

we obtain

$$F_\nu = \frac{\pi\, S_\nu}{2\,\alpha_\nu^2\, D^2}\int_0^{2\,R\,\alpha_\nu} d\tau\;\tau\;[1 - \exp(-\tau)]$$

$$= \frac{\pi\, S_\nu}{2\,\alpha_\nu^2\, D^2}\;[2\,R^2\,\alpha_\nu^2 + 2\,\alpha_\nu\, R\,\exp(-2\,\alpha_\nu\, R) + \exp(-2\,\alpha_\nu\, R) - 1]$$

$$= S_\nu\,\frac{\pi\, R^2}{D^2}\left[1 + \frac{\exp(-2\,\alpha_\nu\, R)}{\alpha_\nu\, R} + \frac{1}{2\,\alpha_\nu^2\, R^2}\;(\exp(-2\,\alpha_\nu\, R) - 1)\right]$$

$$\simeq \begin{cases} \pi\, S_\nu\,\frac{R^2}{D^2} & \tau_{\rm max} = 2\,\alpha_\nu\, R \gg 1 \\ \frac{4\pi}{3}\, j_\nu\,\frac{R^3}{D^2} & \tau_{\rm max} = 2\,\alpha_\nu\, R \ll 1 \end{cases}$$

Let us now consider scattering, which is more difficult to deal with because it depends on the intensity of incoming radiation. Usually we denote a process as scattering, if it changes the direction of radiation, Ω, though in quite a few cases scattering also leads to a change in frequency. A well known example is the Compton (or Thompson) scattering on free electrons. If the frequency does not change, we speak of elastic scattering. In this case we can add a source term and a sink term to the radiation transport equation that derive from an scattering coefficient $\sigma(\Omega_i, \Omega_a)$ for the scattering from the solid angle Ω_i to the element $d\Omega_a$. The sink term then is

$$\sigma(\Omega) = \oint d\Omega' \; \sigma(\Omega, \Omega') \tag{5.13}$$

and the source term

$$j_\nu(\Omega) = \oint d\Omega' \; I_\nu(\Omega') \, \sigma(\Omega', \Omega) \tag{5.14}$$

so the radiation transport equation for scattering writes

$$\frac{dI_\nu(\Omega)}{ds} = \oint d\Omega' \; I_\nu(\Omega') \, \sigma(\Omega', \Omega) - I_\nu(\Omega) \, \sigma(\Omega) \tag{5.15}$$

In the case of isotropic scattering with $\sigma(\Omega_i, \Omega_a) = \sigma/4\pi$ we obtain

$$\oint d\Omega' \; I_\nu(\Omega') \, \sigma(\Omega', \Omega) = \sigma \, J_\nu \qquad \Rightarrow \qquad \frac{dI_\nu(\Omega)}{ds} = \sigma \, (J_\nu - I_\nu) \tag{5.16}$$

Because the transport equation now involves J_ν, the solid-angle integral of the intensity, we have to simultaneously solve the radiation transport equation in a variety of directions, which makes treatment of scattering problems awfully difficult. It is obvious, though, that the scattering changes the radiation field toward isotropy, $I_\nu \rightarrow J_\nu$.
So far we have neglected an explicit time dependence of the intensity. We have also used a simplified notation with the tacit understanding that the path length element ds is meant in the direction of radiation, $\vec{e}_k$. In the general sense the radiation transport equation is of the form

$$\frac{dI_\nu}{ds} = \frac{1}{c} \frac{\partial I_\nu}{\partial t} + \vec{e}_k \cdot \vec{\nabla} \, I_\nu = j_\nu - \alpha_\nu \, I_\nu - I_\nu \, \sigma(\Omega) + \oint d\Omega' \; I_\nu(\Omega') \, \sigma(\Omega', \Omega) \tag{5.17a}$$

The sum of the spontaneous absorption coefficient and the scattering coefficient forms the

$$\text{Total opacity} \qquad \kappa_\nu = \alpha_\nu + \sigma \qquad\qquad \tau = \int_{s_0}^{s} \kappa_\nu \; ds \tag{5.17b}$$

and in this case the source function would be

$$\frac{dI_\nu}{d\tau} = S_\nu - I_\nu \qquad \Rightarrow \qquad S_\nu = \frac{1}{\kappa_\nu} \left[j_\nu + \oint d\Omega' \; I_\nu(\Omega') \, \sigma(\Omega', \Omega) \right] \tag{5.17c}$$

In an optically thick situation we again have $I_\nu = S_\nu$.

5.2 Emission and absorption lines

We already noted that in thermodynamic equilibrium the emission spectrum should be a Planckian, $B_\nu(T)$ and the matter will also follow a thermal Maxwellian with temperature T. Thermodynamic equilibrium requires a good coupling between particles and radiation, that is at least optically thick conditions, so $I_\nu = S_\nu = B_\nu(T)$. The source function would therefore be the Planckian. In LTE, the source function is still a Planckian, but the solution to the radiation transport problem is no longer a Planckian, and in the absence of scattering we have

$$j_\nu = \alpha_\nu \, S_\nu = \alpha_\nu \, B_\nu(T) \tag{5.18}$$

Specifying j_ν now boils down to specifying the temperature T, provided α_ν is known. If no background source of emission exists, the solution to the radiation transport equation for a homogeneous medium is

$$I_\nu(\tau) = B_\nu(T) \; [1 - \exp(-\tau)] \tag{5.19}$$

How would a composite of continuum emission and spectral lines look like? If more than one radiation process is operational, then the effective emission and absorption coefficients are the sum of the respective coefficients for the individual processes.

$$j_{\rm tot} = \sum_i j_i \qquad\qquad \alpha_{\rm tot} = \sum_i \alpha_i \tag{5.20}$$

This implies that at the location of a line the total absorption coefficient, and hence the optical depth τ_ν, is larger than at neighboring frequencies, irrespective of the temperature or density of the medium.

If the system is optically thin in continuum (e.g. HI clouds) we would see emission lines. The line intensity would depend on the line absorption coefficient as long as the line is optically thin. In the opposite case of high opacity the line intensity will again be given by the Planckian.

The advantage that lines enjoy over continuum ceases to exist when the optical depth in the continuum approaches and exceeds unity. Let us consider a background source in addition to a homogeneous medium with emission and absorption in LTE.

$$I_\nu(\tau) = I_\nu(0)\exp(-\tau) + B_\nu(T) \; [1 - \exp(-\tau)] = B_\nu(T) + [I_\nu(0) - B_\nu(T)] \; \exp(-\tau) \tag{5.21}$$

If the background intensity is higher than the Planckian $B_\nu(T)$, e.g. due to a higher temperature, then the term in brackets is positive and the emerging line intensity is less than the continuum intensity on account of the larger opacity at the line frequency. We would see absorption lines.

In the solar photosphere the temperature falls off with radius and the continuum is optically thick. Let us approximate this by setting

$$I_\nu(0) = B_\nu(T^*) \qquad\qquad T^* > T \tag{5.22}$$

We now know that absorption lines should be observed from the photosphere as is the case in the optical spectra of the sun and other stars. In the solar corona, however, the temperature is very much higher than in the photosphere, so that lines should be emitted in the ultraviolet where

$$\text{Corona} \qquad I_\nu(0) \ll B_\nu(T \simeq 10^6 \text{ K}) \tag{5.23}$$

Consequently we observe emission lines in the ultraviolet from the solar corona.

5.3 The relation between spontaneous emission and absorption

Let us consider the Einstein coefficients for a two-level system for spontaneous emission A_{21}, spontaneous absorption B_{12}, and stimulated emission B_{21}. In the steady-state we have

$$n_1\, B_{12}\, J = n_2\, A_{21} + n_2\, B_{21}\, J \quad \Rightarrow \quad J = \frac{A_{21}}{\frac{n_1}{n_2} B_{12} - B_{21}} \tag{5.24}$$

In Thermodynamic Equilibrium with $E_1 = E$, $E_2 = E + h\nu$, and

$$\frac{n_1}{n_2} = \frac{g_1\, \exp(-E/kT)}{g_2\, \exp(-(E+h\nu)/kT)} = \frac{g_1}{g_2} \exp(h\nu/kT) \tag{5.25}$$

the intensity spectrum should be that of a black-body. This implies the relations

$$A_{21} = \frac{2h\nu^3}{c^2} B_{21} \qquad g_1 B_{12} = g_2 B_{21} \tag{5.26}$$

Equation 5.26 depends only on atomic properties, not on the temperature or other ensemble properties. Therefore the relation should be valid, whether the system is in thermodynamical equilibrium or not. Furthermore, we have not specified a radiation process yet, thus the relations must be valid whatever the emission process.

The emitted power per particle, P_ν, and the emission coefficient can now be calculated as

$$P_\nu = \sum_{E_1} h\nu A_{21} = \sum_{E_1} \frac{2h\nu^3}{c^2}\, h\nu B_{21} \qquad j_\nu = \frac{1}{4\pi} \sum_{E_2} n(E_2)\, P_\nu \tag{5.27}$$

where $n(E_2)$ is the density of particles at energy state E_2. For the absorption we can calculate the radiation coefficient by writing the absorbed energy as a differential. Then treating stimulated emission as negative absorption we find the energy loss

$$J_\nu = I_\nu \frac{d\Omega}{4\pi} \qquad \text{and} \qquad dV = dA\, ds$$

$$dE = -(n_1 B_{12} - n_2 B_{21}) h\nu\, dV dt\, J_\nu = -(n_1 B_{12} - n_2 B_{21}) \frac{h\nu}{4\pi} I_\nu\, ds\;\, dA d\Omega dt d\nu \tag{5.28}$$

This energy loss corresponds to a change in the intensity of the radiation field

$$dE = dI_\nu\;\, dA d\Omega dt d\nu = -(n_1 B_{12} - n_2 B_{21}) \frac{h\nu}{4\pi} I_\nu\, ds\;\; dA d\Omega dt d\nu$$

$$\Rightarrow \qquad \alpha_\nu = \sum_{E_1}\sum_{E_2} \frac{h\nu}{4\pi}\left[n(E_1)B_{12} - n(E_2)B_{21}\right] = \frac{c^2}{2\,h\nu^3}\sum_{p_2} p_2^2\, P_\nu \left[f(p_2^*) - f(p_2)\right] \tag{5.29}$$

where p_2^* is the momentum corresponding to the energy $E_2 - h\nu$. For continuous distribution we can write the radiation coefficients in the form of integrals.

$$j_\nu = \frac{1}{4\pi}\int dp\; n(p)P_\nu = \frac{1}{4\pi}\int dE\; n(E)P_\nu \qquad\qquad \alpha_\nu = \frac{c^2}{8\pi\, h\,\nu^3}\int d^3p\; P_\nu \left[f(p^*) - f(p)\right] \tag{5.30}$$

Thus all terms in the radiation transport equation are reduced to one basic function, P_ν.
Let us have a closer look at the absorption coefficient α. If the distribution function is inverted, i.e. $\frac{\partial f}{\partial p} > 0$, the absorption coefficient will be negative, implying that stimulated emission is stronger than absorption. In this case we may observe a *Maser* or a *Laser*.