3. Kinetic versus fluid description of plasmas

3.1 Boltzmann's equation

Having established a means to describe the statistical distribution of particles in the six-dimensional phase space, the distribution function $f(\vec{x}, \vec{p}, t)$, we are now looking for a way to quantitatively follow the changes imposed on the distribution function by interactions.
It appears useful to distinguish between two kinds of changes. Collisions will abruptly change the momentum of particles, whereas interactions with, e.g., electromagnetic fields lead to continuous changes that can be describes as a convection of a particle in phase space.

$$\frac{d\vec{x}}{dt} = \vec{v} = \frac{\vec{p}}{m} \tag{3.1}$$

$$\frac{d\vec{p}}{dt} = q\left[\vec{E}(\vec{x}, t) + \frac{1}{c}\,\vec{v} \times \vec{B}(\vec{x}, t)\right] \qquad \text{or} \qquad \frac{d\vec{p}}{dt} = \vec{F}_{\text{grav}} \tag{3.2}$$

The collisions are better described as a catastrophic loss or sink term for the state incoming particle and a source term for the outgoing particle. Then the total rate of change of the distribution function should be

$$\frac{df}{dt} = \text{sources} - \text{sinks} = f_c = \frac{\partial f}{\partial t} + \dot{\vec{x}}\,\frac{\partial f}{\partial \vec{x}} + \dot{\vec{p}}\,\frac{\partial f}{\partial \vec{p}} = \frac{\partial f}{\partial t} + \vec{v}\,\frac{\partial f}{\partial \vec{x}} + q\left[\vec{E}(\vec{x}, t) + \frac{\vec{v}}{c} \times \vec{B}(\vec{x}, t)\right]\frac{\partial f}{\partial \vec{p}} \tag{3.3}$$

This equation is known as Boltzmann's equation. Very often in the astrophysical context we can neglect the collision term f_c and set it to zero, thus deriving Vlasov's equation. In the non-relativistic regime one may use the velocity instead of momentum as second coordinate, so the distribution function would be differential in velocity, not momentum. The Boltzmann equation would then write as

$$f_c(\vec{v}) = \frac{\partial f(\vec{x}, \vec{v})}{\partial t} + \vec{v}\,\frac{\partial f(\vec{x}, \vec{v})}{\partial \vec{x}} + \frac{q}{m}\left[\vec{E}(\vec{x}, t) + \frac{\vec{v}}{c} \times \vec{B}(\vec{x}, t)\right]\frac{\partial f(\vec{x}, \vec{v})}{\partial \vec{v}}$$

The electromagnetic fields that govern the force term in Boltzmann's equation are generated by the collective motion of all the other particles and possibly charges and currents external to the system of interest. The fields derive from the charge and current densities in the plasma, which we define to be

$$\rho_e = \sum_j q_j \int d^3v\; f_j(\vec{x}, \vec{v}, t) \tag{3.4}$$

$$\vec{j} = \sum_j q_j \int d^3v\; \vec{v}\, f_j(\vec{x}, \vec{v}, t) \tag{3.5}$$

The four Maxwell equations therefore read

$$\vec{\nabla} \cdot \vec{B} = 0 \tag{3.6}$$

$$\vec{\nabla} \times \vec{B} = \frac{1}{c}\frac{\partial \vec{E}}{\partial t} + \frac{4\pi}{c}\vec{j}_{\rm int} = \frac{1}{c}\frac{\partial \vec{E}}{\partial t} + \frac{4\pi}{c}\sum_j q_j \int d^3v\ \vec{v}\, f_j \tag{3.7}$$

$$\vec{\nabla} \cdot \vec{E} = 4\pi \sum_j q_j \int d^3v\ f_j \tag{3.8}$$

$$\vec{\nabla} \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t} \tag{3.9}$$

We note the nonlinear coupling between the electromagnetic fields and the particles. The kinetic theory, i.e. using the combined systems of Maxwell's equations and Boltzmann's equation is most powerful for describing plasma dynamics, but it is also most difficult to apply. There are three possible way out of this quandary:

- The test wave approach, in which the particle distribution function is assumed given, so that the resulting electromagnetic fields and their properties can be discussed.
- The test particle approach, in which the fields are assumed given and the response of the particles can be discussed.
- The fluid or magnetohydrodynamical description, in which only moments of the distribution function are used.

3.2 Hydrodynamics of plasmas

Useful though it may appear, the distribution function is not an observable, we can not measure it. What we can measure are moments of the distribution function, average quantities that we derive by integration. We already know the charge density, ρ_e, and the current density, $\vec{j}$. It may be useful to derive equations for the moments of the distribution function and solve these instead of Boltzmann's equation. If one is interested in the global behaviour on large scales and not in single particle effects and small scales, this technique, known as Hydrodynamics, may prove to be much easier to deal with and nevertheless provide sufficient accuracy.

Mass, momentum, and energy should be conserved quantities, so the corresponding moments of the collision term in Boltzmann's equation should vanish.

$$\int d^3v\ \left(m\ /\ m\vec{v}\ /\ \frac{m}{2}v^2\right)\ f_c = 0 \tag{3.10}$$

We can then derive equations for the moments by taking the moments of Boltzmann's equation.

$$\mathcal{D}\,f = f_c \qquad \rightarrow \qquad \int d^3v\ \left(m\ /\ m\vec{v}\ /\ \frac{m}{2}v^2\right)\ \mathcal{D}\,f(\vec{x},\vec{v},t) = 0 \tag{3.11}$$

Be careful with the notation in the following section, where I have to distinguish three different velocities and different pressure tensors, for which all textbooks have their designations. Let us as an example calculate the mass conservation equation. We will make use of the fact that time, location, and velocity or momentum are independent coordinates. Writing as shorthand $\vec{F}$ for the force term we have for each particle species

$$0 = \int d^3v\, m\,\frac{\partial f}{\partial t} + \int d^3v\, m\,\vec{v}\,\frac{\partial f}{\partial \vec{x}} + \int d^3v\, m\,\vec{F}\,\frac{\partial f}{\partial \vec{v}}$$

$$= \frac{\partial}{\partial t}\int d^3v\, m\, f + \frac{\partial}{\partial \vec{x}}\int d^3v\, m\,\vec{v}\, f - \int d^3v\, m\, f\,\frac{\partial \vec{F}}{\partial \vec{v}} = \frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\left(\vec{V}\,\rho\right) \tag{3.12}$$

where we have used the definitions

$$\rho = \int m\, f\, d^3v \qquad \text{mass density} \tag{3.13}$$

$$\vec{V} = \frac{1}{\rho}\int m\,\vec{v}\, f\, d^3v \qquad \text{bulk velocity} \tag{3.14}$$

and also used that the distribution function falls off rapidly for very large velocities or momenta. Using the definitions of a velocity relative to the flow,

$$\text{random velocity} \qquad u_i = v_i - V_i \qquad du = dv \qquad \int u\, f\, d^3u = 0 \tag{3.15}$$

we can formulate two pressure tensors

$$\Pi_{ij} = \int m\, v_i\, v_j\, f\, d^3v \qquad \text{Total pressure tensor} \tag{3.16}$$

$$\pi_{ij} = \int m\, u_i\, u_j\, f\, d^3v = \Pi_{ij} - V_i V_j \rho \qquad \text{Kinetic pressure tensor} \tag{3.17}$$

Further we can define

$$\mathcal{F} = \int \vec{F}\, f\, d^3v \qquad \text{Force density} \tag{3.18}$$

$$\vec{q} = \int \frac{m}{2}\, u^2\, \vec{u}\, f\, d^3v \qquad \text{conduction heat flux} \tag{3.19}$$

and finally

$$\epsilon = \int \frac{1}{2}\, m\, u^2\, f\, d^3v \qquad \text{Thermal energy density} \tag{3.20}$$

which allow us to write down the momentum equation for cartesian component j

$$\rho\left(\frac{\partial V_j}{\partial t} + V_i \frac{\partial V_j}{\partial x_i}\right) + \frac{\partial}{\partial x_i}\pi_{ij} - \mathcal{F}_j = 0 = \rho\left(\frac{\partial \vec{V}}{\partial t} + (\vec{V}\cdot\vec{\nabla})\,\vec{V}\right) + \vec{\nabla}\cdot\vec{\vec{\pi}} - \vec{\mathcal{F}} \tag{3.21}$$

where one needs to some over all i's. and the energy equation

$$\frac{\partial \epsilon}{\partial t} + \frac{\partial (V_i \epsilon)}{\partial x_i} + \pi_{ij} \frac{\partial V_j}{\partial x_i} + \frac{\partial q_i}{\partial x_i} = \mathcal{H} - \mathcal{C} \tag{3.22}$$

where we have added heating and cooling terms on the RHS. The equations thus derived are in fact easier to deal with than is Boltzmann's equation, but we note that we have only five equations to determine 13 variables. We definitely need additional relations to close the system. When f depends only on the absolute value of $\vec{u}$, $|\vec{u}|$, then

$$\pi_{ij} = P\,\delta_{ij} \tag{3.23}$$

and we can close the system, provided an equation of state $P(\rho, \epsilon)$ is known. A particular solution to the energy equation can be given in the absence of heating, cooling, and heat conduction. If the temporal derivative in Eq.3.22 vanishes, the system will behave adiabatically.

$$P\,\rho^{-\gamma} = \text{const.} \qquad \gamma = \text{adiabatic index} \qquad \epsilon = \frac{1}{\gamma - 1}\,P \tag{3.24}$$

For any isotropic (spherically symmetric in $\vec{u}$) distribution function, $f(\vec{x}, u)$, one finds $\epsilon = 3/2\,P$, so $\gamma = 5/3$. For the Maxwellian in particular an integration of (3.17) yields the ideal gas.

3.2 Hydrostatic equilibria

In the hydrostatic limit we assume there are no internal motions. Let us discuss as an examples a star, basically in good approximation a spherically symmetric gas cloud. It makes sense to use spherical coordinates, for which all angular derivatives are zero. The stars are also stationary, so all temporal derivatives are zero. The assumption of a hydrostatic situation means that all velocities are zero, and then the mass conservation equation is trivial. The momentum conservation for an isotropic, scalar pressure P equation reads

$$\frac{\partial P}{\partial r} = -\rho \left(\frac{G\,M(r)}{r^2} \right) \approx -\rho\,\frac{G\,M\,r}{R^3} \qquad M(r) = 4\pi \int_0^r dr'\; r'^2\,\rho(r') \tag{3.25}$$

In the energy equation we have heating by fusion processes in the interior and cooling by radiation at the surface of the star, the so-called photosphere, to be balanced by heat conduction.

$$\frac{1}{r^2}\,\frac{\partial (r^2\,q_r)}{\partial r} = \mathcal{H} - \mathcal{C} \tag{3.26}$$

It turns out that most of the heat transport is actually done by radiation, so on the RHS we would have to add heating by absorption of radiation everywhere inside the star, and cooling by radiation also everywhere inside the star. We would also need to consider the radiation transport inside the star to figure out the internal structure of the star.

The problem becomes simpler when we consider a situation in which the pressure depends only on the density, i.e. the adiabatic index is $\gamma = 1$. An important example of such an equation of state involve systems whose internal pressure is dominated by electron Fermi pressure, such as zero-temperature white dwarfs. In the non-relativistic case the electron degeneracy pressure is

$$P_e = \frac{h^2}{5\, m_e\, m_p} \left(\frac{3}{8\pi\, m_p} \right)^{2/3} \rho^{5/3} = P_0\, \rho^{5/3} \tag{3.27}$$

This allows us to rewrite Eq.(3.25) as

$$\frac{1}{\rho}\frac{\partial P}{\partial r} = \frac{5}{3}\, P_0\, \rho^{-1/3}\, \frac{\partial \rho}{\partial r} = \frac{5}{2}\, P_0\, \frac{\partial \rho^{2/3}}{\partial r} \approx -\rho\, \frac{G\, M\, r}{R^2} \tag{3.28}$$

Integrating this equation from the center to the surface gives

$$\frac{5}{2}\, P_0\, \rho^{2/3}\Big|_0^R = -\frac{G\, M}{2\, R^3}\, r^2 \Big|_0^R \tag{3.29}$$

The left-hand side of (3.29) is dominated by the density in the center of the object, thus giving

$$\frac{5}{2}\, P_0\, \rho(0)^{2/3} = \frac{G\, M}{2\, R} \qquad \text{and} \qquad M = x\, \rho(0)\, R^3 \approx \rho(0)\, R^3 \tag{3.30}$$

The second equation will carry a numerical factor x that can can be determined by proper integration. Because it is a numerical factor, we can still eliminate $\rho(0)$ to derive the mass-radius relationship for low-mass white dwarfs.

$$R \propto M^{-1/3} \tag{3.31}$$

3.3 Hydrodynamic equilibria

Now we search for equilibrium situations in which the gas is allowed to move, thus potentially supporting itself by centrifugal forces or ram pressure. A nice example would be interstellar gas in the Milky Way Galaxy, which here we assume to have an azimuthally symmetric distribution. The Milky Way has the shape of a rotating flat disk, so cylinder coordinates appear advantageous. For simplicity let us assume $v_z = 0$. Again assuming stationary conditions the mass conservation equation reads

$$\frac{\partial (r\, v_r\, \rho)}{\partial r} = 0 \tag{3.32}$$

This equation becomes entirely trivial, if there is no radial flow, $v_r = 0$. Using the potential V instead of the force the momentum conservation equations read

$$\rho\, v_r\, \frac{\partial v_r}{\partial r} - \rho\, \frac{v_\phi^2}{r} + \frac{\partial P}{\partial r} = -\rho\, \frac{\partial V}{\partial r} \tag{3.33a}$$

$$\rho\, v_r\, \frac{\partial v_\phi}{\partial r} + \rho\, \frac{v_\phi\, v_r}{r} = 0 \qquad (3.33b)$$

$$\frac{\partial P}{\partial z} = -\rho\, \frac{\partial V}{\partial z} \qquad (3.33c)$$

Apparently the vertical structure of the gas disk is given by a hydrostatic equilibrium between thermal pressure and gravity. This is a consequence of our assumption $v_z = 0$. The vertical scaleheights in the solar vicinity are about 50 pc or $1.5 \cdot 10^{20}$ cm for cold and dense molecular gas (T=20 K), about 150 pc for cold atomic gas (T=100 K), and around a kiloparsec for dilute hot ionized gas. If the pressure in solar vicinity is so small that it can just balance the small vertical gradient of the gravitational potential, it will be vastly insufficient to balance the radial gradient of the potential which is much larger than the vertical gradient. We can therefore neglect the pressure term in Eq.3.33a.
Equation 3.32 maintains that the radial mass flux be constant. That constant is most likely small, otherwise the Galaxy would be either evacuated or all gas would have concentrated at the center. In a steady-state it will be zero, unless the Galaxy has a source a gas at its center or can accrete from a reservoir at its periphery. Then the radial term on the LHS of Eq.3.33a will also be small and can be neglected. To simplify the treatment we will assume $v_r = 0$, so the angular momentum equation (3.33b) is trivial.
We can then rewrite 3.33a to obtain

$$\frac{v_\phi^2}{r} = \frac{\partial V}{\partial r} = -\frac{\partial}{\partial r}\left(\frac{G\, M(r)}{r}\right) \qquad (3.34)$$

where the last approximation is good only for a spherical mass distribution or a thin axisymmetric disk, otherwise $M(r)$ is a complicated function of r. Generally we would use a Poisson equation to couple the potential to the mass density in this case.
Let us for the position of the sun, $r_s \simeq 8.000$ pc or $2.4 \cdot 10^{22}$ cm away from the Galactic Center, quantitatively determine the gravitationally active mass assuming it acts like a point mass at the Galactic Center. For that purpose, we use the following measurements:

$$v_\phi(r_s) \simeq 200 \text{ km/s} \qquad \rho(z) = \rho_0\, \exp\left(-\frac{|z|}{z_c}\right) \qquad z_c = 50 \text{ pc} \qquad \text{T} \simeq \text{const.} \qquad (3.35)$$

Radially the gas has negligible density variations. The potential and its derivatives are

$$V = -\frac{G\, M}{\sqrt{r^2 + z^2}} \qquad \frac{\partial V}{\partial r} = \frac{G\, M\, r}{(r^2+z^2)^{3/2}} \qquad \frac{\partial V}{\partial z} = \frac{G\, M\, z}{(r^2+z^2)^{3/2}} \simeq \frac{G\, M\, z}{r^3} \quad \text{for } z \ll r \qquad (3.36)$$

because the thickness of the gas distribution is much smaller than r_s. Vertically the disk is only pressure supported. Integrate the pressure-support equation (3.33c) along positive z at r_s to obtain

$$P(z) - P(0) \simeq -\frac{G\, M\, \rho_0}{r_s^3} \int_0^z ds\ s\, \exp\left(-\frac{s}{z_c}\right)$$

$$\Rightarrow \quad P(0) \simeq \frac{G\,M\,\rho_0\,z_c^2}{r_s^3} \qquad \text{for } z_c \ll z \ll r_s \tag{3.37}$$

which is the mid-plane pressure. There is little radial density structure, so

$$\frac{\partial P}{\partial r} \lesssim \frac{\partial P}{\partial z} \quad \text{but} \qquad \frac{\partial V}{\partial r} \gg \frac{\partial V}{\partial z} \tag{3.38}$$

so the pressure term in the radial stability equation must be very much smaller than the other two terms, confirming that it may be neglected. Then equation (3.34) yields

$$\frac{v_\phi^2}{r_s} = \left.\frac{\partial V}{\partial r}\right|_{r_s} \simeq \frac{G\,M}{r_s^2} \qquad \Rightarrow \quad M \simeq \frac{r_s\,v_\phi^2}{G} \simeq 1.7\cdot 10^{44}\ \mathrm{g} \simeq 7\cdot 10^{10}\ M_\odot \tag{3.39}$$

The rotation velocity should therefore scale inversely with the square-root of the galactocentric radius, if most of the mass of the Galaxy is concentrated close to its center ($M(r) \simeq$ const.). One measures a roughly constant rotation velocity out to about twice the solar radius, hence the gravitationally active mass must increase with radius, although most of the stars and interstellar medium are located inside the solar radius. This is evidence for dark matter in the Galaxy.
We can also estimate the implied gas temperature in the solar vicinity using the ideal gas law and equation (3.37).

$$T = \frac{m_p\,P(0)}{\rho_0\,k} \simeq \frac{G\,M\,m_p\,z_c^2}{k\,r_s^3} \simeq \frac{v_\phi^2\,m_p\,z_c^2}{k\,r_s^2} \simeq 200\ \mathrm{K} \tag{3.40}$$

which is a factor of a few larger than the measured temperatures of cold atomic or molecular gas in the Galactic Plane. It turns out the vertical distribution of very cold gas is dominated by the proper motion of individual gas clouds. A temperature of 200 K corresponds to a thermal velocity of about $v_{th} \simeq \sqrt{kT/m_p} \simeq 1$ km/s, which is the same order of magnitude as the random motions of clouds.
Warmer gas will have a larger scaleheight according to (3.40). It takes about 3000 K to maintain a vertical scaleheight of 200 pc, for example. The hot ionized gas has a scaleheight around 1 kpc and a temperature near a million degree.

3.4 Fluid instabilities

An instability of any physical system is given, when the system responds to a perturbation so that it amplifies the perturbation. Conversely, stability is provided, if the backreaction of the system damps away the perturbation. There are numerous instability in gases and plasmas, only one of which I will discuss here as an example.
The convection instability is important for the energy transport in stars, but also in the atmosphere of the Earth. Suppose a gas is in hydrostatic equilibrium in a gravitational field.

Generally the pressure, and with it the temperature, will decrease as one moves up, so we expect a startification in temperature. Consider a small blob of gas immersed into such an atmosphere. We are interested in analyzing whether a small displacement of that blob would induce forces that restore the initial position (stability), or would push the blob further away (instability). In the new environment the pressure may be different, so to establish pressure equilibrium the blob reacts by expansion or compression. If the blob is small and its displacement is effected rapidly, it will not exchange any significant amount of energy with its surroundings, so it reacts adiabatically, i.e. under constant specific entropy s. Therefore the change in density is only related to the change in pressure.

$$d\rho_{\rm blob} = \left(\frac{\partial \rho}{\partial P}\right)_{s=const.} dP \tag{3.41}$$

What is the change in density of the ambient medium? If that has an adiabatic stratification, so s =const., its density at the new position is changed in exactly the same way as that of the blob (3.41). If its change in density in less than that of the blob, $d\rho_{\rm amb.} < d\rho_{\rm blob}$, then buoyancy will accelerate the motion of the blob, thus providing instability. If the density gradient of the ambient medium is larger than the adiabatic scaling, then buoyancy provides a restoring force and stability is achieved.

As we have seen, a hydrostatic atmosphere is largely determined by the pressure profile. Given that profile, a relatively shallow density profile require a steep temperature profile, because $P \propto \rho T$. The convective instability therefore arises in atmospheres with a strong gradient in temperature usually on account of heating from below. In stars this is established by the energy flux coming from the stellar cores, in which nuclear fusion processes operate. In the Earth' atmosphere this arises from heating of the surface by absorption of solar radiation.