17. Hydromagnetic shock waves

17.1 The jump conditions

A magnetized and electrically conducting plasma can carry different types of small-amplitude waves, depending on the angle between the large-scale magnetic field and their direction of propagation or on the values of the Alfvén velocity and the sound speed. We can therefore expect to also find different classes of hydromagnetic shocks.

Let us again consider a steady-state plane-parallel shock in its rest frame in the absence of an external force. Let us further assume ideal MHD, i.e. the conductivity in infinitely large. Then the MHD equations read in Einstein convention (sum over common indices)

$$\frac{\partial}{\partial x_k}(\rho\, V_k) = 0 \tag{17.1}$$

$$\frac{\partial}{\partial x_k}\left[\rho\, V_i\, V_k + P\,\delta_{ik} - \frac{1}{4\pi}\left(B_i B_k - \frac{1}{2}\,|\vec{B}^2|\,\delta_{ik}\right)\right] = 0 \tag{17.2}$$

$$\frac{\partial}{\partial x_k}\left[V_k\left(\frac{\rho}{2}\,V^2 + \frac{\gamma}{\gamma-1}\,P\right) + \frac{1}{4\pi}\,(\vec{B}\times\vec{V})\times\vec{B}\right] = 0 \tag{17.3}$$

$$\frac{\partial}{\partial x_k} B_k = 0 \tag{17.4}$$

$$\vec{\nabla}\times(\vec{B}\times\vec{V}) = 0 \tag{17.5}$$

Calculations show that it is possible to choose a coordinate system, the de Hoffmann-Teller frame, in which the velocity vector $\vec{V}$ and the magnetic field vector $\vec{B}$ on both sides of the shock lie in same plane. In the de Hoffmann-Teller frame the problem is therefore effectively two-dimensional. The corresponding vector components can be indexed with $\perp$ and $\|$, respectively, meaning perpendicular to the shock front or parallel to the shock front (in the shock plane). The integration of the MHD equation across the shock front then yields

$$\rho\, V_\perp = \mathrm{const} \tag{17.6}$$

$$\rho\, V_\perp\, V_\perp + P - \frac{1}{8\pi}\left(B_\perp^2 - B_\|^2\right) = \mathrm{const} \tag{17.7}$$

$$\rho\, V_\perp\, V_\| - \frac{1}{4\pi}\, B_\perp\, B_\| = \mathrm{const} \tag{17.8}$$

$$V_\perp\left(\frac{\rho}{2}\,(V_\perp^2 + V_\|^2) + \frac{\gamma}{\gamma-1}\,P\right) - \frac{1}{4\pi}\, B_\|\,(B_\perp\, V_\| - B_\|\, V_\perp) = \mathrm{const} \tag{17.9}$$

$$B_\perp = \mathrm{const} \tag{17.10}$$

$$B_\perp V_\parallel - B_\parallel V_\perp = \text{const} \tag{17.11}$$

The six equations 17.6 to 17.11 determine the downstream values of the six fluid variables ρ, P, $V_\perp$, $V_\parallel$, $B_\perp$, and $B_\parallel$, if their upstream values are given.
Using 17.6 and 17.10 we can rewrite equation 17.8 as

$$V_\parallel^d - V_\parallel^u = \frac{B_\perp}{4\pi\,\rho\,V_\perp}\left(B_\parallel^d - B_\parallel^u\right) \tag{17.12}$$

A discontinuity occurs in $V_\parallel$ because a current sheet exists in the shock plane on account of $\vec{\nabla} \times \vec{B} \neq 0$.

17.2 Stationary discontinuities ($\rho\, V_\perp = 0$)
Suppose the constant mass flux through the discontinuity (17.6) is actually zero.

$$\rho\, V_\perp = 0 \qquad \Rightarrow \quad V_\perp = 0 \tag{17.13}$$

Then

$$B_\perp = \text{const} \qquad B_\perp V_\parallel = \text{const} \qquad B_\perp B_\parallel = \text{const} \tag{17.14}$$

$$B_\parallel B_\perp V_\parallel = \text{const} \qquad P + \frac{B_\parallel^2}{8\pi} = \text{const} \tag{17.15}$$

Two solutions are possible.
Tangential discontinuity ($B_\perp = 0$):

$$P + \frac{B_\parallel^2}{8\pi} = \text{const} \qquad\qquad V_\parallel \text{ unconstrained} \tag{17.16}$$

Contact discontinuity ($B_\perp \neq 0$):

$$V_\parallel,\ B_\parallel,\ P = \text{const} \qquad\qquad \text{jump in } \rho,\ T \tag{17.17}$$

A boundary between two media of different density in pressure equilibrium. Fluids don't mix!

17.3 Shocks ($\rho\, V_\perp \neq 0$)
We will only discuss the two extreme cases of the magnetic field orientation. The nomenclature is a bit confusing: a parallel shock has the magnetic field parallel to the shock *normal*, so $B_\parallel = 0$ in our notation.
Parallel shocks ($B_\parallel = 0$):

$$V_\parallel,\ B_\perp = \text{const} \tag{17.18}$$

and the magnetic field drops out of the remaining equations, so the remaining jump conditions are the standard Rankine-Hugoniot conditions of purely hydrodynamical shocks.

Perpendicular shocks ($B_\perp = 0$):

$$V_\parallel = \text{const} \qquad \rho\, V_\perp = \text{const} \qquad B_\parallel\, V_\perp = \text{const} \tag{17.19}$$

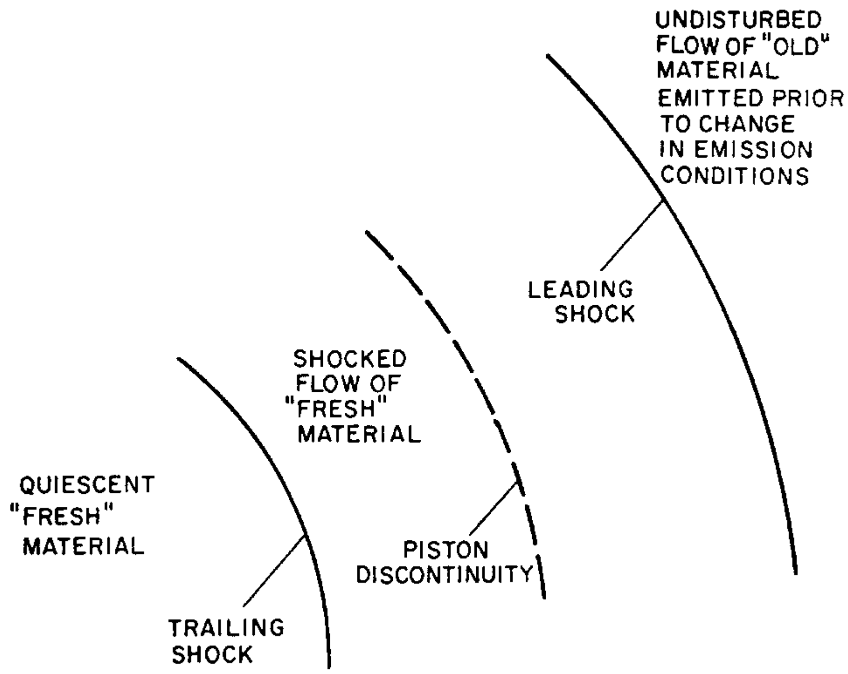
$$\rho\, V_\perp^2 + P + \frac{B_\parallel^2}{8\pi} = \text{const} = \rho \left[V_\perp^2 + \frac{c_s^2}{\gamma} + \frac{V_A^2}{2} \right] \tag{17.20}$$

$$V_\perp^2 + 2\,\frac{c_s^2}{\gamma - 1} + 2\, V_A^2 = \text{const} \tag{17.21}$$

Equation 16.20 indicates that perpendicular MHD shocks are characterized by two Mach numbers, the sonic Mach number $M_s = V_\perp / c_s$ and the Alfvénic Mach number $M_A = V_\perp / V_A$.

17.4 The real structure of blastwaves

So in reality, i.e. allowing for magnetic fields, the blastwave of, e.g. a supernova remnant, has a more complicated structure than given by the purely hydrodynamical Taylor-Sedov solution. The ejecta, i.e. stellar material that is expelled in the course of the supernova explosion, doesn't mix with the interstellar gas, so the two are separated by a contact discontinuity or tangential discontinuity. Both the interstellar gas and the ejecta experience a shock, called the forward shock for the interstellar medium and the reverse shock for the ejecta.



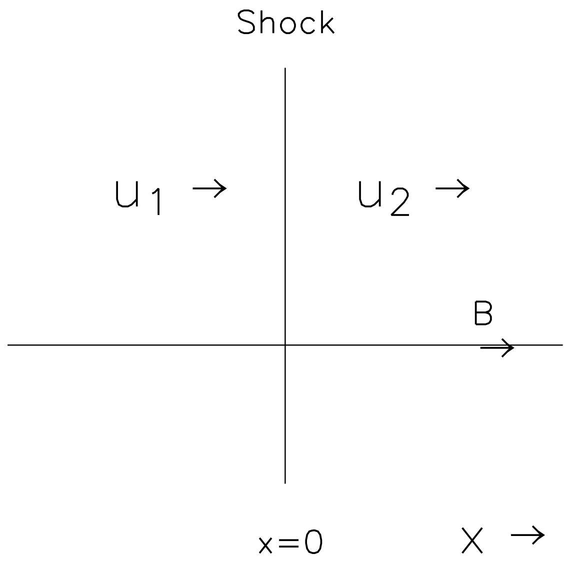
The resulting structure can be clearly seen in the distribution of line (heavy elements: ejecta) and continuum (shock heating) X-ray emission from young supernova remnants.

20. Cosmic rays and shock acceleration

The Milky Way galaxy, and other galaxies, are filled with energetic charged particles called cosmic rays. Most of the cosmic rays are hydrogen nuclei, about 10% are helium nuclei, the rest are heavier nuclei and electrons. We measure cosmic rays with energies up to about 10^{20} eV, that is 16 Joule! Obviously cosmic rays are not in Local Thermodynamical Equilibrium. They play an important role in the Galaxy, as they ionize and heat the interstellar gas. In the Galaxy the energy density of cosmic rays is comparable to that of magnetic field, radiation, and the thermal energy density of the interstellar gas. Highly relativistic particles are also found in active galactic nuclei, very luminous systems that involve supermassive black holes and relativistic outflows.

We now discuss the acceleration of charged particles. The process in question was first proposed by Enrico Fermi and is therefore dubbed Fermi acceleration of type I. Here we study its application to hydromagnetic shocks.

We already know that energetic charged particles propagates diffusively and have an isotropic distribution on account of scattering on magnetic fluctuations or MHD waves. Very often the mean free path for scattering is larger for energetic particles than it is for the bulk of the plasma. Then the energetic particles see the shock as a discontinuity, even if it is collisionless and extends over a few mean free paths of the cold background plasma. The energetic particles will then isotropize on either side of the shock in the respective rest frame of the flow. The systematic difference in flow velocity on both sides of the shock is then used to further accelerate the energetic particles.



Let us study a parallel shock in its rest frame. The density of energetic particles in region 2 (downstream) should follow a continuity equation with diffusion term

$$\frac{\partial n}{\partial t} + \frac{\partial n\, u_2}{\partial x} = \frac{\partial}{\partial x}\left(D(x)\frac{\partial n}{\partial x}\right) \tag{20.1}$$

Let us assume a steady-state, i.e. particles are injected at the shock with the same rate as they convect to the far-downstream region. Then after omitting the $\frac{\partial n}{\partial t}$ term we find

$$n(x) = A + B\exp\left(\int^x \frac{dx'\ u_2}{D(x')}\right) \qquad\qquad A, B = \text{const.} \tag{20.2}$$

If scattering is efficient all over the downstream region, then D has an upper bound and n would approach infinity for $B \neq 0$. A physical solution thus requires $B = 0$ und $n(x)$ =const. The particle convect away from the rate at a rate $u_2\, n$ that we can interpret as an escape rate. The rate for shock crossings to the upstream region is $1/4\, n\, v$ where the random particle velocity $v \gg u$. The ratio of the two rates should be the escape probability, because the particles in the upstream region should all eventually return to the downstream region.

$$\text{escape probability} \qquad \eta = 4\,\frac{u_2}{v} \tag{20.3}$$

The scatterings conserve the particle energy in the rest frame of the scattering centers. For each shock crossing we have to Lorentz-transform the particle energy to the new reference frame, though. A particle, that has already done k cycles and now again propagates from region 1 to region 2 and back, will change its energy to

$$E_{k+1} = E_k\ \left(\frac{1 + v_{k1}(u_1 - u_2)\ c^{-2}\ \cos\theta_{k1}}{1 + v_{k2}(u_1 - u_2)\ c^{-2}\ \cos\theta_{k2}}\right) \tag{20.4}$$

Obviously the particle must go through a very large number of cycles to significantly increase its energy. After $l \gg 1$ cycles the distribution of possible energies will therefore be strongly concentrated around the average energy. Let us assume that the initial energy was already relativistic $v_i \simeq c$ and write

$$\ln\left(\frac{E_l}{E_0}\right) = l\left[\langle\ln\left(1 + (u_1 - u_2)\ c^{-1}\ \cos\theta_{k1}\right)\rangle - \langle\ln\left(1 + (u_1 - u_2)\ c^{-1}\ \cos\theta_{k2}\right)\rangle\right] \tag{20.5}$$

The particle distributions will be isotropic in good approximation on account of $u_i/c \ll 1$. The rate, with which particles propagate through a unit area in the shock plane, is $2\pi\ \cos\theta\, d\cos\theta$ and the averaging over angle yields after Taylor expansion of the logarithm to third order.

$$\ln\left(\frac{E_l}{E_0}\right) = \frac{4}{3}\,l\,\frac{u_1 - u_2}{c}\left(1 + O\left(\frac{(u_1 - u_2)^2}{c^2}\right)\right) \tag{20.6}$$

The probability to go through at least l cycles, and to reach the energy E_l, is

$$\ln P_l = l\,\ln(1-\eta) = l\,\ln\left(1 - \frac{4\,u_2}{c}\right) \approx -l\,\frac{4\,u_2}{c} \approx -\frac{3\,u_2}{u_1 - u_2}\,\ln\left(\frac{E_l}{E_0}\right) \tag{20.7}$$

The differential energy spectrum is then

$$\int_E du\; N(u) = P(E) \approx \left(\frac{E}{E_0}\right)^{-\frac{3\,u_2}{u_1-u_2}} \quad\Rightarrow\quad N(E) \approx \frac{\mu-1}{E_0}\left(\frac{E}{E_0}\right)^{-\mu} \qquad \mu = \frac{2u_2+u_1}{u_1-u_2} \tag{20.8}$$

If the scattering center move only slowly relative to the plasma, we can set the velocities u_1, u_2 equal to the flow velocities and obtain

$$u_1 = v_s \quad u_2 = \frac{v_s}{\chi} \quad\Rightarrow\quad \mu = \frac{2+\chi}{\chi-1} = 2 \qquad \chi = 4 \text{ for a strong Shock} \tag{20.9}$$

where χ is the compression ratio of the shock.

So we have found process that can produce power-law spectra of relativistic particles. Its main characteristica are

– power-law spectra are obtained, if the escape probability is independent of energy.

– the particles gain energy by using the systematic velocity difference on the two sides of the shock.

– the particles thus directly tap the kinetic energy of the bulk flow.

– the energy gain per cycle is small of the order of v_s/c.

– the energetic particles must see he shock as a discontinuity, i.e. must be preaccelerated.

– we made a test-particle calculation. A high energy density in relativistic particles would modify the shock.