16. Wave-particle interaction

16.1 Landau damping

We started our discussion of hydromagnetic waves with simple one-dimensional electrostatic fluctuations, the Langmuir waves. We derived their dispersion relation and their phase velocity as a function of wavenumber. What the MHD treatment did not tell us was the relationship between the waves and the plasma. Can the waves change plasma properties or, vice versa, can plasma excite or damp waves? This question is different from the finite amplitude reasoning that we did for acoustic waves, where we noted that non-linear behaviour occurs for waves of finite amplitude account of the perturbations showing up in the dispersion relation. Here we ask for interactions between particles and waves. We therefore need to embark on an analysis in kinetic theory.

The one-dimensional Boltzmann equation for electrons reads

$$\frac{\partial f}{\partial t} + v\,\frac{\partial f}{\partial x} - \frac{e}{m}\,E\,\frac{\partial f}{\partial v} = 0 \tag{16.1}$$

Suppose there is no large-scale electric field but only perturbations E_1, and the distribution function is the sum of a steady-state term plus a small perturbation, $f = f_0 + f_1$. Then

$$\frac{\partial f_0}{\partial t} + v\,\frac{\partial f_0}{\partial x} = 0 \qquad \Rightarrow \qquad \frac{\partial f_1}{\partial t} + v\frac{\partial f_1}{\partial x} = \frac{e}{m}E_1\frac{\partial f_0}{\partial v} \tag{16.2}$$

Note that v is the velocity coordinate! With the usual wave ansatz for all perturbed quantities we obtain

$$-\imath\omega\, f_1 + v\imath k\, f_1 = \frac{e}{m}E_1\frac{\partial f_0}{\partial v} \tag{16.3}$$

and to close the system the Poisson equation

$$\frac{\partial E_1}{\partial x} = -4\pi\, e\, n_{e,1} \qquad \Rightarrow \qquad \imath k\, E_1 = -4\pi\, e\int_{-\infty}^{\infty} f_1 dv \tag{16.4}$$

Inserting f_1 from 16.3 gives

$$1 = \frac{\omega_p^2}{n_e\, k}\int \frac{\frac{\partial f_0}{\partial v}}{vk - \omega}dv \tag{16.5}$$

So if $\frac{\partial f_0}{\partial v}$ doesn't vanish around the zero of the denumerator, then ω must be complex to satisfy Eq.16.5 and the Langmuir waves are unstable.

We wish to know the sign of the imaginary part of the frequency, ω_I, for

$$E_1 \propto \exp(\imath\, k\, x - \imath\,\omega_R\, t)\,\exp(\omega_I\, t) \qquad \rightarrow \qquad \omega_I \begin{cases} > 0 & \text{wave growth} \\ < 0 & \text{wave damping} \end{cases} \tag{16.6}$$

Setting $\omega = \omega_R + \imath\,\omega_I$ we can rewrite Eq.16.5 as

$$1 = \frac{\omega_p^2}{n_e\,k} \int \frac{\frac{\partial f_0}{\partial v}(vk - \omega_R + \imath\omega_I)}{(vk - \omega_R)^2 + \omega_I^2} dv \tag{16.7}$$

This integral is complicated to solve exactly. An approximate solution can be derived for cold plasma and small wavenumbers, k, so the phase velocity of the waves is much larger than the mean velocity of the electrons $v_\phi = \omega_R/k \gg \bar{v}_e$. Then

$$\omega_R \simeq \omega_p \qquad\qquad \omega_I \simeq \frac{\pi}{2}\,\mathrm{sign}(k)\,\frac{\omega_p^3}{n_e\,k^2}\,\frac{\partial f_0}{\partial v}(v \simeq \frac{\omega_R}{k}) \tag{16.8}$$

Thus Langmuir waves are excited, if the distribution function is inverted, i.e. increases with the absolute value of the velocity. If the distribution function falls off with the absolute value of the velocity, as does a Maxwellian, Langmuir waves would be damped. This process is called Landau damping.

We can summarize

- Waves can be found by solving the MHD equations. A study of wave excitation or damping, however, requires solving the Vlasov or Boltzmann equation.
- The form of the particle distribution function determines, whether a wave is excited, damped, or stable. The excitation of Langmuir waves for example requires $\frac{\partial f}{\partial |v|} > 0$.
- In the case of a Maxwell distribution Langmuir waves undergo Landau damping.

16.2 The relaxation of inverted distribution functions

We have seen that a plasma with the appropriate distribution function can excite waves. The next question would be: can the waves influence the distribution function of the plasma? In other words, is there a backreaction of the waves on the plasma, so plasma and waves would be a truly coupled system?

A wave field is generally a superposition of many individual waves of different frequency, wavenumber, and phase. Only if a particle is in resonance with a wave, i.e. propagates with the same velocity and gyration, it will see the electromagnetic field of the wave not rapidly oscillating for a certain period of time. However, the phase of the wave is generally still random, and so is the sign of the resulting force and acceleration. Therefore the expected change of velocity averages to zero.

$$\lim_{t\to\infty} \frac{\langle \Delta\vec{v} \rangle}{t} = \lim_{t\to\infty} \frac{1}{t} \int_0^t dt'\,\frac{\vec{F}(t')}{m} = 0 \tag{16.9}$$

What is not zero is the expected rate of change of the square of the velocity, which corresponds to a diffusive change of the particles velocity. For our problem of interaction with the electrostatic

Langmuir waves we would arrive at a diffusion equation with diffusion coefficient depending on the energy density spectrum of the waves.

$$\frac{E_\parallel^2}{8\pi} = \int dk\ W(k) \tag{16.9b}$$

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial v_\parallel}\left(D\ \frac{\partial f}{\partial v_\parallel}\right) \qquad D = \lim_{t\to\infty}\frac{\langle \Delta v_\parallel^2\rangle}{t} = 8\pi\ \frac{e^2}{m^2}\int_{+\infty}^{\infty} dk\ W(k)\ \delta\left[k(v_\parallel - v_\phi)\right] \tag{16.10}$$

where the δ-functional is the limit of an infinitely sharp resonance ($\omega_I \to 0$ in Eq. 16.7). So we indeed have a coupled problem of wave excitation and diffusive change of the electron distribution function. Obviously the stable endstate of an initially inverted distribution function is the plateau distribution $\frac{\partial f}{\partial v} = 0$.

16.3 Wave-particle interactions with Alfvén waves

We have seen before that Alfvén waves with phase velocity $v_\phi \ll c$ have much larger magnetic fluctuations than electric fluctuations. Furthermore, in a frame of reference comoving with the wave (with velocity v_A) the electric fields vanishes!

$$\vec{E}_{V_A} = \vec{E} + \frac{V_A}{c}\,\vec{e}_k \times \vec{B} = 0 \tag{16.11}$$

In that frame charged particles will only be elastically scattered, the absolute value of their momentum is conserved and the so-called pitch angle $\phi_B = \angle(\vec{p},\vec{B})$ is changed. The equilibrium state of charged particles in an Alfvén wave bath obviously is an isotropic distribution in the wave frame, which is in most cases similar to the plasma frame on account of the small value of the Alfvén velocity.

Can an anisotropic distribution of particles also excite Alfvén waves, analoguous to the case of Langmuir waves? The answer is yes, the problem is similar, and even the techniques for deriving the coupled equation are similar. The actual calculation is more complicated, though, so here we embark only on a qualitative assessment.

We write the magnetic field as B_0+B_1, where the large-scale homogeneous field B_0 is responsible for the well-known helical motion with gyro or Larmor radius

$$r_g = \frac{p}{ZeB} \simeq (3\cdot 10^{14}\quad \mathrm{cm})\ \ \frac{1}{Z}\left(\frac{E}{\mathrm{TeV}}\right)\left(\frac{B_0}{10\ \mu\mathrm{G}}\right)^{-1} \tag{16.12}$$

The fluctuations B_1 will in general be composed of a spectrum of turbulence with a range of wavelengths λ. If $\lambda \ll r_g$ or $\lambda \gg r_g$, then the effect of the turbulence averaged over one gyration period is effectively zero. For a significant interaction we therefore expect $\lambda \approx r_g$. Over one gyration period the pitch angle ϕ_B will then change approximately by

$$\delta\phi \approx \frac{B_1(r_g)}{B_0} \tag{16.13}$$

The direction of change is random, so we need $N = \delta\phi^{-2}$ steps to change the pitch angle in a significant way, say by one radian. The distance traveled by the particle during that period of time is the mean free path for scattering

$$\lambda_{\rm scat} \simeq N\lambda \approx r_g\,\delta\phi^{-2} \approx r_g \left(\frac{B_0}{B_1(r_g)}\right)^2 \tag{16.14}$$

In the Alfvén wave frame the diffusion equation for the particle distribution function then depends only on the cosine of the particle's pitch angle, $\mu = \cos\phi_B$, and reads

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial\mu}[D_{\mu\mu}\frac{\partial f}{\partial\mu}] \tag{16.15}$$

where for small turbulence level the diffusion coefficient can be approximated by a perfect resonance integral

$$D_{\mu\mu} \simeq \frac{\pi\,\Omega_g^2\,(1-\mu^2)}{2\,B^2}\int_{-\infty}^{\infty} dk\ W(k)\,\delta(\omega_{\rm R} - kv\mu - \Omega_g) \tag{16.16}$$

where $W(k)$ is the energy density spectrum analogous to Eq.16.9b. The system is closed by calculating the growth rate of Alfvén waves, for which one obtains

$$\omega_I = -2\pi^2\,r_e\,V_A\,V\int dv\ v^2\int d\mu\ (1-\mu)\,\frac{\partial f}{\partial\mu}\,\delta(\omega_{\rm R} - kv\mu - \Omega_g) \tag{16.17}$$

The interaction between particles and Alfvén waves (and other electromagnetic waves) has two consequences

- An initially anisotropic distribution function is quickly isotropized.
- Energetic particle cannot freely propagate in a colder background plasma. On account of the pitch-angle diffusion the spatial propagation is also diffusive and preferentially along the large-scale magnetic field. The diffusion coefficient for the spatial propagation is

$$D = \frac{V}{3}\,\lambda_{\rm scat} \tag{16.18}$$

- Recall our discussion on the shock thickness in supernova blast waves. The rapid build-up of magnetic turbulence limits the shock thickness to values small enough that a fluid treatment is justifiable for the evolution of the supernova remnant.