14. Magnetohydrodynamics

14.1 The equations of magnetohydrodynamics

In our preceding considerations we have treated gas independent of its state of ionization, yet we know that an ionized medium can carry and be influenced by electromagnetic fields, that should follow Maxwell's equations.

$$\vec{\nabla} \cdot \vec{E} = 4\pi\, \rho_e \qquad\qquad \vec{\nabla} \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t} \tag{14.1 – 2}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \qquad\qquad \vec{\nabla} \times \vec{B} = \frac{4\pi}{c}\,\vec{j}_e + \frac{1}{c}\frac{\partial \vec{E}}{\partial t} \tag{14.3 – 4}$$

where the charge density and current density are

$$\rho_e = Z\, e\, n_i - e\, n_e \tag{14.5}$$

$$\vec{j}_e = Z\, e\, n_i\, \vec{V}_i - e\, n_e\, \vec{V}_e \tag{14.6}$$

Implicit to Maxwell's equations is the charge conservation, for

$$\frac{\partial \rho_e}{\partial t} = \frac{1}{4\pi}\,\vec{\nabla} \cdot \frac{\partial \vec{E}}{\partial t} = \frac{c}{4\pi}\,\vec{\nabla} \cdot (\vec{\nabla} \times \vec{B}) - \vec{\nabla} \cdot \vec{j}_e$$

$$\Rightarrow \qquad \frac{\partial \rho_e}{\partial t} + \vec{\nabla} \cdot \vec{j}_e = 0 \tag{14.7}$$

The effect of the electromagnetic fields is different depending on the spatial scale on which they arise. Small-scale perturbations of the fields typically lead to electromagnetic waves which we will discuss later. On large scales astrophysical plasma are usual neutral, as expressed in the quasi-neutrality condition

$$\vec{j}_e = \sigma\, \vec{E} \quad \rightarrow \frac{\partial \rho_e}{\partial t} = -\vec{\nabla} \cdot \vec{j}_e = -4\pi\, \sigma\, \rho_e \qquad \Rightarrow\ \rho_e \simeq 0 \qquad \text{on large scales} \tag{14.8}$$

This implies the absence of sources of a large-scale electric field. The conductivity of astrophysical plasmas, σ, is high and consequently any large-scale electric field would be quickly shorted out. Currents can still exist, for only the bulk velocity of electrons and ions must be different, and the Maxwell's fourth equation can be simply written for large scales

$$\vec{\nabla} \times \vec{B} = \frac{4\pi}{c}\,\vec{j}_e \simeq \frac{4\pi}{c}\, e\, n_e\, (\vec{V}_i - \vec{V}_e) \simeq -\frac{4\pi}{c}\, e\, n_e\, \vec{V}_{e,rel} \tag{14.9}$$

with the relative velocity of the electrons with respect to the ions, $\vec{V}_{e,rel} = \vec{V}_e - \vec{V}_i$. A large-scale velocity difference of the positive and negative charges, i.e. not the difference in the gyration

motion around a large-scale magnetic field, will likely be the result of a balance between an electric field and collisional resistance with collision frequency ν_C, so in the rest frame of the ions (upper index "i")

$$\vec{F} = 0 = -e\,\vec{E}^i - m_e\,\nu_C\,\vec{V}_{e,rel} \qquad \Rightarrow \qquad \vec{V}_{e,rel} = -\frac{e}{m_e\,\nu_C}\,\vec{E}^i$$

$$\vec{j}^i = -e\,n_e\,\vec{V}_{e,rel} = \frac{n_e\,e^2}{m_e\,\nu_C}\,\vec{E}^i = \sigma\,\vec{E}^i \tag{14.10}$$

Transforming back from the rest frame of the ions to the laboratory frame we note that the current density involves only a velocity difference and hence is unchanged. Then using the Lorentz transformations for the fields we obtain

$$\frac{v}{c}\,E \ll E \ll B \qquad \vec{j}_e = \vec{j}^i \qquad \vec{B} = \vec{B}^i \qquad \vec{E}^i = \vec{E} + \frac{1}{c}\,\vec{V}_i \times \vec{B} \tag{14.11}$$

which carries two messages:

- the notion of a vanishing electric field is a question of the frame of reference.
- in a general situation we expect a small large-scale electric field that is a factor V/c smaller than the magnetic field.

Equation 14.11 can be written as

$$\vec{E} = \frac{1}{\sigma}\,\vec{j}_e - \frac{1}{c}\,\vec{V}_i \times \vec{B} \simeq \frac{c}{4\pi\,\sigma}\,\vec{\nabla} \times \vec{B} - \frac{1}{c}\,\vec{V}_i \times \vec{B} \tag{14.12}$$

Inserting this expression into Maxwell's second law yields

$$\frac{\partial \vec{B}}{\partial t} = -c\,\vec{\nabla} \times \vec{E} \qquad \frac{\partial \vec{B}}{\partial t} + \vec{\nabla} \times (\vec{B} \times \vec{V}_i) = -\vec{\nabla} \times (\eta\,\vec{\nabla} \times \vec{B}) \qquad \eta = \frac{c^2}{4\pi\,\sigma} \tag{14.13}$$

The resistivity η thus leads to a diffusive spreading and decay of the magnetic fields that is normally slow, however, on account of both the high conductivity (small η) and the large spatial scales.

If the RHS of eq. 14.13 can neglected altogether, one can show that the magnetic flux through any area A that moves with the fluid is conserved.

$$\frac{\partial \vec{B}}{\partial t} + \vec{\nabla} \times (\vec{B} \times \vec{V}_i) = 0 \qquad \Rightarrow \qquad \frac{d\Phi}{dt} = 0 \qquad \Phi \equiv \int_A d\vec{A}\;\vec{B} \tag{14.14}$$

The magnetic field is said to be frozen in. In the same limit, $\eta \to 0$ or $\sigma \to \infty$, the change in the energy density of electromagnetic fields is given only by the pointing flux. If the conductivity σ is finite, we must modify the energy balance equation in Magnetohydrodynamics. The momentum conservation equation must always contain Maxwell's stress tensor.

$$T_{ik} = \frac{1}{4\pi}\left(B_i B_k - \frac{1}{2}\,|\vec{B}^2|\,\delta_{ik}\right) \tag{14.15}$$

15. Hydromagnetic waves

15.1 Electrostatic waves

Let us start with the simplest possible case of a vanishing external electric and magnetic field, and a homogeneous (ρ =const.) cold ($T = 0$) fluid at rest ($V = 0$). We are interested in longitudinal waves, so our ansatz is

$$\vec{E}_1 = E_1\,\vec{e}_k\,\exp(\imath\,\vec{k}\,\vec{x} - \imath\,\omega\,t) \qquad \Rightarrow \vec{\nabla}\times\vec{E}_1 = 0 = \frac{\partial\vec{B}}{\partial t} \qquad \Rightarrow\ \vec{B} = 0\ \forall\ t \tag{15.1}$$

and we are dealing with electrostatic waves, and the problem is essentially one-dimensional. Let us further consider frequencies so high that only the electrons can follow. The relevant equations are

$$\vec{\nabla}\cdot\vec{E} = 4\pi\,\rho_e = 4\pi\,e\,(Z\,n_i - n_e) \tag{15.2}$$

$$4\pi\,\vec{j}_e = -\frac{\partial\vec{E}}{\partial t} \simeq -4\pi\,e\,n_e\,\vec{V}_e \tag{15.3}$$

$$\frac{\partial n_e}{\partial t} = -\vec{\nabla}\cdot(n_e\,\vec{V}_e) \tag{15.4}$$

$$m_e\,n_e\left[\frac{\partial\vec{V}_e}{\partial t} + (\vec{V}_e\cdot\nabla)\vec{V}_e\right] = -e\,n_e\,\vec{E} \tag{15.5}$$

Suppose all variables are composed of the equilibrium value plus a small, wavelike perturbation according to Eq.15.1, so all derivatives turn into multiplications with $\imath\,k$ or $-\imath\,\omega$. Then retaining only terms linear in the perturbation we obtain

$$\begin{aligned} \imath\,k\,E_1 &= -\,4\pi\,e\,n_{e,1} \qquad \wedge \qquad \imath\,\omega\,E_1 = -4\pi\,e\,v_1\,n_{e,0} \\ \omega\,n_{e,1} &= n_{e,0}\,k\,v_1 \qquad \wedge \qquad n_{e,0}\,\imath\,\omega\,v_1 = e\,\frac{n_{e,0}}{m_e}\,E_1 \end{aligned} \tag{15.6}$$

This set of equations is solved by the dispersion relation

$$\omega^2 = 4\pi\,\frac{e^2\,n_{e,0}}{m_e} = \omega_{p,e}^2 \tag{15.7}$$

where we recover the electron plasma frequency $\omega_{p,e}$ that we had used earlier. The longitudinal waves that follow this dispersion relation are called electron plasma waves or Langmuir waves. The phase velocity and group velocity of Langmuir waves are

$$v_\Phi = \frac{\omega}{k} = \frac{\omega_{p,e}}{k} \qquad\qquad v_{gr} = \frac{\partial\omega}{\partial k} = 0 \tag{15.8}$$

so the wave energy does not propagate in the cold plasma limit.

15.2 Electromagnetic waves

Now we are considering transverse waves, for which the magnetic components cannot be neglected. For the moment we will still assume that an external magnetic field does not exist. From Maxwell's equations we find

$$c\,\vec{\nabla}\times\vec{E}_1 = -\frac{\partial\vec{B}_1}{\partial t} \qquad \wedge \qquad c\,\vec{\nabla}\times\vec{B}_1 = \frac{\partial\vec{E}_1}{\partial t} + 4\pi\vec{j}_1 \tag{15.9}$$

Combining these two expressions yields

$$c^2\left[\vec{\nabla}\times(\vec{\nabla}\times\vec{E}_1)\right] = -\frac{\partial^2\,\vec{E}_1}{\partial t^2} - 4\pi\frac{\partial\vec{j}_1}{\partial t} \tag{15.10}$$

which with our wave ansatz writes

$$k^2\vec{E}_1 - \vec{k}(\vec{k}\cdot\vec{E}_1) = \frac{\omega^2}{c^2}\vec{E}_1 + \imath\frac{4\pi\,\omega}{c^2}\vec{j}_1 \tag{15.11}$$

Electromagnetic waves are transverse, i.e. $\vec{E}_1 \perp \vec{k}$, so the second term on the LHS of 15.11 vanishes. Let us again consider high-frequency waves, so the currents are carried only by the electrons.

$$\vec{j}_1 = -e\,n\,\vec{v} = -e(n_0\vec{v}_1 + n_1\vec{v}_0) = -e\,n_0\,\vec{v}_1 \qquad \text{since} \quad v_0 = 0 \tag{15.12}$$

with the equation of motion

$$m\frac{\partial}{\partial t}\vec{v}_1 = -e\,\vec{E}_1 \qquad \Rightarrow \qquad \vec{v}_1 = -\imath\frac{e}{m\omega}\vec{E}_1 \tag{15.13}$$

Inserting this in 15.12 and 15.11 gives

$$\vec{E}_1\left(k^2 - \frac{\omega^2}{c^2}\right) = \imath\frac{4\pi\,\omega}{c^2}(-e\,n_0)\frac{e}{\imath\,m\,\omega}\vec{E}_1 \qquad \Rightarrow \qquad \omega^2 = k^2c^2 + \omega_{p,e}^2 \tag{15.14}$$

Thus the dispersion relation for electromagnetic waves in a plasma differs at low frequencies from that for the vacuum case. At frequencies lower than the electron plasma frequencies electromagnetic waves cannot propagate.

At frequencies higher than the plasma frequency the phase velocity is always larger and the group velocity is always smaller than the speed of light, c.

$$v_\Phi = \frac{\omega}{k} = \sqrt{c^2 + \frac{\omega_p^2}{k^2}} > 1 \qquad\qquad v_g = \frac{\partial\omega}{\partial k} = \frac{k\,c^2}{\omega} = \frac{c^2}{v_{ph}} \tag{15.15}$$

This has a number of consequences:

• The low-frequency radio emission from the sun comes from a spherical shell in the corona, where the propagation of electromagnetic waves just starts to become possible. The apparent size of the sun in radio observations therefore depends on the observing frequency!
• The earth ionosphere is impenetrable for radio waves with a frequency below ~ 1 MHz. Radio waves that enter the ionosphere at a shallow angle will be diffracted back, i.e. be reflected. This explains the extremely high range of the long-wavelength radio communications.
• Plasma structures in the interstellar medium of galaxies act like dispersing lenses. As static structures they delay the radio pulses from pulsars. If the plasma structures undergo turbulent motion perpendicular to the line-of-sight, the arrival times of the radio pulses from pulsars will fluctuate and scintillation will be observed in compact radio sources such as active galactic nuclei.

15.3 Electromagnetic waves in a magnetized medium

What would change in the dispersion relation, if the plasma carried a homogeneous magnetic field? We consider the case $\vec{k} \parallel \vec{B}_0$ with the coordinates

$$\vec{E}_1 = E_x \vec{e}_x + E_y \vec{e}_y \qquad \vec{k} = k\vec{e}_z \qquad \vec{B}_0 = B_0 \vec{e}_z$$

The equation of motion for the electrons (15.13) is now

$$-\imath\, m\omega \vec{v}_1 = -e\vec{E}_1 - \frac{e}{c}\vec{v}_1 \times \vec{B}_0$$

$$\Rightarrow \quad v_x = -\frac{\imath\, e}{m\,\omega}\,(E_x + \frac{v_y}{c}B_0) \qquad \wedge \qquad v_y = -\frac{\imath\, e}{m\,\omega}\,(E_y - \frac{v_x}{c}B_0)$$

$$\Rightarrow \quad v_x = -\frac{e}{m\,\omega}\,\frac{\imath E_x + \frac{\omega_g}{\omega}E_y}{1 - \frac{\omega_g^2}{\omega^2}} \qquad \wedge \qquad v_y = -\frac{e}{m\,\omega}\,\frac{\imath E_y - \frac{\omega_g}{\omega}E_x}{1 - \frac{\omega_g^2}{\omega^2}} \tag{15.16}$$

with the electron gyrofrequency

$$\omega_g = \frac{eB_0}{mc} \tag{15.17}$$

Inserting this into the expression for the current density (15.11 and 15.12) yields

$$(k^2c^2 - \omega^2)\; E_x = \imath\, 4\pi\, \omega\, j_x = \frac{\omega_p^2}{1 - \frac{\omega_g^2}{\omega^2}}\left(\imath\frac{\omega_g}{\omega}E_y - E_x\right)$$

$$(k^2c^2 - \omega^2)\; E_y = -\frac{\omega_p^2}{1 - \frac{\omega_g^2}{\omega^2}}\left(\imath\frac{\omega_g}{\omega}E_x + E_y\right) \tag{15.18}$$

This set of equations is solved by two dispersion relations

$$\omega^2 - k^2c^2 = \frac{\omega_p^2}{1 \pm \frac{\omega_g}{\omega}} \tag{15.19}$$

Inserting both solutions into 15.18 gives us the phase relation between E_x and E_y, which indicate that the two dispersion relations correspond to left and right circular polarized waves.

$$\omega^2 - k^2c^2 = \frac{\omega_p^2}{1-\frac{\omega_g}{\omega}} \qquad \Rightarrow \qquad E_y = \imath\, E_x \qquad \text{right}-\text{handed} \tag{15.20}$$

$$\omega^2 - k^2c^2 = \frac{\omega_p^2}{1+\frac{\omega_g}{\omega}} \qquad \Rightarrow \qquad E_y = -\imath\, E_x \qquad \text{left}-\text{handed} \tag{15.21}$$

The right-handed wave propagates apparently a tiny bit faster than the left-handed wave. After passing a length L a phase difference will build up.

$$\Phi = k\,L = \frac{\omega}{v_{ph}}\,L \qquad\qquad \delta\Phi = \omega\,L\left(\frac{1}{v_{ph_l}} - \frac{1}{v_{ph_r}}\right) \tag{15.22}$$

We can imagine a linearly polarized wave as being composed of two oppositely handed circularly polarized waves. The plane of polarization of the linearly polarized wave would rotate by $\delta\Theta = \delta\Phi/2$, exactly half of the phase difference (15.22). After a Taylor expansion of the dispersion relation for $\omega_g, \omega_p \ll \omega$ we obtain

$$\delta\Theta \simeq \frac{L\,\omega_p^2\,\omega_g}{2\,c\,\omega^2} \qquad \Rightarrow \qquad \Theta \propto \int dz\; n_e\, B_0 \tag{15.23}$$

This is the famous Faraday rotation, which is used as a diagnostic of interstellar plasma, but is a burden in polarization studies of the magnetic field structures in sources.
The correction term in the dispersion relation is of first order in ω_g/ω. In the case $\vec{k} \perp \vec{B}_0$ the corrections are of second order and can thus generally be neglected.

15.4 Alfvén waves

We are still considering waves with $\vec{k} \parallel \vec{B}_0$ in a homogeneous magnetic field B_0. In contrast to the preceding case we now intend to study low frequencies below the proton gyrofrequency $\Omega_g = eB_0/m_pc$. Hence the motion of the protons can no longer be neglected, when calculating the current density. The proton velocity should be

$$V_x = \frac{e}{m_p\,\omega}\,\frac{\imath E_x - \frac{\Omega_g}{\omega}E_y}{1-\frac{\Omega_g^2}{\omega^2}} \qquad \wedge \qquad V_y = \frac{e}{m_p\,\omega}\,\frac{\imath E_y + \frac{\Omega_g}{\omega}E_x}{1-\frac{\Omega_g^2}{\omega^2}} \tag{15.24}$$

The current density is carried by the protons ($\vec{V}$) and the electrons ($\vec{v}$).

$$\vec{j}_1 = e\,n_0\,(\vec{V} - \vec{v}) \tag{15.25}$$

Now the condition $\omega \ll \Omega_g \ll \omega_g$ implies that the velocities are proportional to the particle mass and thus the proton or ion velocity is much higher than the electron velocity and that the problem effectively becomes one-dimensional.

$$\vec{j}_1 \simeq -\imath \frac{\omega\, \Omega_p^2}{4\pi\, \Omega_g^2}\, \vec{E}_1 \left(1 + \frac{\Omega_g}{\omega_g}\right) \tag{15.26}$$

where $\Omega_p = 4\pi e^2 n_0/m_p$ denotes the proton plasma frequency.
So we can neglect the electrons in this case, and obtain after inserting 15.26 into 15.11

$$k^2c^2 - \omega^2 \simeq \frac{\Omega_p^2 \omega^2}{\Omega_g^2} \tag{15.27}$$

and from there the dispersion relation

$$\omega = kc \frac{1}{\sqrt{1 + \frac{c^2}{v_A^2}}} \simeq k\, v_A \quad \text{wenn } v_A \ll c \qquad\qquad v_A = \frac{B_0}{\sqrt{4\pi\, \rho_0}} = c\frac{\Omega_g}{\Omega_p} \tag{15.28}$$

The quantity v_A is named Alfvén velocity after Hannes Alfvén who is one of the founding fathers of plasma astrophysics.
In systems with finite temperature and for arbitrary direction of wave propagation the dispersion relation also contains the sound speed. One then distinguishes between Alfvén waves, slow-mode waves and fast-mode waves.
If we take a peek at Maxwell's equations again, we note for electromagnetic waves

$$c\, \vec{\nabla} \times \vec{E}_1 = -\dot{\vec{B}}_1 \;\Rightarrow\; c\, k\, E_1 = -\imath\, \omega\, B_1 \;\Rightarrow\; B_1^2 = \frac{c^2}{v_\phi^2}\, E_1^2 = n\, E_1^2 \tag{15.29}$$

For Alfvén waves we usually have $v_\phi \ll c$, and therefore the magnetic fluctuations are much stronger than the electric fluctuations. The opposite is true for Langmuir waves, for which the electric fluctuations dominate. This has consequences for the interaction of these waves with energetic particles.