10. Accretion disks

10.1 Viscosity

Experience shows that we have overlooked some interactions, when we wrote down the force terms in the hydrodynamic equations. What is missing is the ability of gases and fluids to sustain viscous stress. In homework we saw that for an isotropic distribution function, e.g. as established in LTE, the kinetic pressure tensor collapses to a scalar isotropic pressure. However, we must add stress terms on account of viscosity.

$$\pi_{ij} = P\,\delta_{ij} - P_{ij} \tag{10.1}$$

with

$$P_{ij} = +\mu\, D_{ij} + \beta\,(\vec{\nabla}\cdot\vec{V})\delta_{ij} \qquad \nu = \frac{\mu}{\rho} \qquad D_{ij} = \frac{\partial V_i}{\partial x_j} + \frac{\partial V_j}{\partial x_i} - \frac{2}{3}\,(\vec{\nabla}\cdot\vec{V})\delta_{ij} \tag{10.2}$$

where μ and β are respectively called the shear and bulk coefficients of viscosity. The kinematic viscosity ν is often constant or only dependent on the temperature. The shear effect corresponds to a randomization of bulk velocity gradients, i.e. the transfer of relative bulk kinetic energy to thermal energy. Viscosity plays an important role in contractive flows around central masses, such as the accretion of gas by galaxies, galaxy clusters, stars or black holes.

Normally any mass point, that is falling toward a massive central object, will carry a non-vanishing angular momentum, $\vec{l} = m\,\vec{r}\times\vec{V}$, with respect to the central mass. The centrifugal force then scales as

$$\vec{F} \propto \frac{|\vec{l}|^2}{m\,r^3} \qquad V_\phi \propto \frac{|\vec{l}|}{m\,r} \tag{10.3}$$

Two conclusions can be drawn:

• The bulk velocity of the material will soon be very high, so the material could be radially supported by centrifugal forces. Support in other directions must be effected by pressure, which would be weaker if $v_{th} < v_\phi$. This explains why many structures have the geometry of a disk.

• The material must loose angular momentum or it can not reach the central object.

10.2 Angular momentum transport

Let us study a cold accretion flow around a gravitationally dominant central object in cylindrical coordinate (r, ϕ, z), for it will have the shape of a thin disk. As useful quantity is the surface mass density, σ, that we can obtain by integration of the volume mass density. For an axisymmetric system

$$\sigma(r,t) = \int_{-\infty}^{\infty} dz\ \rho(r,z,t) \tag{10.4}$$

In the thin disk the velocity will depend, if at all, only weakly on z. We can therefore integrate the fluid equations over z and only use the mid-plane value of the velocity $\vec{V} = \vec{V}(z = 0)$. Neglecting pressure, the radial force equation (momentum conservation equation) then yields the classical centrifugal balance

$$\frac{V_\phi^2}{r} = r\,\Omega^2 = r\,\frac{4\,\pi^2}{T^2} = \frac{G\,M}{r^2} \tag{10.5}$$

where we recover Kepler's third law under the limit

$$2\pi \int_{r_{\rm min}}^{r_{\rm max}} dr\ r\,\sigma(r,t) \ll M \tag{10.6}$$

Such a Keplerian disk possesses a shear

$$r\,\frac{d\Omega}{dr} = -\frac{3}{2}\,\Omega \tag{10.6}$$

and consequently a shear stress

$$-\pi_{r\phi} = P_{r\phi} = \mu\,r\,\frac{d\Omega}{dr} \tag{10.7}$$

A rigidly rotating disk would thus not generate any stress. In arbitrarily rotating disk we find for the stress-related force density in ϕ-direction

$$F_\phi = -\frac{\partial P_{r\phi}}{\partial r} = -\mu\,\frac{\partial\left(r\,\frac{d\Omega}{dr}\right)}{\partial r} \overset{\rm Kepler}{=} -\,\frac{9}{4}\,\mu\,\frac{\Omega}{r} \tag{10.8}$$

For $\mu > 0$ a differentially rotating disk with $d\Omega/dr < 0$ would generate a force in negative ϕ-direction, i.e. decelerating the rotation of the flow. Since the total angular momentum must be conserved, there must be an inward transport of mass and/or an outward transport of angular momentum.

Let us denote the angular momentum per unit mass of the fluid as j

$$j = r^2\,\Omega \tag{10.9}$$

Then the ϕ-component of the momentum conservation equation can be written as

$$\rho\,\left[\frac{\partial j}{\partial t} + V_r\,\frac{\partial j}{\partial r} + V_z\,\frac{\partial j}{\partial z}\right] = \frac{1}{r}\,\frac{\partial}{\partial r}\,(r^2\,P_{r\phi}) \tag{10.10}$$

Our assumption of a thin disk around a gravitationally dominating central object imply that j has only minimal variation with z. Let us further assume a steady-state situation. Then integrating Eq.10.10 over z yields

$$V_r\,r\,\frac{\partial j}{\partial r}\,\int_{-\infty}^{\infty} dz\ \rho = \frac{\partial}{\partial r}\,\left(r\,\int_{-\infty}^{\infty} dz\ r\,P_{r\phi}\right) \tag{10.11}$$

which can be written as

$$\dot{M}_r \frac{\partial j}{\partial r} = -\frac{\partial T_\phi}{\partial r} \tag{10.12}$$

using

$$\dot{M}_r = -2\pi\, r\, V_r\, \sigma(r) \qquad \text{mass} - \text{accretion rate at radius } r \tag{10.13}$$

$$T_\phi = 2\pi\, r \int_{-\infty}^{\infty} dz\ r\, P_{r\phi} \qquad \text{viscous torque at } r \tag{10.14a}$$

Using the (hopefully constant over z) kinematic viscosity $\nu = \mu/\rho$, Eq.10.14a reduces to

$$T_\phi = 2\pi\, r \int_{-\infty}^{\infty} dz\ r^2\, \nu\, \rho\, \frac{d\Omega}{dr} = 2\pi\, \sigma\, \nu\, r^3\, \frac{d\Omega}{dr} \tag{10.14b}$$

The surface density must obey a continuity equation

$$\frac{\partial \sigma}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}\,(r\, V_r\, \sigma) = \frac{\partial \sigma}{\partial t} - \frac{1}{2\pi\, r}\,\frac{\partial \dot{M}_r}{\partial r} = 0 \tag{10.15}$$

The equations 10.12, 10.14, and 10.15 constitute a closed set to solve for the accretion rate, the torque, and the surface density.

10.3 Anomalous viscosity

We can eliminate $\dot{M}_r$ and T_ϕ using Eqs. 10.12 and 10.14 to obtain for Eq.10.15

$$\frac{\partial \sigma}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}\left[\left(\frac{\partial j}{\partial r}\right)^{-1} \frac{\partial}{\partial r}\left(\sigma\, \nu\, r^3\, \frac{d\Omega}{dr}\right)\right] = 0 = \frac{\partial \sigma}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}\left[r\left(V\,\sigma - D\,\frac{\partial \sigma}{\partial r}\right)\right] \tag{10.16}$$

which is a diffusion-convection equation with

$$V = \frac{1}{r}\left(\frac{\partial j}{\partial r}\right)^{-1} \frac{\partial}{\partial r}\left(\nu\, r^3\, \frac{d\Omega}{dr}\right) \tag{10.17}$$

$$D = -\nu\, r^2\, \frac{d\Omega}{dr}\left(\frac{\partial j}{\partial r}\right)^{-1} \tag{10.18}$$

How fast are these processes? Let us as an example assume a Keplerian disk with $\Omega \propto r^{-3/2}$ and a constant kinematic viscosity ν. Then

$$D = 3\,\nu \quad \Rightarrow\ \tau_{\rm diff} = \frac{r^2}{D} \approx \frac{r^2}{\nu} \qquad\qquad V = -\frac{3}{2}\,\frac{\nu}{r} \quad \Rightarrow\ \tau_{\rm conv} = \frac{r}{|V|} \approx \frac{r^2}{\nu} \tag{10.19}$$

so both processes operate with approximately the same efficacy. Whatever the underlying process of the viscosity is, as an order of magnitude it scales with the thermal velocity of particles and the mean free path of the interaction process, $\nu \sim v_{\rm th}\, l$, with $l_x = 1/(n\,\sigma_x)$.

At high densities two-body (Coulomb) collisions are dominant and hence their cross section should determine the viscosity. A comparison with observation shows that in real accretion disks around protostars or compact objects such as neutron stars or black holes the accretion is many orders of magnitude more efficient than what one calculates for two-body collisions. In other words, massive black holes could not have grown within the age of the universe, if was not anomalous diffusion that is much more efficient than ordinary two-body collisions. It is probable that magnetic fields and small-scale turbulence play an important role in providing anomalous viscosity.

10.4 Steady-state situations

In a steady-state $\dot{M}_r$ must be independent of r or the surface density σ would be explicitely time-dependent.

$$\frac{\partial \dot{M}_r}{\partial r} = 0 \tag{10.20}$$

We can then integrate Eq.10.12 to yield

$$\dot{M}_r \frac{\partial j}{\partial r} = -\frac{\partial T_\phi}{\partial r} \qquad \Rightarrow \quad T_\phi = -\dot{M}_r \,(j - j_0) \tag{10.21}$$

where j_0 is an integration constant, which is determined by the angular momentum at the radius, where the torque vanishes.

$$j_0 = j(r_c) = r_c^2\, \Omega(r_c) \qquad T_\phi(r_c) \propto \frac{d\Omega}{dr}(r_c) = 0 \tag{10.22}$$

This corresponds to the inner edge of the accretion disk, where the rotation turns over from its Keplerian behaviour to rigid rotation or even slower flows expected near the central objects.

10.5 Radiation and spectral properties

Viscosity leads to additional terms in the energy conservation equation, that correspond to heating, i.e. the transfer of bulk kinetic energy to thermal energy. The heating rate per volume is

$$\Psi = P_{r\phi}\, r\, \frac{d\Omega}{dr} \tag{10.23}$$

The total heating rate at a given radius is then

$$\int_{-\infty}^{\infty} dz\ \Psi = r\,\frac{d\Omega}{dr} \int_{-\infty}^{\infty} dz\ P_{r\phi} = \frac{1}{2\pi\, r}\, T_\phi\, \frac{d\Omega}{dr} \tag{10.24}$$

In a steady state that heat must be transported away at the same rate, which for thin accretion disks requires radiation. Let us assume the upper and lower surfaces radiate in thermodynamic equilibrium, so the Stefan-Boltzmann law applies.

$$2\,b_{\rm SB}\,T_{eff}^4 = \frac{1}{2\pi\,r}\,T_\phi\,\frac{d\Omega}{dr} = -\frac{1}{2\pi\,r}\,\dot M_r\,\left(r^2\,\Omega(r) - r_c^2\,\Omega(r_c)\right)\,\frac{d\Omega}{dr} = \frac{3\,\dot M_r\,G\,M}{4\pi\,r^{7/2}}\,\left(\sqrt{r}-\sqrt{r_c}\right) \tag{10.25}$$

assuming a Keplerian disk. Far from the inner boundary we can use $r \gg r_c$

$$T_{eff} \simeq \left(\frac{3\,\dot M_r\,G\,M}{8\pi\,b_{\rm SB}}\right)^{1/4}\,r^{-3/4} = T_0\,r^{-3/4} \tag{10.26}$$

so the effective temperature does not follow the naively expected linear scaling with the virialized kinetic energy.

To derive the total emission spectrum, we have to integrate the Planckian with radius-dependent temperature over the total surface, that is starting at some minimum radius $r_{min} \gtrsim r_c$ that allows us to use approximation (10.26). For an observer who sees the accretion disk at an aspect angle θ the apparent surface area is reduced by the factor $\cos\theta$, so

$$L_\nu = 2\pi\,\cos\theta\int_{r_{min}}^\infty dr\;r\,B_\nu(T) = \frac{4\pi\,\cos\theta\,h\,\nu^3}{c^2}\int_{r_{min}}^\infty dr\;r\,\frac{1}{\exp\left(\frac{\nu\,r^{3/4}}{\nu_0}\right)-1} \qquad \nu_0 = \frac{k\,T_0}{h} \tag{10.27}$$

We can transform the integration variable as

$$x = \frac{\nu\,r^{3/4}}{\nu_0} \qquad r = \left(\frac{\nu_0}{\nu}\right)^{4/3}\,x^{4/3} \qquad dr = \frac{4}{3}\,\left(\frac{\nu_0}{\nu}\right)^{4/3}\,x^{1/3}\,dx$$

so the integral 10.27 writes

$$\begin{aligned} L_\nu &= \frac{16\pi\,\cos\theta\,k^{8/3}\,T_0^{8/3}}{3\,c^2\,h^{5/3}}\,\nu^{1/3}\int_{\frac{\nu}{\nu_0}\,r_{min}^{3/4}}^\infty dx\;x^{5/3}\,\frac{1}{\exp(x)-1} \\ &\simeq 1.93\,\frac{16\pi\,\cos\theta\,k^{8/3}\,T_0^{8/3}}{3\,c^2\,h^{5/3}}\,\nu^{1/3} \qquad\qquad \text{for } \nu \ll \nu_0\,r_{min}^{-3/4} = \frac{k\,T_{\rm max}}{h} \end{aligned} \tag{10.28}$$

The observer should thus see a power-law spectrum up to a frequency corresponding to the temperature of the inner edge of the disk.

Let us consider three example: a protostar, i.e a forming star that is still accreting matter, a neutron star, and a supermassive black hole. We will set $r_{\rm min} = 3\,R_*$ and $\cos\theta = 1$.

star	mass $[M_\odot]$	radius [km]	$\dot M$ $[M_\odot/\mathrm{yr}]$	$\nu_{\rm max}$ [Hz]	$\nu\,L_\nu(\nu_{\rm max})$ [erg/sec]
protostar	1	10^6	10^{-7}	10^{14}	10^{33}
neutron star	3	10	10^{-8}	$6\cdot10^{17}$	$8\cdot10^{37}$
supermassive black hole	10^8	$3\cdot10^8$	1	$2\cdot10^{16}$	10^{46}

Neutron stars (and stellar-mass black holes) are therefore most likely found in the X-ray band. In most cases the mass accretion rate, e.g. by overflow from a companion star, is a bit lower than assumed here. The accretion disks around supermassive black holes most likely radiate in the UV, but can be very luminous, thus outshining an entire galaxy. For comparison, the total luminosity of the Milky Way galaxy is around 10^{46} erg/sec peaking in the near-infrared.