17. Hydromagnetic shock waves

Reading: Shu, Vol.II, Ch. 25

17.1 The jump conditions

A magnetized and electrically conducting plasma can carry different types of small-amplitude waves, depending on the angle between the large-scale magnetic field and their direction of propagation or on the values of the Alfvén velocity and the sound speed. We can therefore expect to also find different classes of hydromagnetic shocks.

Let us again consider a steady-state plane-parallel shock in its rest frame in the absence of an external force. Let us further assume ideal MHD, i.e. the conductivity in infinitely large. Then the MHD equations read in Einstein convention (sum over common indices)

$$\frac{\partial}{\partial x_k}(\rho\, V_k) = 0 \tag{17.1}$$

$$\frac{\partial}{\partial x_k}\left[\rho\, V_i\, V_k + P\,\delta_{ik} - \frac{1}{4\pi}\left(B_i B_k - \frac{1}{2}\,|\vec{B}^2|\,\delta_{ik}\right)\right] = 0 \tag{17.2}$$

$$\frac{\partial}{\partial x_k}\left[V_k\,\left(\frac{\rho}{2}\,V^2 + \frac{\gamma}{\gamma-1}\,P\right) + \frac{1}{4\pi}\,(\vec{B}\times\vec{V})\times\vec{B}\right] = 0 \tag{17.3}$$

$$\frac{\partial}{\partial x_k} B_k = 0 \tag{17.4}$$

$$\vec{\nabla}\times(\vec{B}\times\vec{V}) = 0 \tag{17.5}$$

Calculations show that it is possible to choose a coordinate system, the de Hoffmann-Teller frame, in which the velocity vector $\vec{V}$ and the magnetic field vector $\vec{B}$ on both sides of the shock lie in same plane. In the de Hoffmann-Teller frame the problem is therefore effectively two-dimensional. The corresponding vector components can be indexed with $\perp$ and $\|$, respectively, meaning perpendicular to the shock front or parallel to the shock front (in the shock plane). The integration of the MHD equation across the shock front then yields

$$\rho\, V_\perp = \text{const} \tag{17.6}$$

$$\rho\, V_\perp\, V_\perp + P - \frac{1}{8\pi}\,\left(B_\perp^2 - B_\|^2\right) = \text{const} \tag{17.7}$$

$$\rho\, V_\perp\, V_\| - \frac{1}{4\pi}\, B_\perp\, B_\| = \text{const} \tag{17.8}$$

$$V_\perp\,\left(\frac{\rho}{2}\,(V_\perp^2 + V_\|^2) + \frac{\gamma}{\gamma-1}\,P\right) - \frac{1}{4\pi}\, B_\|\,(B_\perp\, V_\| - B_\|\, V_\perp) = \text{const} \tag{17.9}$$

$$B_\perp = \text{const} \tag{17.10}$$

$$B_\perp V_\parallel - B_\parallel V_\perp = \text{const} \tag{17.11}$$

The six equations 17.6 to 17.11 determine the downstream values of the six fluid variables ρ, P, $V_\perp$, $V_\parallel$, $B_\perp$, and $B_\parallel$, if their upstream values are given.
Using 17.6 and 17.10 we can rewrite equation 17.8 as

$$V_\parallel^d - V_\parallel^u = \frac{B_\perp}{4\pi\,\rho\,V_\perp}\left(B_\parallel^d - B_\parallel^u\right) \tag{17.12}$$

A discontinuity occurs in $V_\parallel$ because a current sheet exists in the shock plane on account of $\vec{\nabla} \times \vec{B} \neq 0$.

17.2 Stationary discontinuities ($\rho\,V_\perp = 0$)

Suppose the constant mass flux through the discontinuity (17.6) is actually zero.

$$\rho\,V_\perp = 0 \qquad \Rightarrow \quad V_\perp = 0 \tag{17.13}$$

Then

$$B_\perp = \text{const} \qquad B_\perp V_\parallel = \text{const} \qquad B_\perp B_\parallel = \text{const} \tag{17.14}$$

$$B_\parallel B_\perp V_\parallel = \text{const} \qquad P + \frac{B_\parallel^2}{8\pi} = \text{const} \tag{17.15}$$

Two solutions are possible.
Tangential discontinuity ($B_\perp = 0$)**:**

$$P + \frac{B_\parallel^2}{8\pi} = \text{const} \qquad\qquad V_\parallel \text{ unconstrained} \tag{17.16}$$

Contact discontinuity ($B_\perp \neq 0$)**:**

$$V_\parallel,\ B_\parallel,\ P = \text{const} \qquad\qquad \text{jump in } \rho,\ T \tag{17.17}$$

A boundary between two media of different density in pressure equilibrium. Fluids don't mix!

17.3 Shocks ($\rho\,V_\perp \neq 0$)

We will only discuss the two extreme cases of the magnetic field orientation. The nomenclature is a bit confusing: a parallel shock has the magnetic field parallel to the shock *normal*, so $B_\parallel = 0$ in our notation.
Parallel shocks ($B_\parallel = 0$)**:**

$$V_\parallel,\ B_\perp = \text{const} \tag{17.18}$$

and the magnetic field drops out of the remaining equations, so the remaining jump conditions are the standard Rankine-Hugoniot conditions of purely hydrodynamical shocks.

Perpendicular shocks ($B_\perp = 0$):

$$V_\parallel = \text{const} \qquad \rho\, V_\perp = \text{const} \qquad B_\parallel\, V_\perp = \text{const} \tag{17.19}$$

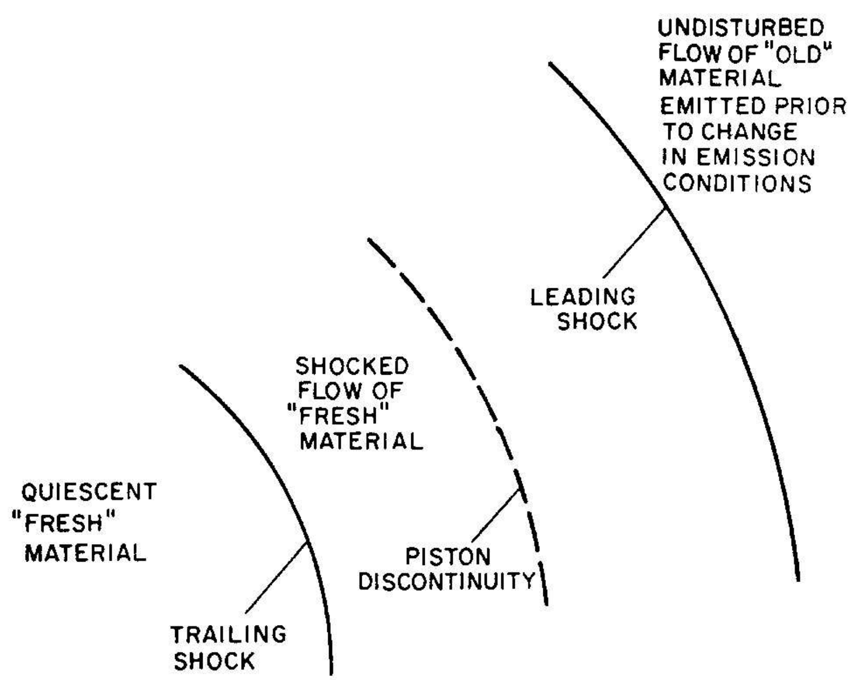
$$\rho\, V_\perp^2 + P + \frac{B_\parallel^2}{8\pi} = \text{const} = \rho \left[V_\perp^2 + \frac{c_s^2}{\gamma} + \frac{V_A^2}{2} \right] \tag{17.20}$$

$$V_\perp^2 + 2\,\frac{c_s^2}{\gamma - 1} + 2\, V_A^2 = \text{const} \tag{17.21}$$

Equation 16.20 indicates that perpendicular MHD shocks are characterized by two Mach numbers, the sonic Mach number $M_s = V_\perp / c_s$ and the Alfvénic Mach number $M_A = V_\perp / V_A$.

17.4 The real structure of blastwaves

So in reality, i.e. allowing for magnetic fields, the blastwave of, e.g. a supernova remnant, has a more complicated structure than given by the purely hydrodynamical Taylor-Sedov solution. The ejecta, i.e. stellar material that is expelled in the course of the supernova explosion, doesn't mix with the interstellar gas, so the two are separated by a contact discontinuity or tangential discontinuity. Both the interstellar gas and the ejecta experience a shock, called the forward shock for the interstellar medium and the reverse shock for the ejecta.



The resulting structure can be clearly seen in the distribution of line (heavy elements: ejecta) and continuum (shock heating) X-ray emission from young supernova remnants.