

$O(\alpha_s^3)$ Contributions to the Heavy Flavor DIS Wilson Coefficients at general Values of N

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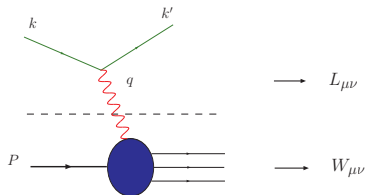


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Introduction



- kinematic quantities: $Q^2 := -q^2$, $x := \frac{Q^2}{2pq}$, $\nu := \frac{Pq}{M}$
- differential cross-section: $\frac{d\sigma}{dQ^2 dx} \sim W_{\mu\nu} L^{\mu\nu}$

$$W_{\mu\nu}(q, P, s) = \frac{1}{4\pi} \int d^4\xi \exp(iq\xi) \langle P, s | [J_\mu^{em}(\xi), J_\nu^{em}(0)] | P, s \rangle$$

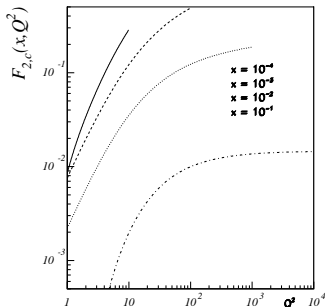
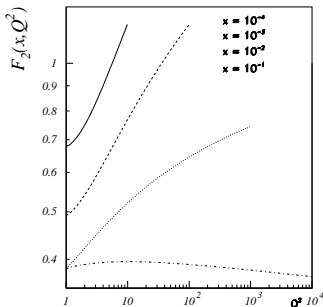
$$\text{unpol.} \left\{ \begin{aligned} &= \frac{1}{2x} \left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right) F_L(x, Q^2) + \frac{2x}{Q^2} \left(P_\mu P_\nu + \frac{q_\mu P_\nu + q_\nu P_\mu}{2x} - \frac{Q^2}{4x^2} g_{\mu\nu} \right) F_2(x, Q^2) \end{aligned} \right.$$

$$\text{pol.} \left\{ \begin{aligned} &- \frac{M}{2Pq} \varepsilon_{\mu\nu\alpha\beta} q^\alpha \left[s^\beta g_1(x, Q^2) + \left(s^\beta - \frac{sq}{Pq} p^\beta \right) g_2(x, Q^2) \right] . \end{aligned} \right.$$

Structure Functions: $F_{2,L}$
contain light and heavy quark contributions



Heavy flavor contributions to F_2



LO charm contributions: PDFs from [\[Alekhin, Melnikov, Petriello, 2006.\]](#)

→ different scaling violations,

→ massive contributions at lower values of x are of order 20%-35%.

Hence for the prediction of cross sections at the LHC the precise knowledge of all

PDFs and the exact value of $\alpha_s(M_Z^2)$ is needed. $\alpha_s(M_Z^2) = 0.1135 \pm 0.0014$



Representation for F_2 at $Q^2 > 10m^2$

- in the asymptotic region F_L is known for general values of N to NNLO

[Blümlein, De Freitas, van Neerven, Klein, 2006.]

- F_2 for N_F massless and one heavy quark flavor:

[Bierenbaum, Blümlein, Klein, 2009.]

$$\begin{aligned}
 F_{(2,L)}^{Q\bar{Q}}(x, N_F + 1, Q^2, m^2) &= \sum_{k=1}^{N_F} e_k^2 \left\{ L_{q,(2,L)}^{NS} \left(x, N_F + 1, \frac{Q^2}{m^2}, \frac{m^2}{\mu^2} \right) \otimes [f_k(x, \mu^2, N_F) + \bar{f}_k(x, \mu^2, N_F)] \right. \\
 &+ \frac{1}{N_F} \left[L_{q,(2,L)}^{PS} \left(x, N_F + 1, \frac{Q^2}{m^2}, \frac{m^2}{\mu^2} \right) \otimes \Sigma(x, \mu^2, N_F) + L_{g,(2,L)}^S \left(x, N_F + 1, \frac{Q^2}{m^2}, \frac{m^2}{\mu^2} \right) \otimes G(x, \mu^2, N_F) \right] \Big\} \\
 &+ e_Q^2 \left[H_{q,(2,L)}^{PS} \left(x, N_F + 1, \frac{Q^2}{m^2}, \frac{m^2}{\mu^2} \right) \otimes \Sigma(x, \mu^2, N_F) + H_{g,(2,L)}^S \left(x, N_F + 1, \frac{Q^2}{m^2}, \frac{m^2}{\mu^2} \right) \otimes G(x, \mu^2, N_F) \right]
 \end{aligned}$$

- \otimes denotes the Mellin convolution

$$[A \otimes B](x) = \int_0^1 \int_0^1 dx_1 dx_2 \delta(x - x_1 x_2) A(x_1) B(x_2),$$

- The asymptotic representation for $F_2(x, Q^2)$ becomes effective at $Q^2 \geq 10 \cdot m^2$



Heavy flavor Wilson Coefficients

- In this limit the massive Wilson coefficients up to $O(a_s^3)$ read

$$\begin{aligned}
 L_{q,(2,L)}^{\text{NS}}(N_F+1) &= a_s^2 \left[A_{qq,Q}^{(2),\text{NS}}(N_F+1) \delta_2 + \tilde{C}_{q,(2,L)}^{(2),\text{NS}}(N_F) \right] \\
 &+ a_s^3 \left[A_{qq,Q}^{(3),\text{NS}}(N_F+1) \delta_2 + A_{qq,Q}^{(2),\text{NS}}(N_F+1) C_{q,(2,L)}^{(1),\text{NS}}(N_F+1) + \tilde{C}_{q,(2,L)}^{(3),\text{NS}}(N_F) \right] \\
 L_{q,(2,L)}^{\text{PS}}(N_F+1) &= a_s^3 \left[A_{qq,Q}^{(3),\text{PS}}(N_F+1) \delta_2 + A_{qq,Q}^{(2)}(N_F) N_F \tilde{C}_{g,(2,L)}^{(1)}(N_F+1) + N_F \tilde{C}_{q,(2,L)}^{(3),\text{PS}}(N_F) \right] \\
 L_{g,(2,L)}^{\text{S}}(N_F+1) &= a_s^2 \left[A_{gg,Q}^{(1)}(N_F+1) N_F \tilde{C}_{g,(2,L)}^{(1)}(N_F+1) + a_s^3 \left[A_{qq,Q}^{(3)}(N_F+1) \delta_2 \right. \right. \\
 &+ A_{gg,Q}^{(1)}(N_F+1) N_F \tilde{C}_{g,(2,L)}^{(2)}(N_F+1) + A_{gg,Q}^{(2)}(N_F+1) N_F \tilde{C}_{g,(2,L)}^{(1)}(N_F+1) \\
 &+ A_{Qg}^{(1)}(N_F+1) N_F \tilde{C}_{q,(2,L)}^{(2),\text{PS}}(N_F+1) + N_F \tilde{C}_{g,(2,L)}^{(3)}(N_F) \left. \right], \\
 H_{q,(2,L)}^{\text{PS}}(N_F+1) &= a_s^2 \left[A_{Qq}^{(2),\text{PS}}(N_F+1) \delta_2 + \tilde{C}_{q,(2,L)}^{(2),\text{PS}}(N_F+1) \right] + a_s^3 \left[A_{Qq}^{(3),\text{PS}}(N_F+1) \delta_2 \right. \\
 &+ \tilde{C}_{q,(2,L)}^{(3),\text{PS}}(N_F+1) + A_{qq,Q}^{(2)}(N_F+1) \tilde{C}_{g,(2,L)}^{(1)}(N_F+1) \\
 &+ A_{Qq}^{(2),\text{PS}}(N_F+1) C_{q,(2,L)}^{(1),\text{NS}}(N_F+1) \left. \right], \\
 H_{g,(2,L)}^{\text{S}}(N_F+1) &= a_s \left[A_{Qg}^{(1)}(N_F+1) \delta_2 + \tilde{C}_{g,(2,L)}^{(1)}(N_F+1) \right] + a_s^2 \left[A_{Qg}^{(2)}(N_F+1) \delta_2 \right. \\
 &+ A_{Qg}^{(1)}(N_F+1) C_{q,(2,L)}^{(1),\text{NS}}(N_F+1) + A_{gg,Q}^{(1)}(N_F+1) \tilde{C}_{g,(2,L)}^{(1)}(N_F+1) \\
 &+ \tilde{C}_{g,(2,L)}^{(2)}(N_F+1) \left. \right] + a_s^3 \left[A_{Qg}^{(3)}(N_F+1) \delta_2 + A_{Qg}^{(2)}(N_F+1) C_{q,(2,L)}^{(1),\text{NS}}(N_F+1) \right. \\
 &+ A_{gg,Q}^{(2)}(N_F+1) \tilde{C}_{g,(2,L)}^{(1)}(N_F+1) + A_{Qg}^{(1)}(N_F+1) \left\{ C_{q,(2,L)}^{(2),\text{NS}}(N_F+1) \right. \\
 &+ \left. \left. \tilde{C}_{q,(2,L)}^{(2),\text{PS}}(N_F+1) \right\} + A_{gg,Q}^{(1)}(N_F+1) \tilde{C}_{g,(2,L)}^{(2)}(N_F+1) + \tilde{C}_{g,(2,L)}^{(3)}(N_F+1) \left. \right]
 \end{aligned}$$



- Fixed moments $N = 2 \dots 10$ (12, 14) are known [Bierenbaum, Blümlein, Klein, 2009]
- The renormalization of this problem has been worked out by [Bierenbaum, Blümlein, Klein, 2009]
- Through the renormalization the general structure of the **unrenormalized** OME's is known
- Example:

$$\begin{aligned}
 \hat{A}_{Qg}^{(3)} = & \left(\frac{\hat{m}^2}{\mu^2}\right)^{3\epsilon/2} \left[\frac{\hat{\gamma}_{qg}^{(0)}}{6\epsilon^3} \left((N_F + 1) \gamma_{gq}^{(0)} \hat{\gamma}_{qg}^{(0)} + \gamma_{qq}^{(0)} [\gamma_{qq}^{(0)} - 2\gamma_{gg}^{(0)} - 6\beta_0 - 8\beta_{0,Q}] + 8\beta_0^2 \right. \right. \\
 & \left. \left. + 28\beta_{0,Q} \beta_0 + 24\beta_{0,Q}^2 + \gamma_{gg}^{(0)} [\gamma_{gg}^{(0)} + 6\beta_0 + 14\beta_{0,Q}] \right) + \frac{1}{6\epsilon^2} \left(\hat{\gamma}_{qg}^{(1)} [2\gamma_{qq}^{(0)} - 2\gamma_{gg}^{(0)} \right. \right. \\
 & \left. \left. - 8\beta_0 - 10\beta_{0,Q}] + \hat{\gamma}_{qg}^{(0)} [\hat{\gamma}_{qq}^{(1),PS} \{1 - 2N_F\} + \gamma_{qq}^{(1),NS} + \hat{\gamma}_{qq}^{(1),NS} + 2\hat{\gamma}_{gg}^{(1)} - \gamma_{gg}^{(1)} - 2\beta_1 \right. \right. \\
 & \left. \left. - 2\beta_{1,Q}] + 6\delta m_1^{(-1)} \hat{\gamma}_{qg}^{(0)} [\gamma_{gg}^{(0)} - \gamma_{qq}^{(0)} + 3\beta_0 + 5\beta_{0,Q}] \right) + \frac{1}{\epsilon} \left(\frac{\hat{\gamma}_{qg}^{(2)}}{3} - N_F \frac{\hat{\gamma}_{qg}^{(2)}}{3} \right. \\
 & \left. + \hat{\gamma}_{qg}^{(0)} [a_{gg,Q}^{(2)} - N_F a_{qq}^{(2),PS}] + a_{Qg}^{(2)} [\gamma_{qq}^{(0)} - \gamma_{gg}^{(0)} - 4\beta_0 - 4\beta_{0,Q}] + \frac{\hat{\gamma}_{qg}^{(0)} \zeta_2}{16} [\gamma_{gg}^{(0)} \{2\gamma_{qq}^{(0)} \right. \right. \\
 & \left. \left. - \gamma_{gg}^{(0)} - 6\beta_0 + 2\beta_{0,Q}\} - (N_F + 1) \gamma_{gq}^{(0)} \hat{\gamma}_{qg}^{(0)} + \gamma_{qq}^{(0)} \{-\gamma_{qq}^{(0)} + 6\beta_0\} - 8\beta_0^2 \right. \right. \\
 & \left. \left. + 4\beta_{0,Q} \beta_0 + 24\beta_{0,Q}^2 + \frac{\delta m_1^{(-1)}}{2} [-2\hat{\gamma}_{qg}^{(1)} + 3\delta m_1^{(-1)} \hat{\gamma}_{qg}^{(0)} + 2\delta m_1^{(0)} \hat{\gamma}_{qg}^{(0)}] \right. \right. \\
 & \left. \left. + \delta m_1^{(0)} \hat{\gamma}_{qg}^{(0)} [\gamma_{gg}^{(0)} - \gamma_{qq}^{(0)} + 2\beta_0 + 4\beta_{0,Q}] - \delta m_2^{(-1)} \hat{\gamma}_{qg}^{(0)} \right) + a_{Qg}^{(3)} \right]
 \end{aligned}$$



$O(\alpha_S^3 T_F^2 N_F C_{A,F})$: Contributing diagrams I

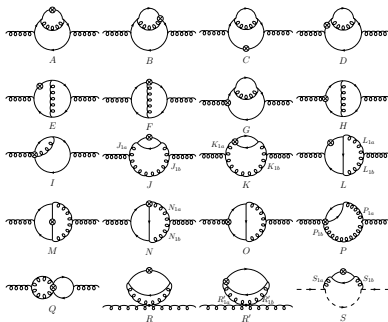


Figure: Generating 2-loop diagrams

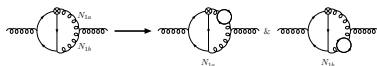


Figure: Gluons are decorated with quark bubbles



Contributing diagrams II

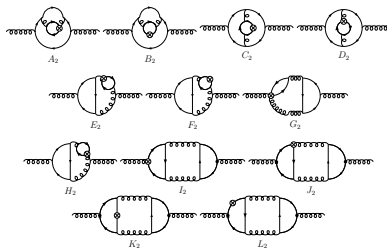


Figure: Further diagrams

- 289 Diagrams $\propto N_F T_F^2$ contribute
- 167 Diagrams $\propto T_F^2$ contribute
- due to symmetry some diagrams are identical

Evaluation of Feynman integrals

- Typical Feynman parameter integral after momentum integration

$$I_1 = \int_0^1 \int_0^1 \int_0^1 \int_0^1 dx_1 dx_2 dx_4 dx_5 x_1^{2+\varepsilon} x_2^{1-\varepsilon/2} x_5^{1-\varepsilon} (1-x_1)^{\varepsilon/2} (1-x_5)^2 (x_4 - x_5 x_4 + x_2 x_5)^N \times \left(1 - x_5 \left(1 - \frac{1}{1-x_1}\right)\right)^{3/2\varepsilon}$$

- Performing the integral yields a linear combination of sums over B -functions and Hypergeometric ${}_pF_Q$ s

$$I_1 = \frac{\Gamma(1-\varepsilon)\Gamma(3+\varepsilon)}{6(N+1)} \left\{ \sum_{j=1}^{N+1} \binom{1+N}{j} (-1)^j B(2-\varepsilon+j, 2) B(1+j, 2-\varepsilon/2) {}_3F_2 \left[\begin{matrix} -3/2\varepsilon, 2, 3+\varepsilon \\ 4+j-\varepsilon, 4 \end{matrix} ; 1 \right] + B(3+N-\varepsilon, 2) B(1, 3+N-\varepsilon/2) {}_3F_2 \left[\begin{matrix} -3/2\varepsilon, 2, 3+\varepsilon \\ 5-\varepsilon, 4 \end{matrix} ; 1 \right] \right\}$$

- where the ${}_pF_Q$ is defined by

$${}_pF_Q \left[\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_Q \end{matrix} ; z \right] = \sum_{i=0}^{\infty} \frac{(a_1)_i \dots (a_p)_i}{(b_1)_i \dots (b_Q)_i} \frac{z^i}{\Gamma(i+1)}$$



Mathematical structures

- Now: perform a series expansion in ε and evaluate the remaining sums
- Up to 4 (in)finite sums occur, which are computed using modern summation methods encoded in SIGMA [C. Schneider, 2007]
- results are given in terms of ζ_2 , ζ_3 and harmonic Sums $S_{\vec{a}}(N)$

$$S_{a_1, \dots, a_m}(N) = \sum_{n_1=1}^N \sum_{n_2=1}^{n_1} \dots \sum_{n_m=1}^{n_{m-1}} \frac{(\text{sign}(a_1))^{n_1}}{n_1^{|a_1|}} \frac{(\text{sign}(a_2))^{n_2}}{n_2^{|a_2|}} \dots \frac{(\text{sign}(a_m))^{n_m}}{n_m^{|a_m|}}$$

- in intermediary steps also generalized harmonic Sums occur

$$\tilde{S}_{m_1, \dots, (x_1, \dots; N)} = \sum_{i_1}^N \frac{x_1^{i_1}}{i_1^{m_1}} \sum_{i_2=1}^{i_1-1} \frac{x_2^{i_2}}{i_2^{m_2}} \tilde{S}_{m_3, \dots, (x_3, \dots; i_2)} + \tilde{S}_{m_1+m_2, m_3, \dots, (x_1 \cdot x_2, x_3, \dots; N)}$$

[Moch, Uwer, Weinzierl, 2002]

- algebraic and structural relations for these sums have been worked out [Ablinger, Blümlein, Schneider, 2011]



Results for the contributions $\propto N_F T_F^2 C_{F,A}$

$$\begin{aligned}
\hat{a}_{Q_9}^{(3),0} = & N_F T_F^2 C_A \left\{ \frac{16(N^2 + N + 2)}{27N(N+1)(N+2)} \left[108S_{-2,1,1} - 78S_{2,1,1} - 90S_{-3,1} + 72S_{2,-2} - 6S_{3,1} \right. \right. \\
& - 108S_{-2,1}S_1 + 42S_{2,1}S_1 - 6S_{-4} + 90S_{-3}S_1 + 118S_3S_1 + 120S_4 + 18S_{-2}S_2 + 54S_{-2}S_1^2 \\
& \left. \left. + 33S_2S_1^2 + 15S_2^2 + 2S_1^4 + 18S_{-2}\zeta_2 + 9S_2\zeta_2 + 9S_1^2\zeta_2 - 42S_1\zeta_3 \right] \right. \\
& + 32 \frac{5N^4 + 14N^3 + 53N^2 + 82N + 20}{27N(N+1)^2(N+2)^2} \left[6S_{-2,1} - 5S_{-3} - 6S_{-2}S_1 \right] \\
& - \frac{64(5N^4 + 11N^3 + 50N^2 + 85N + 20)}{27N(N+1)^2(N+2)^2} S_{2,1} - \frac{16(40N^4 + 151N^3 + 544N^2 + 779N + 214)}{27N(N+1)^2(N+2)^2} S_2S_1 \\
& - \frac{32(65N^6 + 429N^5 + 1155N^4 + 725N^3 + 370N^2 + 496N + 648)}{81(N-1)N^2(N+1)^2(N+2)^2} S_3 \\
& - \frac{16(20N^4 + 107N^3 + 344N^2 + 439N + 134)}{81N(N+1)^2(N+2)^2} S_1^3 + \frac{Q_1(N)}{81(N-1)N^3(N+1)^3(N+2)^3} S_2 \\
& + \frac{32(47N^6 + 278N^5 + 1257N^4 + 2552N^3 + 1794N^2 + 284N + 448)}{81N(N+1)^3(N+2)^3} S_{-2} \\
& + \frac{8(22N^6 + 271N^5 + 2355N^4 + 6430N^3 + 6816N^2 + 3172N + 1256)}{81N(N+1)^3(N+2)^3} S_1^2 \\
& + \frac{Q_2(N)}{243(N-1)N^2(N+1)^4(N+2)^4} S_1 + \frac{448(N^2 + N + 1)(N^2 + N + 2)}{9(N-1)N^2(N+1)^2(N+2)^2} \zeta_3 \\
& - \frac{16(5N^4 + 20N^3 + 59N^2 + 76N + 20)}{9N(N+1)^2(N+2)^2} S_1\zeta_2 - \frac{Q_3(N)}{9(N-1)N^3(N+1)^3(N+2)^3} \zeta_2 \\
& \left. - \frac{Q_4(N)}{243(N-1)N^5(N+1)^5(N+2)^5} \right\}
\end{aligned}$$



$$\begin{aligned}
& + N_F T_F^2 C_F \left\{ \frac{16(N^2 + N + 2)}{27N(N+1)(N+2)} \left[144S_{2,1,1} - 72S_{3,1} - 72S_{2,1}S_1 + 48S_4 - 16S_3S_1 \right. \right. \\
& - 24S_2^2 - 12S_2S_1^2 - 2S_1^4 - 9S_1^2\zeta_2 + 42S_1\zeta_3 \left. \right] + 32 \frac{10N^3 + 49N^2 + 83N + 24}{81N^2(N+1)(N+2)} \left[3S_2S_1 + S_1^3 \right] \\
& - \frac{128(N^2 - 3N - 2)}{3N^2(N+1)(N+2)} S_{2,1} - \frac{Q_5(N)}{81(N-1)N^3(N+1)^3(N+2)^2} S_3 \\
& + \frac{Q_6(N)}{27(N-1)N^4(N+1)^4(N+2)^3} S_2 - \frac{32(10N^4 + 185N^3 + 789N^2 + 521N + 141)}{81N^2(N+1)^2(N+2)} S_1^2 \\
& - \frac{16(230N^5 - 924N^4 - 5165N^3 - 7454N^2 - 10217N - 2670)}{243N^2(N+1)^3(N+2)} S_1 \\
& + \frac{16(5N^3 + 11N^2 + 28N + 12)}{9N^2(N+1)(N+2)} S_1\zeta_2 - \frac{Q_7(N)}{9(N-1)N^3(N+1)^3(N+2)^2} \zeta_3 \\
& \left. + \frac{Q_8(N)}{9(N-1)N^4(N+1)^4(N+2)^3} \zeta_2 + \frac{Q_9(N)}{243(N-1)N^6(N+1)^6(N+2)^5} \right\}
\end{aligned}$$



$L_{2,q}^{PS}$

$$\begin{aligned}
a_{qq,Q}^{(3),PS} = & C_F T_F^2 N_F \left\{ \frac{128}{27} \frac{(N^2 + N + 2)^2}{(-1 + N) N^2 (1 + N)^2 (2 + N)} S_1^3 - \frac{64}{27} \frac{(266 N^4 + 181 N^5 + 269 N^3 + 230 N^2 + 74 N^6 + 16 N^7 + 44 N - 24)}{N^3 (-1 + N) (2 + N)^2 (1 + N)^3} S_1^2 \right. \\
& + \frac{128}{9} \frac{(N^2 + N + 2)^2}{(-1 + N) N^2 (1 + N)^2 (2 + N)} S_1 S_2 + \frac{64}{81} \frac{P_1(N)}{(-1 + N) N^4 (1 + N)^4 (2 + N)^3} + \frac{32}{3} \frac{(N^2 + N + 2)^2}{(-1 + N) N^2 (1 + N)^2 (2 + N)} \zeta_2 S_1 \\
& - \frac{64}{27} \frac{(266 N^4 + 181 N^5 + 269 N^3 + 230 N^2 + 74 N^6 + 16 N^7 + 44 N - 24)}{N^3 (-1 + N) (2 + N)^2 (1 + N)^3} S_2 + \frac{256}{27} \frac{(N^2 + N + 2)^2}{(-1 + N) N^2 (1 + N)^2 (2 + N)} S_3 \\
& \left. - \frac{32}{243} \frac{P_2(N)}{N^5 (-1 + N) (2 + N)^4 (1 + N)^5} - \frac{16}{9} \frac{P_3(N)}{N^3 (-1 + N) (2 + N)^2 (1 + N)^3} \zeta_2 + \frac{224}{9} \frac{(N^2 + N + 2)^2}{(-1 + N) N^2 (1 + N)^2 (2 + N)} \zeta_3 \right\}
\end{aligned}$$

$$P_1(N) = 10101 N^7 + 4737 N^8 + 14923 N^6 + 17085 N^5 + 14133 N^4 + 5944 N^3 + 144 - 48 N + 568 N^2 + 1352 N^9 + 181 N^{10}$$

$$P_2(N) = (311482 N^{10} + 105173 N^{11} + 636490 N^9 + 966828 N^8 + 1126568 N^7 + 968818 N^6 + 550813 N^5 + 169250 N^4 - 864 - 1008 N + 12104 N^3 - 3408 N^2 + 21728 N^{12} + 2074 N^{13})$$

$$P_3(N) = 266 N^4 + 181 N^5 + 269 N^3 + 230 N^2 + 74 N^6 + 16 N^7 + 44 N - 24$$

(complete OME)



$$\begin{aligned}
\gamma_{qq}^{(2)} = & \frac{N_F^2 T_F^2}{(N+1)(N+2)} \left\{ C_A \left[(N^2 + N + 2) \left(\frac{128}{3N} S_{2,1} + \frac{128}{3N} S_{-3} + \frac{64}{9N} S_3 + \frac{32}{9N} S_1^3 \right. \right. \right. \\
& - \left. \frac{32}{3N} S_2 S_1 \right) - \frac{128(5N^2 + 8N + 10)}{9N} S_{-2} - \frac{64(5N^4 + 26N^3 + 47N^2 + 43N + 20)}{9N(N+1)(N+2)} S_2 \\
& - \frac{64(5N^4 + 20N^3 + 41N^2 + 49N + 20)}{9N(N+1)(N+2)} S_1^2 + \frac{64P_1(N)}{27N(N+1)^2(N+2)^2} S_1 \\
& \left. \left. \left. + \frac{16P_2(N)}{27(N-1)N^4(N+1)^3(N+2)^3} \right] \right. \\
& + C_F \left[\frac{32}{9} \frac{N^2 + N + 2}{N} \{ 10S_3 - S_1^3 - 3S_1 S_2 \} \right. \\
& + \frac{32(5N^2 + 3N + 2)}{3N^2} S_2 + \frac{32(10N^3 + 13N^2 + 29N + 6)}{9N^2} S_1^2 \\
& \left. \left. - \frac{32(47N^4 + 145N^3 + 426N^2 + 412N + 120)}{27N^2(N+1)} S_1 + \frac{4P_3(N)}{27(N-1)N^5(N+1)^4(N+2)^3} \right] \right\}
\end{aligned}$$

in agreement with [\[Moch, Vermaseren, Vogt 2004\]](#)

- furthermore the $N_F T_F^2 C_F$ and $N_F T_F^2 C_A$ -contributions to the OMEs $A_{qq,Q}^{NS,(3)}$, $A_{Qq}^{PS,(3)}$, and $A_{qq,Q}^{NS,Trans,(3)}$ have been computed
- this holds also for contributions to the 3-loop anomalous dimensions γ_{qq}^{PS} , γ_{qq}^{NS} and $\gamma_{qq}^{NS,Trans}$



Cyclotomic Harmonic Sums and Polylogarithms

- The usual harmonic polylogarithms are extended allowing for more letters in the alphabet :

$$\frac{1}{x}, \frac{1}{1-x}, \frac{1}{1+x}, \frac{1}{1+x+x^2}, \frac{1}{1+x^2}, \frac{1}{1+x+x^2+x^3+x^4}, \frac{1}{1-x+x^2}, \dots$$

- Their iterated integrals are the cyclotomic harmonic polylogarithms (CHPL).
- The Mellin transforms of the CHPLs are spanned by the cyclotomic harmonic sums (CHS).
- The cyclotomic harmonic numbers are the CHSs at $N \rightarrow \infty$.
- They are related to the MZVs of higher roots of unity and the motivic numbers.
- Quantities of this kind emerge in massive Feynman diagram calculations (simplest example: sums containing 6th root of unity numerators).
- All formalism is again incorporated into the package **HarmonicSums**

J. Ablinger, J.B., C. Schneider, Journ. Math. Phys., in print arXiv:1105.6063 [math-ph].



First contributions $\propto T_F^2 C_{A,F}$, $m_1 = m_2$

- Feynman integrals could not be mapped directly onto higher functions
- \rightarrow Mellin-Barnes representation is introduced
- this yields Meijer G-functions which can be expanded into hypergeometric functions



The flavor non-singlet contributions

$$\hat{a}_{qq,Q}^{(3),NS} = T_F^2 C_F \left\{ \frac{128}{27} S_4 - \frac{1024}{27} \zeta_3 S_1 + \frac{64}{9} \zeta_2 S_2 + \frac{256(3N^2 + 3N + 2)}{27N(N+1)} \zeta_3 - \frac{320}{27} \zeta_2 S_1 - \frac{640}{81} S_3 \right. \\ \left. + \frac{8(3N^4 + 6N^3 + 47N^2 + 20N - 12)}{27N^2(N+1)^2} \zeta_2 + \frac{1856}{81} S_2 - \frac{19424}{729} S_1 \right. \\ \left. - \frac{4(417N^8 + 1668N^7 - 4822N^6 - 12384N^5 - 6507N^4 + 740N^3 + 216N^2 + 144N + 432)}{729N^4(N+1)^4} \right\}$$

$$\hat{\gamma}_{qq}^{(2),NS} = C_F T_F^2 \left(\frac{128S_3}{9} - \frac{640S_2}{27} - \frac{128S_1}{27} + \frac{8(51N^6 + 153N^5 + 57N^4 + 35N^3 + 96N^2 + 16N - 24)}{27N^3(N+1)^3} \right)$$

in agreement with [\[Moch, Vermaseren, Vogt 2004\]](#)

$$\hat{a}_{qq,Q}^{(3),TR} = C_F T_F^2 \left\{ \frac{128}{27} S_4 - \frac{1024}{27} \zeta_3 S_1 + \frac{64}{9} S_2 \zeta_2 + \frac{256}{9} \zeta_3 - \frac{320}{27} S_1 \zeta_2 - \frac{640}{81} S_3 \right. \\ \left. + \frac{8}{9} \zeta_2 + \frac{1856}{81} S_2 - \frac{19424}{729} S_1 - \frac{4(139N^4 + 278N^3 - 101N^2 + 48N + 144)}{243N^2(N+1)^2} \right\}$$

$$\hat{\gamma}_{qq}^{(2),TR} = C_F T_F^2 \left\{ \frac{128S_3}{9} - \frac{640S_2}{27} - \frac{128S_1}{27} + \frac{8(17N^2 + 17N - 8)}{9N(N+1)} \right\}$$

in agreement with [\[Moch, Vermaseren, Vogt 2004\]](#)



The flavor pure-singlet contributions

$$\hat{a}_{Qq}^{(3),PS} = \frac{T_F^2 C_F}{(N-1)N^2(N+1)^2(2+N)} \left\{ (N^2 + N + 2)^2 \left(\frac{32}{27} S_1^3 - \frac{512}{27} S_3 + \frac{128}{3} S_{2,1} - \frac{1024}{9} \zeta_3 - \frac{160}{9} S_2 S_1 \right) \right. \\ \left. + \frac{32}{3} \zeta_2 S_1 \right) - \frac{32 P_1(N)}{9N(N+2)} \zeta_2 + \frac{32 P_2(N)}{27N(N+2)(N+3)(N+4)(N+5)} S_2 - \frac{32 P_3(N)}{27N(N+1)(N+2)(N+3)(N+4)(N+5)} S_1^2 \\ \left. + \frac{64 P_4(N)}{81N^2(N+1)^2(N+2)^2(N+3)(N+4)(N+5)} S_1 - \frac{64 P_5(N)}{243N^3(N+1)^2(N+2)^3(N+3)(N+4)(N+5)} \right\}.$$

$$\hat{\gamma}_{qq}^{(3),PS} = \frac{T_F^2 C_F}{(N-1)N^2(N+1)^2(2+N)} \left\{ -\frac{32}{3} (N^2 + N + 2)^2 (S_1^2 + S_2) \right. \\ \left. + \frac{64 P_6(N)}{9N(N+1)(N+2)} S_1 - \frac{64 P_7(N)}{27N^2(N+1)^2(N+2)^2} \right\},$$

with

$$P_6(N) = 8N^7 + 37N^6 + 68N^5 - 11N^4 - 86N^3 - 56N^2 - 104N - 48,$$

$$P_7(N) = 52N^{10} + 392N^9 + 1200N^8 + 1353N^7 - 317N^6 - 1689N^5 - 2103N^4 \\ - 2672N^3 - 1496N^2 - 48N + 144.$$



$$m_1 \neq m_2$$

- $m_c/m_b \simeq 1.3\text{GeV}/4.2\text{GeV} \rightarrow x^3 := (m_c/m_b)^6 \simeq 0.0001 \rightarrow$ expand in masses
- for fixed values of N the diagrams can be mapped onto tadpole diagrams by projection operators [Bierenbaum, Blümlein, Klein 2009.]
- e.g. $N = 2$

$$\Pi_{\mu\nu} = \frac{1}{d-1} \left(\frac{-g_{\mu\nu}}{p^2} + d \frac{p^{(\mu} p^{\nu)}}{p^4} \right)$$

- more complex structures occur for higher Moments
- expansion in masses was performed using EXP [Harlander, Seidensticker, Steinhauser 1998, Seidensticker 1999]
- 120 out of 256 diagrams have been computed for general values of N
- These diagrams cannot be re-interpreted in the VFNS either for c - or b -distributions.
- Ergo: the VFNS ends at 2 loops.



$N = 4$

$$\begin{aligned}
a_{Q_g}^{(3)} = & T_F^2 C_A \left\{ \frac{4887988511}{24300000} - \frac{47146}{2025} \zeta_3 + \frac{5807}{180} \zeta_2 + \frac{496855133}{7441875} x + \frac{2510388298}{468838125} x^2 + \frac{250077164867}{5616211899375} x^3 \right. \\
& + \ln\left(\frac{m_2^2}{\mu^2}\right) \left[\frac{47956573}{810000} + \frac{17963}{450} \zeta_2 - \frac{1877399}{70875} x - \frac{3269548}{1488375} x^2 + \frac{156082853}{1620840375} x^3 \right] \\
& + \ln^2\left(\frac{m_2^2}{\mu^2}\right) \left[\frac{532373}{16200} + \frac{707}{1350} x - \frac{284}{525} x^2 - \frac{744283}{935550} x^3 \right] + \ln^2\left(\frac{m_2^2}{\mu^2}\right) \frac{74657}{4050} + \ln^2\left(\frac{m_2^2}{\mu^2}\right) \ln\left(\frac{m_1^2}{\mu^2}\right) \frac{17963}{675} \\
& + \ln\left(\frac{m_2^2}{\mu^2}\right) \ln\left(\frac{m_1^2}{\mu^2}\right) \left[\frac{62893}{2025} - \frac{707}{675} x + \frac{568}{525} x^2 + \frac{744283}{467775} x^3 \right] + \ln\left(\frac{m_2^2}{\mu^2}\right) \ln^2\left(\frac{m_1^2}{\mu^2}\right) \frac{7579}{675} \\
& + \ln\left(\frac{m_1^2}{\mu^2}\right) \left[\frac{384762007}{2430000} \frac{17963}{450} \zeta_2 + \frac{1877399}{70875} x + \frac{3269548}{1488375} x^2 - \frac{156082853}{1620840375} x^3 \right] \\
& + \ln^2\left(\frac{m_1^2}{\mu^2}\right) \left[\frac{532373}{16200} + \frac{707}{1350} x - \frac{284}{525} x^2 - \frac{744283}{935550} x^3 \right] + \ln^3\left(\frac{m_1^2}{\mu^2}\right) \frac{3817}{162} \left. \right\} \\
& + T_F^2 C_F \left\{ \left[-\frac{33406758667}{1093500000} + \frac{260414}{10125} \zeta_3 + \frac{1473641}{202500} \zeta_2 - \frac{119314474}{4134375} x + \frac{582667691}{37507050} x^2 + \frac{46049137562}{44929695195} x^3 \right] \right. \\
& + \ln\left(\frac{m_2^2}{\mu^2}\right) \left[\frac{76621423}{4050000} - \frac{18601}{2250} \zeta_2 - \frac{368428}{39375} x - \frac{2876423}{297675} x^2 - \frac{570093292}{324168075} x^3 \right] \\
& + \ln^2\left(\frac{m_2^2}{\mu^2}\right) \left[\frac{530371}{81000} + \frac{4456}{1125} x + \frac{27101}{4725} x^2 + \frac{1759616}{467775} x^3 \right] - \ln^3\left(\frac{m_2^2}{\mu^2}\right) \frac{130207}{20250} - \ln^2\left(\frac{m_2^2}{\mu^2}\right) \ln\left(\frac{m_1^2}{\mu^2}\right) \frac{18601}{3375} \\
& + \ln\left(\frac{m_2^2}{\mu^2}\right) \ln\left(\frac{m_1^2}{\mu^2}\right) \left[\frac{442267}{50625} - \frac{8912}{1125} x - \frac{54202}{4725} x^2 - \frac{3519232}{467775} x^3 \right] + \ln\left(\frac{m_2^2}{\mu^2}\right) \ln^2\left(\frac{m_1^2}{\mu^2}\right) \frac{18601}{3375} \\
& + \ln\left(\frac{m_1^2}{\mu^2}\right) \left[-\frac{37307959}{2430000} - \frac{18601}{2250} \zeta_2 + \frac{368428}{39375} x + \frac{2876423}{297675} x^2 + \frac{570093292}{324168075} x^3 \right] \\
& + \ln^2\left(\frac{m_1^2}{\mu^2}\right) \left[\frac{530371}{81000} + \frac{4456}{1125} x + \frac{27101}{4725} x^2 + \frac{1759616}{467775} x^3 \right] - \ln^3\left(\frac{m_1^2}{\mu^2}\right) \frac{204611}{20250} \left. \right\} + \mathcal{O}(x^4 \ln^3(x))
\end{aligned}$$



Example: Diagram D_{872}

$$\begin{aligned}
 D_{872} = & C_A T_F^2 \left[\left\{ \delta(N-2) [\dots] + \delta(N-3) [\dots] + \delta(N-4) [\dots] \right. \right. \\
 & + \theta(N-5) \left[\frac{1}{\epsilon^3} \frac{1256(N-3)(2N^2+N-4)}{9N^2(N+1)^2} - \frac{1}{\epsilon^2} \left\{ \frac{64(2N^5+33N^4-13N^3+87N^2-283N+354)}{(N-1)N^2(N+1)^2(N+2)} \right. \right. \\
 & + \left. \frac{64(N-3)(2N^2+N-4)}{3N^2(N+1)^2} \ln(x) \right\} + \frac{1}{\epsilon} \left\{ \frac{32Q_1}{81(N-1)^2 N^4 (N+1)^4 (2+N)^2} \right. \\
 & + \left. \frac{32(N-3)(2N^2+N-4)}{3N^2(N+1)^2} (C_2 + S_2 + \ln^2(x)) \right\} + \frac{8Q_2}{243(N-1)^3 N^4 (N+1)^5 (N+2)^3} \\
 & - \frac{8(2N^5+33N^4-13N^3+87N^2-283N+354)}{9(N-1)N^2(N+1)^2(N+2)} C_2 \\
 & - \frac{8(N-3)(2N^2+N-4)}{9N^2(N+1)^2} (4C_3 + 9C_2 + 6S_2 + 12S_3) \\
 & + \frac{4Q_3}{225(N-2)(N-1)^2 N^3 (N+1)^3 (N+2)(N+3)^2 (N+4)^2 x^2} \\
 & + \frac{Q_4}{7350(N-3)^2 (N-2)^2 (N-1)N(N+1)^3 (N+2)^3 (N+3)^3 (N+4)^3 (N+5)^3 (N+6)^3} x^2 \\
 & + \frac{2Q_5}{893025(N-4)^2 (N-3)(N-2)(N-1)N(N+1)^2 (N+2)^2 (N+3)^3 (N+4)^3 (N+5)^3 (N+6)^3 (N+7)^3 (N+8)^3} x^3 \\
 & - \frac{8(2N^5+33N^4-13N^3+87N^2-283N+354)}{9(N-1)N^2(N+1)^2(N+2)} \ln^2(x) \\
 & + \frac{4Q_6}{(N+1)(N+2)(N+3)(N+4)(N+5)(N+6)} x^2 \ln^2(x) + \frac{32Q_7}{9(N+3)(N+4)(N+5)(N+6)(N+7)(N+8)} x^3 \ln^2(x) \\
 & + \frac{32(N-3)(2N^2+N-4)}{9N^2(N+1)^2} \ln^3(x) - \frac{8Q_8}{9(N-1)N^3(N+1)^3(N+2)} S_2 \\
 & + \frac{8Q_6}{(N+1)(N+2)(N+3)(N+4)(N+5)(N+6)} S_2 x^2 + \frac{64Q_7}{9(N+3)(N+4)(N+5)(N+6)(N+7)(N+8)} S_2 x^3 \\
 & + \frac{8Q_9}{27(N-1)^2 N^4 (N+1)^4 (N+2)^2} \ln(x) + \frac{8Q_{10}}{15(N-1)N^2(N+1)^2(N+3)(N+4)} x \ln(x) \\
 & - \frac{Q_{11}}{35(N-3)(N-2)(N-1)N(N+1)^2(N+2)^2(N+3)^2(N+4)^2(N+5)^2(N+6)^2} x^2 \ln(x) \\
 & + \left. \frac{8Q_{12}}{2835(N-4)(N-3)(N-2)(N-1)N(N+1)^2(N+2)^2(N+3)^2(N+4)^2(N+5)^2(N+6)^2(N+7)^2(N+8)^2} x^3 \ln(x) \right]
 \end{aligned}$$



3-loop ladder diagrams: up to 6 massive propagators



$$\begin{aligned}
 I_{\text{fin}} = & \frac{1}{(N+3)(N+4)} \left\{ \left[\frac{-4(N^3+3N^2-N-5)}{(N+1)(N+2)(N+3)} S_1 + 2S_2^2 + \frac{4(-1)^N S_1 + 4S_{-2}}{N+3} \right. \right. \\
 & + 2(2N+5)S_4 + \frac{4(-1)^N(2N^3+7N^2+4N-3)}{(N+1)^2(N+2)^2(N+3)} S_1^2 + \frac{4(6N^3+34N^2+63N+39)}{(N+1)^2(N+2)^2(N+3)} \left. \right] \frac{1}{s^2} \\
 & + \left[\frac{-4N^4-25N^3-30N^2+49N+76}{(N+1)(N+2)(N+3)(N+4)} S_1^2 - \frac{4(2N^4+14N^3+27N^2+5N-16)}{(N+1)(N+2)(N+3)(N+4)} S_{-2} \right. \\
 & + \frac{10N^4+73N^3+158N^2+73N-32}{(N+1)(N+2)(N+3)(N+4)} S_{-2}^2 \\
 & + \frac{2(-1)^N(12N^5+127N^4+538N^3+1177N^2+1354N+648)}{(N+1)^2(N+2)^2(N+3)^2(N+4)} S_1 \\
 & - \frac{2(8N^5+51N^4-72N^3-1330N^2-4062N^2-5151N-2436)}{(N+1)^2(N+2)^2(N+3)^2(N+4)} S_1 \\
 & + S_2^2 + \frac{(-1)^N}{N+3} (S_1^2 - S_2) + 4S_{-2}S_1 - 5S_2S_1 + 2(4N+15)S_{-2} + 2(N-1)S_1 \\
 & - 12S_{-2,1} + 9(N+4)S_{-1} \\
 & + \frac{2(-1)^N(11N^4+60N^3-160N^2-1837N^2-5095N^2-5801N-2598)}{(N+1)^2(N+2)^2(N+3)^2(N+4)} \\
 & + \left. \frac{2(70N^4+893N^3+4640N^2+12026N^2+19074N^2+15269N+5100)}{(N+1)^2(N+2)^2(N+3)^2(N+4)} \right] \frac{1}{s^2} \\
 & + \frac{7}{24} S_1^2 + \frac{(-10N^4-61N^3-68N^2+129N+188)}{6(N+1)(N+2)(N+3)(N+4)} S_1^2 \\
 & + \frac{(-1)^N(12N^5+127N^4+538N^3+1177N^2+1354N+648)}{2(N+1)^2(N+2)^2(N+3)^2(N+4)} S_1^2 \\
 & + \frac{P_{20}}{2(N+1)^2(N+2)^2(N+3)^2(N+4)} S_1^2 + \frac{3}{4} \zeta_2 S_1^2 - 4S_{-2} S_1^2 - \frac{13}{4} S_2 S_1^2 \\
 & + \frac{(-1)^N P_{20}}{(N+1)^2 P_{20}} + \frac{P_{20}}{(N+1)^2(N+2)^2(N+3)^2(N+4)^2} S_1 \\
 & + \frac{2(N^3+3N^2-N-5)}{(N+1)(N+2)(N+3)} \zeta_2 S_1 - 2S_{-2} S_1 \\
 & - \frac{4(4N^4+41N^3+155N^2+254N+148)}{(N+1)(N+2)(N+3)(N+4)} S_{-2} S_1 \\
 & + \frac{(-1)^N}{N+3} \left(-4S_{-2} S_1 + \frac{9}{2} S_2 S_1 + \frac{3}{2} S_2 S_1 + \frac{10}{3} S_1 + 2S_{2,1} + 12S_{-2,1} \right) \\
 & + \frac{(-14N^4-201N^3-936N^2-1715N-1044)}{2(N+1)(N+2)(N+3)(N+4)} S_{-2} S_1 - \frac{119}{3} S_2 S_1 \\
 & - 12S_{-2,1} S_1 + 22S_{2,1} S_1 - 2S_{2,2}^2 + \frac{1}{6} (32N+119) S_1^2 \\
 & + \frac{(-1)^N P_{20}}{(N+1)^2(N+2)^2(N+3)^2(N+4)^2} + \frac{P_{20}}{(N+1)^2(N+2)^2(N+3)^2(N+4)^2}
 \end{aligned}$$

$$\begin{aligned}
 & + \frac{3(-1)^N(2N^3+7N^2+4N-3)}{2(N+1)^2(N+2)^2(N+3)} \zeta_2 + \frac{2(6N^3+34N^2+63N+39)}{2(N+1)^2(N+2)^2(N+3)} \zeta_2 \\
 & + (8N+30)S_{-2} + \frac{2(8N^5+108N^4+558N^3+1365N^2+1553N+640)}{(N+1)(N+2)(N+3)(N+4)} S_{-2} \\
 & - \frac{4(-1)^N(2N^3+7N^2+4N-3)}{(N+1)^2(N+2)^2(N+3)} S_{-2} - \frac{4P_{20}}{(N+1)^2(N+2)^2(N+3)^2(N+4)^2} S_{-2} \\
 & + \frac{3}{2} \zeta_2 S_{-2} + \frac{(-1)^N(8N^5+79N^4+186N^3-279N^2-1426N-1224)}{2(N+1)^2(N+2)^2(N+3)^2(N+4)} S_1 \\
 & + \frac{P_{20}}{2(N+1)^2(N+2)^2(N+3)^2(N+4)^2} S_1 + \frac{3}{4} (2N+5) \zeta_2 S_1 + 8S_{-2} S_1 \\
 & + \frac{(-18N^4-229N^4-1498N^3-5558N^2-10017N-6460)}{3(N+1)(N+2)(N+3)(N+4)} S_1 \\
 & + \frac{1}{4} (20N-29) S_1 - 14S_{-1,1} + \frac{4(4N^4+22N^3+11N^2-85N-96)}{(N+1)(N+2)(N+3)(N+4)} S_{-2,1} \\
 & - 14S_{-2,2} + \frac{2(11N^4+107N^3+397N^2+640N+361)}{(N+1)(N+2)(N+3)} S_{2,1} + 2(N+36) S_{1,1} \\
 & + 28S_{-2,1,1} + 2(2N-7) S_{2,1,1} \Big\} + O(\epsilon), \quad (1)
 \end{aligned}$$

$$P_{22} = -6N^8 - 164N^7 - 1613N^6 - 7762N^5 - 19526N^4 - 22888N^3 - 2137N^2 + 19968N + 13264, \quad (2)$$

$$P_{23} = 119N^8 + 2250N^7 + 18755N^6 + 90365N^5 + 275464N^4 + 542281N^3 + 668958N^2 + 469072N + 142112, \quad (3)$$

$$P_{24} = 163N^{11} + 448N^{10} + 5568N^9 + 41171N^8 + 204092N^7 + 730291N^6 + 1858328N^5 + 35604939N^4 + 4712624N^3 + 4272331N^2 + 2335932N + 581072, \quad (4)$$

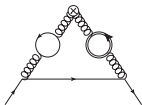
$$P_{25} = 78N^8 + 937N^7 + 2466N^6 - 17638N^5 - 155141N^4 - 538674N^3 - 1047495N^2 - 1197445N - 757472N - 206296, \quad (5)$$

$$P_{26} = 568N^9 + 11297N^8 + 98332N^7 + 492027N^6 + 1561688N^5 + 3266831N^4 + 4516420N^3 + 2994885N^2 + 2061840N + 475824, \quad (6)$$

$$P_{27} = 4N^9 + 96N^8 + 942N^7 + 4995N^6 + 15753N^5 + 30351N^4 + 34903N^3 + 21844N^2 + 5648, \quad (7)$$

$$P_{28} = -32N^9 - 730N^8 - 7180N^7 - 40057N^6 - 139918N^5 - 317434N^4 - 466820N^3 - 426421N^2 - 216416N - 45040, \quad (8)$$



First 3 loop contributions VFNS OMEs: $A_{gq,Q}^{(3)}(N)$ 

$$\begin{aligned}
 A_{gq,Q,C_F T_F^2 n_f}^{(3),\overline{\text{MS}}} &= C_{Fn_f} T_F^2 \frac{1 + (-1)^N}{2} \left\{ \frac{88(N^2 + N + 2)}{27(N-1)N(N+1)} \log^3 \left(\frac{m^2}{\mu^2} \right) \right. \\
 &+ \left(\frac{8(N^2 + N + 2)}{9(N-1)N(N+1)} S_1 - \frac{8(8N^3 + 13N^2 + 27N + 16)}{27(N-1)N(N+1)^2} \right) \log^2 \left(\frac{m^2}{\mu^2} \right) \\
 &+ \left[\frac{16(N^2 + N + 2)}{9(N-1)N(N+1)} \left(S_1^2 + S_2 - \frac{1}{2} \zeta_2 \right) - \frac{32(8N^3 + 13N^2 + 27N + 16)}{27(N-1)N(N+1)^2} S_1 \right. \\
 &+ \left. \frac{16(179N^4 + 489N^3 + 913N^2 + 925N + 358)}{81(N-1)N(N+1)^3} \right] \log \left(\frac{m^2}{\mu^2} \right) \\
 &+ \left[\frac{16(N^2 + N + 2)}{9(N-1)N(N+1)} \left(S_1^3 + 3S_2 S_1 + 2S_3 + 5S_1 \zeta_2 + \frac{11}{2} \zeta_3 \right) \right. \\
 &- \frac{16(8N^3 + 13N^2 + 27N + 16)}{9(N-1)N(N+1)^2} \left(S_1^2 + S_2 + \frac{5}{3} \zeta_2 \right) \\
 &+ \frac{32(160N^4 + 408N^3 + 827N^2 + 845N + 320)}{81(N-1)N(N+1)^3} S_1 \\
 &\left. - \frac{32(1189N^5 + 4276N^4 + 9248N^3 + 12289N^2 + 8668N + 2378)}{243(N-1)N(N+1)^4} \right] \left. \right\}
 \end{aligned}$$



Conclusion

- The present status of the calculation of the $O(\alpha_s^3)$ massive Wilson Coefficients in DIS for general values of the Mellin variable N has been presented.
- All logarithmic contributions $O(\alpha_s^3 \ln^k(Q^2/m^2))$, $k = 1, 2, 3$ to all the OMEs have been computed.
- The $O(\alpha_s^3 N_F T_F^2 C_{A,F})$ contributions to all the OMEs A_{ij} which contribute to the nucleonic structure function $F_2(x, Q^2)$ and transversity for general values of the Mellin variable N were calculated.
- These results constitute first complete expressions for two color factors contributing to the heavy flavor Wilson Coefficients for $F_2(x, Q^2)$ at $O(a_s^3)$ for the Wilson Coefficients $L_{qq,Q}^{PS}$ and $L_{qg,Q}^S$.
- Along with the computation of the massive OMEs we obtained the corresponding parts of the **3-loop anomalous dimensions** and confirmed results given in the literature analytically, partly for the first time.
- Results have been obtained for the $O(\alpha_s^3 T_F^2 C_{A,F})$ terms of the NS and PS OMEs resulting from the graphs with two massive lines with equal and non-equal masses.
- For the OME A_{Qg} fixed moments have been generated for the case of two non-equal masses. Many diagrams have already been computed for general values of N .



Conclusion

- The ladder graphs can be calculated by the technologies being available to us.
- The major charged current DIS heavy flavor Wilson Coefficients to $O(\alpha_s^2)$ were calculated.
- Along with the calculation a lot of new technologies to perform analytic massless and massive 3-loop computations are developed and are improved significantly.
- Summation technologies in product and difference fields are further refined along with other discrete algorithms.
- New higher transcendental function spaces are coined and explored (Generalized harmonic sums, cyclotomic harmonic sums and polylogarithms, hyperlogarithms, higher hypergeometric functions).

