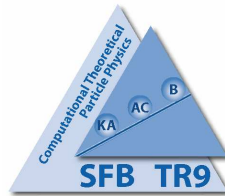


Recent Results on Finite Harmonic Sums in Quantum Field Theory

J. Blümlein, DESY



- Structural Relations of Harmonic Sums up to $w = 6$
- From Moments to Functions in HO QCD

Based on :

arXiv:0901.3106, arXiv:0901.0837;

arxiv:0902.4091 [hep-ph], CPC in print with M. Kauers (RISC), S. Klein (DESY), and C. Schneider (RISC)

1. Introduction

Why are zero- and single scale quantities in Quantum Field Theory related to ζ -values, nested harmonic sums and related objects ?

- The former quantities can be obtained from the latter putting the Mellin variable N either to fixed values or $N \rightarrow \infty$.

Perturbation Theory [fixed order]

Scalar Propagators:

$$\frac{i}{p^2 + i\epsilon}$$

Combine Momenta using the Feynman Trick

$$\frac{1}{A \cdot B} = \int_0^1 \frac{dx}{[xA + (1-x)B]^2}$$

Feynman parameter integrals substitute all non-trivial angular integrals in $D = 4 + \varepsilon$ -dimensions.

Introduction

Momentum Integrals yield rational functions of Γ -functions

$$\frac{\Gamma(n_1 + \alpha_1 \varepsilon) \dots \Gamma(n_k + \alpha_k \varepsilon)}{\Gamma(m_1 + \beta_1 \varepsilon) \dots \Gamma(m_l + \beta_l \varepsilon)} \Big|_{m_i, n_i \in \mathbf{Z}, \alpha_i, \beta_i \in \mathbf{Q}}$$

- The scale-ratio in the diagrams factors form the Feynman parameter integrals for single scale processes

The Feynman parameter integrals can be transformed into Mellin-Barnes Integrals

$$\frac{1}{(A+B)^q} = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} d\sigma A^\sigma B^{-q-\sigma} \frac{\Gamma(-\sigma)\Gamma(q+\sigma)}{\Gamma(q)}$$

Then all Feynman parameter integrals can again be integrated \implies rational function of Γ -functions

One or more Feynman parameters contain N as power x_i^N, \dots in the numerators. \implies the Γ -functions contain N as argument.

The Mellin-Barnes Integrals can be carried out using the Residue Theorem. \implies several infinite sums over the rational functions of Γ -functions.

Introduction

- Seek compact representations for these in terms of (Generalizations of) generalized hypergeometric functions.
- The Mellin variable N is a discrete quantity in the first place for physical reasons
 \implies Light-cone expansion; cut vertex method + dispersion relations
- Perform the ε -expansion.
- The respective coefficients obey Difference Equations of finite order.
- The ε -expansion of Pochhammer-Symbols & Γ -functions leads to products of single finite harmonic sums and MZV's.
- The infinite sums over the Mellin-Barnes parameters lead to the respective Nesting.
- Observation: Most of the sums occurring are Nested Finite Harmonic Sums.
- However, other related sums are possible too for individual Feynman diagrams. [Vermaseren et al. (2005)]
- General solution formalisms like, Sigma, will reveal this uniquely. cf. C. Schneider.

Introduction

- Single scale processes in massless Quantum Field Theories or being considered in the limit $m^2/Q^2 \rightarrow 0$ exhibit **significant simplifications** when calculated in Mellin space.
- This is, to some extent, due to structure of **Feynman parameter integrals** which possess a **Mellin symmetry**.
- **Harmonic sums** form the appropriate language to derive **compact expressions** in the respective calculations.
- We will line out the relations of the harmonic sums, resp. their continuations to $N \in \mathbf{Q}, \mathbf{R}, \mathbf{C}$.

x-space results :

Nielsen-type integrals, resp. harmonic polylogarithms (E. Remiddi and J. Vermaseren (1999))

$$S_{n,p,q}(x) = \frac{(-1)^{n+p+q-1}}{\Gamma(n)p!q!} \int_0^1 \frac{dz}{z} \ln^{(n-1)}(z) \ln^p(1-zx) \ln^q(1+zx)$$

2 Loop Wilson Coefficients

W.L. van Neerven et al.: (1992) 79 functions 80 objects would be maximal.

- The high complexity is partly caused applying the the IBP–Method.
- x –space usually is not the best space to work in.

3 Loop Anomalous Dimensions & Wilson Coefficients

- \implies Harmonic Sums in linear representation.
- Still high complexity of terms.
- Compactification possible applying algebraic and structural relations.
- Observation : In all single scale calculations the same Basic Functions occur in the resp. weight.
- \implies Derive these Universal Functions and their complex analysis.

2. Algebraic Relations

cf. J.Blümlein, Comput. Phys. Commun. **159** (2004) 19

Number of harmonic sums up to weight w : 3^{w-1} .

Harmonic sums form a quasi-shuffle algebra through . (M.E. Hoffman, J. Algebraic Combin. **11** (2000) 49)

$$\begin{aligned} S_{a_1, a_2} S_{a_3, a_4} &= S_{a_1, a_2, a_3, a_4} + S_{a_1, a_3, a_2, a_4} + S_{a_1, a_2, a_4, a_2} \\ &+ S_{a_3, a_4, a_1, a_2} + S_{a_3, a_1, a_4, a_2} + S_{a_3, a_1, a_2, a_4} \quad \text{etc.} \end{aligned}$$

Solve all the linear equations possible for the harmonic sums \implies algebraic basis.

Let $\{a, a, a, \dots, b, b, \dots, \dots, z, z\}$ a set of n_1 a 's, n_2 b 's etc. The number of basis elements corresponding to all words formed by ALL the above letters is:

$$l_n(n_1, \dots, n_q) = \frac{1}{n} \sum_{d|n_i} \mu(d) \frac{(n/d)!}{(n_1/d)! \dots (n_d/d)!}, \quad \sum_i n_i = n$$

(E. Witt, 1937) \implies # Lyndon words

w	1	2	3	4	5	6
$\#_c$	2	8	26	80	242	728
$\#_r$	2	5	13	31	79	195

Observation in Quantum Field Theory :

At least up to $O(\alpha_s^3)$ the contributing harmonic sums exhibit never any index $a_k = -1$ applying a compact representation.

The number of sums of this type is

$$N_{\neg\{-1\}}(w) = \frac{1}{2} \left[\left(1 - \sqrt{2}\right)^w + \left(1 + \sqrt{2}\right)^w \right]$$

$$N_{\neg\{-1\}}^{\text{basic}}(w) = \frac{2}{w} \sum_{d|w} \mu\left(\frac{w}{d}\right) N_{\neg\{-1\}}^{\text{basic}}(d) .$$

w	1	2	3	4	5	6
$\#_c$	1	4	11	28	69	168
$\#_r$	1	3	7	14	30	60

- Here $\#_c$ is smaller than $\#_r$ in the general case.

Algebraic Relations

Remark:

Harmonic, Generalized Harmonic Polylogarithms and Multiple Polylogarithms also form **shuffle algebras**. As shuffle algebras are **sub-sets** of the quasi-shuffle algebra studied above, the respective algebraic relations can be derived **directly**.

- Form the **index alphabet**.
- Solve the **shuffle-relations** \implies Basis

As the relations in J.B., Comput. Phys. Commun. **159** (2004) 19 are of **arbitrary weight** (**general alphabet**) and **depth** $d \leq 6$ the corresponding relations can be read off there.

Algorithms to extend this scenario are available.

3. Structural Relations

w = 1:

$$\frac{1}{1-x} \quad \& \quad \frac{1}{1+x}$$

$$\frac{1}{1-x^2} = \frac{1}{2} \left[\frac{1}{1-x} + \frac{1}{1+x} \right]$$

$$\mathbf{M} \left[\left(\frac{1}{1-x} \right)_+ \right] \left(\frac{N}{2} \right) = \mathbf{M} \left[\left(\frac{1}{1-x} \right)_+ \right] (N) + \mathbf{M} \left[\frac{1}{1+x} \right] (N) + \ln(2)$$

$$-\psi \left(\frac{N}{2} \right) - \gamma_E = -\psi(N) - \gamma_E + \beta(N) + \ln(2); \quad \beta(N) = \frac{1}{2} \left[\psi \left(\frac{N+1}{2} \right) - \psi \left(\frac{N}{2} \right) \right]$$

- $S_{-1}(N)$ depends on $S_1(N)$ for $N \in \mathbf{Q}$

Structural Relations

$N \in \mathbf{R}$:

$$S_2(N) = -\frac{d}{dN} S_1(N) + \zeta_2 \quad (\text{etc.})$$

For $N \in \mathbf{R}$: only one independent single sum occurs.

$$S_1(N) = \sum_{k=1}^N \frac{1}{k} = \psi(N+1) + \gamma_E$$

Harmonic sums $\cup \zeta_{k_1, \dots, k_n}$ are closed under differentiation.

$w = 2$:

$$\mathbf{M} \left[\frac{\ln(1-x)}{1+x} \right] (N) = -\mathbf{M} \left[\frac{\ln(1+x)}{1+x} \right] (N) - [\psi(N) + \gamma_E + \ln(2)]\beta(N) + \beta'(N)$$

$$F_1(N) := \mathbf{M} \left[\frac{\ln(1+x)}{1+x} \right] (N) \rightarrow S_{1,-1}(N)$$

Structural Relations

The relations for $w = 2$ were explored by N. Nielsen (1906).

$$\begin{aligned} \xi(N) &= \mathbf{M} \left[\left(\frac{\ln(1-x)}{1+x} \right)_+ \right] (N); & \eta(N) &= \mathbf{M} \left[\frac{\ln(1+x) - \ln(2)}{1-x} \right] (N) \\ \xi_1(N) &= \mathbf{M} \left[\frac{\ln(1+x)}{1+x} \right] (N); & -\xi_2(N) &= \mathbf{M} \left[\frac{\ln(1-x)}{1+x} \right] (N) \end{aligned}$$

$$\begin{aligned} [\psi(z) + \gamma_E][\psi(1-z) + \gamma_E] &= 2\zeta_2 - \xi(z) - \xi(1-z) \\ \beta(z)[\psi(z) + \gamma_E] &= \beta'(z) + \beta(z) \ln(2) - \xi_1(z) + \xi_2(z) \\ \beta(z)\beta(1-z) &= \eta(z) + \eta(1-z) \\ \beta^2(z) &= \psi'(z) - 2\eta(z) \end{aligned}$$

Structural Relations

If half-integer arguments in N are allowed

$\mathbf{M}[\text{Li}_k(-x)/(x \pm 1)](N)$ are not independent functions :

$$\frac{1}{2^{k-2}} \frac{\text{Li}_k(x^2)}{1-x^2} = \frac{\text{Li}_k(x)}{1-x} + \frac{\text{Li}_k(x)}{1+x} + \frac{\text{Li}_k(-x)}{1-x} + \frac{\text{Li}_k(-x)}{1+x} \rightarrow \frac{\text{Li}_k(-x)}{1-x}$$

- There always exists another IBP relation to express also $\text{Li}_k(-x)/(1+x)$

$$\begin{aligned} (-1)^N \mathbf{M} \left[\frac{\text{Li}_2(-x)}{1+x} \right] (N) &= -S_{2,-1}(N) - \ln(2)[S_2(N) - S_{-2}(N)] \\ &\quad - \frac{1}{2} \zeta_2 S_{-1}(N) + \frac{1}{4} \zeta_3 - \frac{1}{2} \zeta_2 \ln(2) \end{aligned}$$

$$\begin{aligned} (-1)^N \mathbf{M} \left[\frac{-\text{Li}_2(x) - \ln(x) \ln(1-x) + \zeta_2}{1+x} \right] (N) &= -S_{-1,2}(N) + \zeta_2 S_{-1}(N) - \zeta_3 + \frac{3}{2} \zeta_2 \ln(2) \\ S_{-1,2}(N) + S_{2,-1}(N) &= S_{-1}(N) S_2(N) + S_{-3}(N) \end{aligned}$$

Structural Relations

$$(-1)^{(N+1)} \mathbf{M} \left[\frac{\text{Li}_3(-x)}{1+x} \right] (N) = -S_{3,-1}(N) - \ln(2)[S_3(N) - S_{-3}(N)] \\ - \frac{1}{2} \zeta_2 S_{-2}(N) + \frac{3}{4} \zeta_3 S_{-1}(N) - \frac{1}{8} \zeta_2^2 + \frac{3}{4} \ln(2) \zeta_3$$

$$(-1)^N \mathbf{M} \left[\frac{S_{1,2}(1-x)}{1+x} \right] (N) = -S_{-1,3}(N) + \zeta_3 S_{-1}(N) - \frac{19}{40} \zeta_2^2 + \frac{7}{4} \zeta_3 \ln(2)$$

$$S_{1,2}(1-x) = -\text{Li}_3(x) + \log(x) \text{Li}_2(x) + \frac{1}{2} \log(1-x) \log^2(x) + \zeta_3$$

$$S_{-1,3}(N) + S_{3,-1}(N) = S_{-1}(N) S_3(N) + S_{-4}(N)$$

- At even w there exists an algebraic relation

$$S_{w/2, w/2}(N) = \frac{1}{2} \left[S_{w/2}^2(N) + S_w(N) \right]$$

which yields an additional relation for $\text{Li}_k(x)/(1+x)$.

$w = 3$:

$$\rightarrow \frac{\text{Li}_2(x)}{x \pm 1}, \quad \frac{\ln^2(1+x)}{x \pm 1}$$

Double Sums in General

- Applying differential operators one may show :

For $N \in \mathbf{R}$ double harmonic sums can always be represented by one basic function for even weight and two basic functions for odd weight.

$$\implies \frac{\text{Li}_k(x)}{1+x}, \quad \frac{\text{Li}_k(x)}{1 \pm x}$$

Examples, which reduce :

$$S_{2,3}(N) = \mathbf{M} \left[\left(\frac{\ln(x) [S_{1,2}(1-x) - \zeta_3] + 3 [S_{1,3}(1-x) - \zeta_4]}{x-1} \right)_+ \right] (N) + 3\zeta_4 S_1(N)$$

$$S_{-4,-2}(N) = -\mathbf{M} \left[\left(\frac{4\text{Li}_5(-x) - \ln(x)\text{Li}_4(-x)}{x-1} \right)_+ \right] (N) \\ + \frac{1}{2}\zeta_2 [S_4(N) - S_{-4}(N)] - \frac{3}{2}\zeta_3 S_3(N) + \frac{21}{8}\zeta_4 S_2(N) - \frac{15}{4}\zeta_5 S_1(N)$$

$$S_{1,3}(1-x) = -\text{Li}_4(x) + \log(x)\text{Li}_3(x) - \frac{1}{2}\log^2(x)\text{Li}_2(x) - \frac{1}{6}\log^3(x)\log(1-x) + \zeta_4$$

Structural Relations

$w = 4; i \neq -1$:

$$\frac{\text{Li}_3(x)}{x+1}, \quad \frac{S_{1,2}(x)}{x \pm 1}$$

The Mellin transform of

$$\left(\frac{\text{Li}_3(x)}{x-1} \right)_+$$

reads

$$\mathbf{M} \left[\left(\frac{\text{Li}_3(x)}{x-1} \right)_+ \right] (N) = \frac{1}{2} \left\{ \frac{d}{dN} \mathbf{M} \left[\left(\frac{\text{Li}_2(x) + \zeta_2}{x-1} \right)_+ \right] (N) \right. \\ \left. - S_{2,2}(N-1) + \zeta_2 S_2(N-1) + 2\zeta_3 S_1(N-1) \right\}$$

and can be traced back to that of $(\text{Li}_2(x)/(x-1))_+$

$w = 5; i \neq -1$:

$$\begin{array}{ccccc}
 \frac{\text{Li}_4(x)}{x \pm 1} & \frac{S_{1,3}(x)}{x + 1} & \frac{S_{2,2}(x)}{x \pm 1} & \frac{\text{Li}_2^2(x)}{x + 1} & \frac{\ln(x)S_{1,2}(-x) - \text{Li}_2^2(-x)/2}{x \pm 1} \\
 \frac{\text{Li}_2^2(x)}{x - 1} & \frac{S_{1,3}(x)}{x - 1} & & &
 \end{array}$$

[Occur in the 3 – loop Wils. Coeff. only]

$w = 6; i \neq -1$:

$$\begin{array}{ccccc}
 \frac{\text{Li}_5(x)}{x + 1} & \frac{S_{3,2}(x)}{x \pm 1} & \frac{S_{2,3}(x)}{x \pm 1} & \frac{S_{1,4}(x)}{x \pm 1} & \frac{\text{Li}_2(x)\text{Li}_3(x)}{x \pm 1} \\
 \frac{S_{1,2}(x)\text{Li}_2(x)}{x + 1} & \frac{A_1(x)}{x + 1} & \frac{A_2(x)}{x \pm 1} & \frac{A_3(x)}{x + 1} & \frac{H_{0,-1,0,1,1}(x)}{x \pm 1} \\
 \frac{H_{0,0,-1,0,1}(x)}{x \pm 1} & \frac{A_1(-x) + 2S_{3,2}(-x) - 2S_{2,2}(-x)\ln(x)}{x \pm 1} & & & \\
 \frac{A_1(-x) + 2S_{3,2}(-x) - 2S_{2,2}(-x)\ln(x) + \text{Li}_2^2(-x)\ln(x)/4 - \text{Li}_3(-x)\text{Li}_2(-x)}{x - 1} & & & &
 \end{array}$$

New numerator functions :

$$A_1(x) = \int_0^x \frac{dy}{y} \text{Li}_2^2(y), \quad A_2(x) = \int_0^x \frac{dy}{y} \ln(1 - y)S_{1,2}(y), \quad A_3(x) = \int_0^x \frac{dy}{y} [\text{Li}_4(1 - y) - \zeta_4]$$

Harmonic Polylogarithms

- iterated integrals over the alphabet

$$f_a(x) = \frac{1}{x}, \quad \frac{1}{1-x}, \quad \frac{1}{1+x}$$

$$H_0(x) = \int_0^x \frac{dx}{x}, \quad H_1(x) = \int_0^x \frac{dx}{1-x}, \quad H_{-1}(x) = \int_0^x \frac{dx}{1+x}$$

$$H_{a,\vec{b}}(x) = \int_0^x dz f_a(z) H_{\vec{b}}(z)$$

$w = 5; i \neq -1$:

$$\begin{array}{ccccc}
 \frac{\text{Li}_4(x)}{x \pm 1} & \frac{S_{1,3}(x)}{x + 1} & \frac{S_{2,2}(x)}{x \pm 1} & \frac{\text{Li}_2^2(x)}{x + 1} & \frac{\ln(x)S_{1,2}(-x) - \text{Li}_2^2(-x)/2}{x \pm 1} \\
 \frac{\text{Li}_2^2(x)}{x - 1} & \frac{S_{1,3}(x)}{x - 1} & & &
 \end{array}$$

[Occur in the 3 – loop Wils. Coeff. only]

$w = 6; i \neq -1$:

$$\begin{array}{ccccc}
 \frac{\text{Li}_5(x)}{x + 1} & \frac{S_{3,2}(x)}{x \pm 1} & \frac{S_{2,3}(x)}{x \pm 1} & \frac{S_{1,4}(x)}{x \pm 1} & \frac{\text{Li}_2(x)\text{Li}_3(x)}{x \pm 1} \\
 \frac{S_{1,2}(x)\text{Li}_2(x)}{x + 1} & \frac{A_1(x)}{x + 1} & \frac{A_2(x)}{x \pm 1} & \frac{A_3(x)}{x + 1} & \frac{H_{0,-1,0,1,1}(x)}{x \pm 1} \\
 \frac{H_{0,0,-1,0,1}(x)}{x \pm 1} & \frac{A_1(-x) + 2S_{3,2}(-x) - 2S_{2,2}(-x)\ln(x)}{x \pm 1} & & & \\
 \frac{A_1(-x) + 2S_{3,2}(-x) - 2S_{2,2}(-x)\ln(x) + \text{Li}_2^2(-x)\ln(x)/4 - \text{Li}_3(-x)\text{Li}_2(-x)}{x - 1} & & & &
 \end{array}$$

New numerator functions :

$$A_1(x) = \int_0^x \frac{dy}{y} \text{Li}_2^2(y), \quad A_2(x) = \int_0^x \frac{dy}{y} \ln(1 - y)S_{1,2}(y), \quad A_3(x) = \int_0^x \frac{dy}{y} [\text{Li}_4(1 - y) - \zeta_4]$$

Representation of Observables

- Unpolarized and Polarized Drell-Yan and Higgs-Boson Production Cross Section $O(\alpha_s^2)$,
 $w = 4$ JB and V. Ravindran, Nucl. Phys. **B716** (2005) 128.
- Unpolarized and Polarized Time-like Anomalous Dimensions and Wilson Coefficients
 $O(\alpha_s^2)$, $w = 4$ JB and V. Ravindran, Nucl. Phys. **B749** (2006) 1.
- Anomalous Dimensions and Wilson Coefficients $O(\alpha_s^3)$, $w = 5, 6$,
 from: S. Moch, J. Vermaseren, A. Vogt, Nucl. Phys. **B688** (2004) 101; **691** (2004) 129; **B724** (2005) 3 → J.B., DESY 07-042
- Polarized and Unpolarized Wilson Coefficients $O(\alpha_s^2)$, $w = 4$ J.B. and S. Moch
- Polarized and Unpolarized asymptotic Heavy Flavor Wilson Coefficients $O(\alpha_s^{2(3)})$, $w = 4, 5$,
 J.B., A. de Freitas, W. van Neerven, S. Klein, Nucl. Phys. **B755** (2006) 272; I. Bierenbaum, J.B., S. Klein, DESY 07-026,
 DESY 07-027; DESY-08-029;
- Virtual and soft corrections to Bhabha Scattering $O(\alpha^2)$, $w = 4$,
 J.B. and S. Klein, arXiv:0706.2426 [hep-ph]

Example: Bhabha s+v

$$\begin{aligned}
 T_0 = & \frac{248 + 15 N^2 + N^4}{2(N-2)(N-1)N(N+1)(N+2)} S_{1,1,1,1}(N) + \frac{-2}{(N-1)(N+1)} \mathbf{S}_{2,1,1}(N) \\
 & + \frac{-340 + 120 N + 17 N^2 + 18 N^3 - 31 N^4}{2(N-2)(N-1)N(N+1)(N+2)} S_{3,1}(N) + \frac{1344 - 502 N - 69 N^2 - 2 N^3 + 57 N^4}{8(N-2)(N-1)N(N+1)(N+2)} S_4(N) \\
 & + \frac{304 - 328 N - 500 N^2 + 330 N^3 - 6 N^4 + 6 N^5 - 2 N^6 + 4 N^7}{(N-2)^2(N-1)^2 N^2(N+1)(N+2)} \mathbf{S}_{2,1}(N) \\
 & + \frac{-112 - 4 N^2 - 4 N^4}{(N-2)(N-1)N(N+1)(N+2)} S_{2,1}(N) \mathbf{S}_1(N) + \frac{-48 + 8 N + 6 N^2 + 7 N^3}{(N-1)N(N+1)(N+2)} S_3(N) S_1(N) \\
 & + \frac{-1840 + 292 N + 5532 N^2 + 827 N^3 - 1978 N^4 - 274 N^5 + 36 N^6 + 19 N^7 - 22 N^8}{4(N-2)^2(N-1)^2 N^2(N+1)^2(N+2)} S_{1,1,1}(N) \\
 & + \frac{128 - 56 N - 252 N^2 + 54 N^3 + 177 N^4 - 91 N^5 + 19 N^6 + 9 N^7}{2(N-2)(N-1)^2 N^2(N+1)^2(N+2)} S_3(N) \\
 & + \frac{4032 - 2048 N - 14200 N^2 + 5036 N^3 + 23610 N^4 + 2521 N^5 - 12342 N^6}{4(N-2)^3(N-1)^3 N^3(N+1)^3(N+2)} S_{1,1}(N) \\
 & + \frac{-3365 N^7 + 2148 N^8 + 903 N^9 + 14 N^{10} - 167 N^{11} + 50 N^{12}}{4(N-2)^3(N-1)^3 N^3(N+1)^3(N+2)} S_{1,1}(N) \\
 & + \frac{-124 + 16 N + 24 N^2 - 4 N^3 - 14 N^4}{(N-2)(N-1)N(N+1)(N+2)} S_{1,1}(N) \zeta(2) + \frac{424 - 118 N + 9 N^2 - 2 N^3 + 23 N^4}{4(N-2)(N-1)N(N+1)(N+2)} S_2(N) S_{1,1}(N) \\
 & + \frac{224 + 144 N - 1216 N^2 - 56 N^3 + 1786 N^4 + 641 N^5 - 406 N^6}{4(N-2)^2(N-1)^3 N^3(N+1)^3(N+2)} S_2(N) \\
 & + \frac{+17 N^7 - 308 N^8 + 141 N^9 - 56 N^{10} + N^{11}}{4(N-2)^2(N-1)^3 N^3(N+1)^3(N+2)} S_2(N) + \frac{58 + 21 N + N^2 + 15 N^3 + 10 N^4}{(N-2)(N-1)N(N+1)(N+2)} S_2(N) \zeta(2)
 \end{aligned}$$

Example: Bhabha s+v

$$\begin{aligned}
 & + \frac{232 - 384 N^2 - 17 N^3 + 286 N^4 - 128 N^5 - 14 N^6 + N^7}{4(N-2)(N-1)^2 N^2 (N+1)^2 (N+2)} S_2(N) S_1(N) \\
 & + \frac{-560 - 26 N - 31 N^2 - 10 N^3 - 33 N^4}{8(N-2)(N-1)N(N+1)(N+2)} S_2(N)^2 \\
 & + \frac{576 + 1088 N - 3280 N^2 - 5136 N^3 + 11764 N^4 + 20392 N^5 - 17385 N^6 - 30114 N^7}{4(N-2)^3 (N-1)^4 N^4 (N+1)^4 (N+2)} S_1(N) \\
 & + \frac{+5984 N^8 + 17228 N^9 - 1228 N^{10} - 2754 N^{11} - 112 N^{12} - 8 N^{13} + 33 N^{14} - 24 N^{15}}{4(N-2)^3 (N-1)^4 N^4 (N+1)^4 (N+2)} S_1(N) \\
 & + \frac{-56 + 336 N + 522 N^2 + 424 N^3 - 53 N^4 - 500 N^5 + 60 N^6 + 28 N^7 - 5 N^8}{2(N-2)^2 (N-1)^2 N^2 (N+1)^2 (N+2)} S_1(N) \zeta(2) \\
 & + \frac{64 + 6 N^2 + N^3}{(N-2)(N-1)N(N+1)} S_1(N) \zeta(3) + \frac{2112 + 608 N + 76 N^2 - 140 N^3 + 107 N^4}{10(N-2)(N-1)N(N+1)(N+2)} \zeta(2)^2 \\
 & + \frac{-224 - 136 N + 1688 N^2 + 1290 N^3 - 1998 N^4 - 1997 N^5 + 198 N^6}{2(N-1)^3 N^3 (N-2)^2 (N+2)(N+1)^3} \zeta(2) \\
 & + \frac{+405 N^7 + 376 N^8 - 119 N^9 + 56 N^{10} + 5 N^{11}}{2(N-1)^3 N^3 (N-2)^2 (N+2)(N+1)^3} \zeta(2) \\
 & + \frac{-552 + 144 N + 1654 N^2 - 370 N^3 - 361 N^4 + 19 N^5 + 35 N^6 - 25 N^7}{2(N-2)^2 (N-1)^2 N^2 (N+1)^2} \zeta(3) \\
 & + \frac{320 - 64 N - 1920 N^2 + 1600 N^3 + 6524 N^4 - 14872 N^5 - 19036 N^6 + 31543 N^7 - 43960 N^8 - 13935 N^9}{16(N-1)^5 (N+1)^5 (N-2)^3 N^5 (N+2)} \\
 & + \frac{+65372 N^{10} + 26822 N^{11} - 44576 N^{12} - 9558 N^{13} + 9840 N^{14} + 339 N^{15} + 428 N^{16} - 371 N^{17} + 128 N^{18}}{16(N-1)^5 (N+1)^5 (N-2)^3 N^5 (N+2)} \\
 & + 4 \frac{N^4 - N^2 + 12}{(N-2)(N-1)N(N+1)(N+2)} f_{0,2} + (-2) \frac{N^4 - N^2 + 12}{(N-2)(N-1)N(N+1)(N+2)} f_{0,1}^2
 \end{aligned}$$

z-space: A. Penin, (2005)
 \implies 3 basic sums only; no alternating sums.

5. Factorial Series

Consider

$$\Omega(z) = \int_0^1 dt t^{z-1} \varphi(t); \quad \varphi(1-t) = \sum_{k=0}^{\infty} a_k t^k$$

$$\operatorname{Re}(z) > 0, \quad \Omega(z) = \sum_{k=0}^{\infty} \frac{a_{k+1} k!}{z(z+1)\dots(z+k)}$$

- $\Omega(z)$ is meromorphic in $z \in \mathbf{C}$, obeys a recursion $z \rightarrow z + 1$ and has an analytic asymptotic representation.
- The poles are situated at the non-positive integers.

Examples:

$$F_5(z) = \mathbf{M} \left[\frac{\operatorname{Li}_2(z)}{1+z} \right] (z)$$

$$F_5(z+1) = -F_5(z) + \frac{1}{z} \left[\zeta_2 - \frac{\psi(z+1) + \gamma_E}{z} \right]$$

$$\text{Asymp. ser. : } \operatorname{Li}_2(z) \rightarrow -\operatorname{Li}_2(1-z) - \ln(z) \ln(1-z) + \zeta_2$$

$$\mathbf{M} \left[\frac{\operatorname{Li}_2(1-z)}{1+z} \right] (N) \sim \frac{1}{2N^2} + \frac{1}{4N^3} - \frac{7}{24} \frac{1}{N^4} - \frac{1}{3} \frac{1}{N^5} + \frac{73}{120} \frac{1}{N^6} \dots$$

Factorial Series

$$F_{13}(z) = \mathbf{M} \left[\left(\frac{\text{Li}_2^2(z)}{z-1} \right)_+ \right] (z)$$

$$F_{13}(z+1) = -F_{13}(z) + \frac{\zeta_2^2}{z} + \frac{4\zeta_3}{z^2} + \frac{2\zeta_2}{z^2} S_1(z) + \frac{2S_{2,1}(z)}{z^2} + \frac{2}{z^3} [S_1^2(z) + S_2(z)]$$

$$\text{Asymp. ser. : } \text{Li}_2^2(z) \rightarrow \text{Li}_2^2(1-z) + \ln^2(z) \ln^2(1-z) + \zeta_2^2 + 2\text{Li}_2(1-z) \ln(z) \ln(1-z) + \dots$$

$$\begin{aligned} \mathbf{M} \left[\left(\frac{\text{Li}_2^2(1-z)}{z-1} \right)_+ \right] (N) &\sim \frac{1}{z^2} - \frac{7}{24} \frac{1}{z^4} + \frac{1}{12} \frac{1}{z^5} + \frac{223}{1080} \frac{1}{z^6} - \frac{7}{45} \frac{1}{z^7} - \frac{3767}{15120} \frac{1}{z^8} + \frac{38}{105} \frac{1}{z^9} \\ &+ \frac{14327}{31500} \frac{1}{z^{10}} - \frac{198}{175} \frac{1}{z^{11}} - \frac{138673}{118800} \frac{1}{z^{12}} + \frac{3263}{693} \frac{1}{z^{13}} + \frac{5265804043}{1324323000} \frac{1}{z^{14}} \\ &- \frac{13399637}{525525} \frac{1}{z^{15}} - \frac{143341487}{8408400} \frac{1}{z^{16}} + \frac{25092}{143} \frac{1}{z^{17}} + \frac{34809672614}{402026625} \frac{1}{z^{18}} \\ &- \frac{5749693892}{3828825} \frac{1}{z^{19}} + O\left(\frac{1}{z^{20}}\right) \end{aligned}$$

6. The Basis

$w = 1$	$1/(x - 1)_+$		
$w = 2$	$\ln(1 + x)/(x + 1)$		
$w = 3$	$\text{Li}_2(x)/(x \pm 1)$		
$w = 4$	$\text{Li}_3(x)/(x + 1)$	$S_{1,2}(x)/(x \pm 1)$	
$w = 5$	$\text{Li}_4(x)/(x \pm 1)$	$S_{1,3}(x)/(x \pm 1)$	$S_{2,2}(x)/(x \pm 1)$
	$\text{Li}_2^2(x)/(x \pm 1)$	$[\ln(x)S_{1,2}(-x) - \text{Li}_2^2(-x)/2]/(x \pm 1)$	
$w = 6$	$\text{Li}_5(x)/(x + 1)$	$S_{1,4}(x)/(x \pm 1)$	$S_{2,3}(x)/(x \pm 1)$
	$S_{3,2}(x)/(x \pm 1)$	$\text{Li}_2(x)\text{Li}_3(x)/(x \pm 1)$	
	$A_1(x)/(x + 1)$	$A_2(x)/(x \pm 1)$	$A_3(x)/(x + 1)$
	$H_{0,-1,0,1,1}(x)/(x \pm 1)$	$H_{0,0,-1,0,1}(x)/(x \pm 1)$	

$$\frac{[A_1(-x) + 2S_{3,2}(-x) - 2S_{2,2}(-x) \ln(x)]}{x \pm 1}$$

$$\frac{A_1(-x) + 2S_{3,2}(-x) - 2S_{2,2}(-x) \ln(x) + \text{Li}_2^2(-x) \ln(x)/4 - \text{Li}_3(-x)\text{Li}_2(-x)}{x - 1}$$

- $O(\alpha)$ Wilson Coefficients/anom. dim. #1
- $O(\alpha^2)$ Anomalous Dimensions #2
- $O(\alpha^2)$ Wilson Coefficients # ≤ 5
- $O(\alpha^3)$ Anomalous Dimensions #15
- $O(\alpha^3)$ Wilson Coefficients #35

7. Conclusions - A

- We considered mathematical structures which determine **no scale** and **single scale** quantities in **Quantum Field Theories**.
- The former correspond to **integrated cross sections**, expansion coefficients of the β -function, or **anomalous dimensions at fixed moments**, etc.
- The latter correspond to **differential scattering cross sections of one variable**, N -dependent **anomalous dimensions**, coefficient functions, etc.
- The single-scale quantities in **Quantum Field Theories to 3 Loop Order** $\Leftrightarrow w = 6$ can be represented in a polynomial ring spanned by **a few Mellin transforms** of the above **basic functions**, which are the same for all known processes. This points to their general nature.
- The **basic Mellin transforms** are meromorphic functions with single poles at the non-positive integers.
- The total amount of harmonic sums reduces due to **algebraic relations** [index structure], and **structural relations** $N \in \mathbf{Q}$, $N \in \mathbf{R}$.

- They can be represented in terms of **factorial series** up to simple “soft components”. This allows an exact **analytic continuation**.
- Up to **w = 6** physical (pseudo-) observables are free of harmonic sums with **index = {-1}**. Up to **w = 5** all numerator functions are **Nielsen integrals**.

8. From Moments to Functions in QCD

- Higher order calculations in Quantum Field Theories easily become tedious due to the larger number of terms and the sophistication of the Feynman parameter integrals.
- This even applies to **Zero Scale** and **Single Scale** Quantities.
- Even more this is the case for **higher scale** problems.
- While in the latter case the mathematical structure of the solution for the Feynman Integrals is widely unknown, it is explored to **a certain extent** for **Zero Scale** and **Single Scale** quantities.
- They can be expressed by **rational numbers** and certain **special numbers** as **multiple ζ -values** and related quantities.

- **Single Scale** quantities depend on a scale $z \in [0, 1]$, with z a ratio of Lorentz invariants. One may perform a **Mellin Transform** over z

$$\int_0^1 dz z^{N-1} f(z) = M[f](N)$$

- Here one assumes $N \in \mathbf{N}, N > 0$. Due to this the problem on hand becomes **discrete**.
- One may seek a description in terms of **difference equations**.
- **Zero Scale** problems are obtained from **Single Scale** problems treating N as a fixed integer or considering the limit $N \rightarrow \infty$.
- Can one reconstruct the general formula for **Single Scale** quantities out of a **finite number** of fixed moments ? This is possible for recurrent quantities.
- At least up to **3-loop order**, presumably to higher orders, single scale quantities belong to this class.
- Goal : design a general formalism to solve the problem.

9. Single Scale Feynman Integrals as Recurrent Quantities

- Can one reconstruct the general formula for **Single Scale** quantities out of a **finite number** of fixed moments ?
- **Polynomials and Nested Harmonic Sums** obey recurrence relations, so do their polynomials.
- Example: Harmonic Sums or linear combinations thereof:

$$F(N + 1) - F(N) = \frac{\text{sign}(a)^{N+1}}{(N + 1)^{|a|}}$$

is solved by $S_a(N)$; and similarly for deeper nested sums

$$S_{a,\vec{b}}(N) = \sum_{k=1}^N \frac{(\text{sign}(a))^k}{k^{|a|}} S_{\vec{b}}(k)$$

.

Single Scale Feynman Integrals as Recurrent Quantities

- Feynman integrals have often a form like

$$\int_0^1 dz \frac{z^{N-1} - 1}{1 - z} H_{\vec{a}}(z), \quad \int_0^1 dz \frac{(-z)^{N-1} - 1}{1 + z} H_{\vec{a}}(z)$$

- This structure leads to recurrences.
- It is very likely that single scale Feynman diagrams do always obey difference equations.

10. Establishing and Solving Recurrences

- One seeks the relation

$$\sum_{k=0}^l \left[\sum_{i=0}^d c_{i,k} N^i \right] F(N+k) = 0 .$$

- The corresponding linear system is dense.
- Rational number arithmetics is **not applicable** for the large systems to be determined; $C_{2,q,C_F}^{(3)}$ would require 11 Tb of memory.
- Use arithmetic in **finite fields** together with **Chinese remaindering**
 \implies few Gb of memory
- The linear system approximately minimizes for $l \approx d$.
- Join different recurrences found to reduce l to a minimal value.

Establishing and Solving Recurrences

- For the solution of the recurrence **low degrees** are clearly preferred.
- The linear difference equation of order l with **polynomial coefficients** is equivalent to a linear system in l variables.
- It is solved in **$\Pi - \Sigma$ fields**.
- Apply **advanced symbolic summation** methods: telescoping, creative telescoping and its refinement. Code: **sigma**.
- The solutions are found as linear combinations of rational terms in N combined with functions, which cannot be further reduced in the **$\Pi - \Sigma$ fields**. In the present application they turn out all to be harmonic sums $S_{\vec{b}}(N)$.
- Other or higher order applications may consist of **other sums** too, which are **uniquely found** by the algorithm.

11. Application to 3-Loop Anomalous Dimensions and Wilson coefficients

- We apply the method for the unfolding of the unpolarized **anomalous dimensions** and **Wilson coefficients** up to **3-loop** order.
- \implies analyze for individual color factors; **141** contributions from **1 – 3 loops**
- Input: Moch, Vermaseren, Vogt, 2004/05. The expressions are given in terms of harmonic sums.
- Calculate the moments (**rational numbers**) recursively through recursions for the harmonic sums; **MAPLE** code.
- Establish the corresponding difference equation by a **recurrency finder**; build a difference equation of **minimal** order possible; test the recurrency.
- Solve the difference equation order by order with the summation package **sigma** C. Schneider.; most complicated cases: 4 weeks @ $\leq 10\text{Gb}$, 2 GHz Proc.

Input

C2qq3CF³

N=3:

#11 digits / #10 digits

-98268084191 / 1166400000

N=500:

#1262 digits / #1256 digits

1641840770424196780953020619176376506284303544481262083057197600746507008493793994
4224110323441591630311482222058287688942209570859151121677307585313995100978363179
2518952817622034037186132846974627021672678012913675099511203807811938593043910803
5044345920218696052588332036355325089998361354226882367322149037631053761764348772
5403810874264968729520075619227285471802419403727207822473765999900236383740315299
2050533601633484348249454757555344664210814111140065475391136798689167410065076749
3578709478683390573977410013520894494463909291327425815766566386397276158317387748
5945471392646089700875157445075073192328542890965462004805711998748144414379386093
9937361798029044425789953726133675199790523770427298500510063464061985840066296071
3372543015648919155964069606994597363886301185067827291937065300754786947063672848
9382081926871078600328628131936766057475970450896556667622163365895808773428119721
5352792131089063577045069693962213061198894057033606068695607123271969726981060056
0115846094360239986233917872260722277322690450132376836253549152130116645670565045
9666945920164586023958060271746606798898861360772333088030741775605546518788793327
2264368297071217405654474375844238250889238538974548421298170425909521742559494728
72017877003947396562261659860366839154407853462338171648227013134266795320251847
/

3057444614247225372882570514367358697278130741348282122206492932820352440850471902
7491046962105336645563654873675690796713906565688820365601907263710863954826386081
3227580037879361869941003802807590860358894142891046776447162895908787986423254678
5776778283337231702130612499429819559798501074020676282769289102955679421885795867
1982932998601320344971927374905889934059987271939760212836368619501189238215442366
3805773701929509268157747992859384837403751183019423692868569168206789710047557452
5131217382272060267681480496298975522467614707848639773185909858278799786637303834
1017166676276847525704755493166263297079720470719813623901545811953853986456533543
9994182050551827959988760121168490745476969259468454613431624179198860751513076481
0304734205926703138519418575731315944374873897873646706993620825697218523316375559
4068222004765962715924208526106008109740402380126260947524640509361283802755722132
4856690051525724685919792641506082307567956962328560073471086799287131287564668441
6256698083504233897436484702002471314330803421467773925541151273924985946178771189
231243716221343813770389606473498715702080141315355435311326719739117599044341913
5922693587373856609594245948237469293148702516714038297077639382332251255360181047
496586232475091126597629976797375278827111116774593003520000000000000000

N=5114:

#13388 digits / #13381 digits

Table 1: Run parameters for the unfolding of the non-singlet anomalous dimensions

	number of terms needed	order of recurrence	degree of recurrence	total time [sec]	length of recurrence [kbyte]	number of harm. sums a [b]	solution time [sec]
$P_{NS,0}$	14	2	3	0.05	0.087	1 [1]	0.55
$P_{NS,1,C_F^2}^-$	142	5	31	3.32	4.666	6 [10]	7.45
$P_{NS,1,C_A C_F}^-$	109	4	24	1.91	2.834	6 [7]	6.28
$P_{NS,1,C_F N_F}^-$	24	2	7	0.13	0.271	2 [2]	0.92
$P_{NS,1,C_F^2}^+$	142	5	31	3.35	4.707	6 [10]	7.45
$P_{NS,1,C_A C_F}^+$	109	4	23	1.88	2.703	6 [7]	6.23
$P_{NS,1,C_F N_F}^+$	24	2	7	0.09	0.271	2 [2]	0.89
$P_{NS,2,C_F^3}^-$	1079	16	192	3152.19	529.802	25 [68]	1194.41
$P_{NS,2,C_F^3 \zeta_3}^-$	48	3	11	0.49	0.643	1 [1]	1.56
$P_{NS,2,C_A C_F^2}^-$	974	15	181	1736.08	450.919	25 [62]	1194.41
$P_{NS,2,C_A C_F^2 \zeta_3}^-$	48	3	11	0.53	0.643	1 [1]	1.53
$P_{NS,2,C_A^2 C_F}^-$	749	12	147	1004.12	242.892	25 [62]	1100.88
$P_{NS,2,C_A^2 C_F \zeta_3}^-$	48	3	11	0.56	0.643	1 [1]	1.56
$P_{NS,2,C_F N_F^2}^-$	39	2	11	0.31	0.369	3 [3]	1.20
$P_{NS,2,C_F^2 N_F}^-$	377	8	68	76.34	33.946	12 [24]	72.22
$P_{NS,2,C_F^2 N_F \zeta_3}^-$	14	2	3	0.12	0.101	1 [1]	0.53
$P_{NS,2,C_A C_F N_F}^-$	356	7	62	65.25	23.830	12 [20]	52.67
$P_{NS,2,C_A C_F N_F \zeta_3}^-$	14	2	3	0.12	0.101	1 [1]	0.55
$P_{NS,2,C_F^3}^+$	1079	16	192	4713.27	527.094	25[68]	1165.22
$P_{NS,2,C_F^3 \zeta_3}^+$	48	3	11	0.55	0.643	1[1]	1.562
$P_{NS,2,C_A C_F^2}^+$	974	15	178	1715.03	442.031	25[62]	889.047
$P_{NS,2,C_A C_F^2 \zeta_3}^+$	48	3	11	0.61	0.643	1[1]	1.531
$P_{NS,2,C_A^2 C_F}^+$	749	12	146	991.22	240.325	25[50]	516.812
$P_{NS,2,C_A^2 C_F \zeta_3}^+$	48	3	11	0.61	0.643	1[1]	1.593
$P_{NS,2,C_F^2 N_F}^+$	377	8	69	111.38	33.872	12[24]	71.235
$P_{NS,2,C_F^2 N_F \zeta_3}^+$	14	2	3	0.15	0.101	1[1]	0.531
$P_{NS,2,C_A C_F N_F}^+$	307	7	61	48.62	23.808	12[24]	71.235
$P_{NS,2,C_A C_F N_F \zeta_3}^+$	14	2	3	0.15	0.101	1[1]	0.547
$P_{NS,2,C_F N_F^2}^+$	39	2	11	0.40	0.369	3[3]	1.172
$P_{NS,2,N_F^{d_{abc}}}^-$	39	2	11	0.55	0.369	3 [3]	1.19

Table 2: Run parameters for the unfolding of the unpolarized quarkonic Wilson Coefficients for the structure function $F_2(x, Q^2)$.

	number of terms needed	order of recurrence	degree of recurrence	total time [sec]	length of recurrence [kbyte]	number of harm. sums a [b]	solution time [sec]
$C_{2,q,C_F}^{(1)}$	35	3	7	0.26	0.429	2[3]	1.13
$C_{2,q,C_F^2}^{(2)}$	689	11	137	1134.10	177.806	13[39]	258.24
$C_{2,q,C_A C_F}^{(2)}$	545	10	121	413.33	127.893	12[35]	178.73
$C_{2,q,C_F^2\zeta_3}^{(2)}$	15	2	3	0.27	0.100	1[1]	0.54
$C_{2,q,C_A C_F \zeta_3}^{(2)}$	15	2	3	0.27	0.112	1[1]	0.55
$C_{2,q,N_F C_F}^{(2)}$	71	4	16	2.68	1.655	4[10]	3.95
$C_{2,q,C_F^3}^{(3)}$	5114	35	938	1.78886×10^6	30394.173	58[289]	0.50924×10^6
$C_{2,q,C_F^3\zeta_3}^{(3)}$	284	8	64	31.02	32.363	7 [18]	27.60
$C_{2,q,C_F^3\zeta_4}^{(3)}$	19	2	5	0.08	0.163	1 [1]	0.47
$C_{2,q,C_F^3\zeta_5}^{(3)}$	19	2	5	0.08	0.163	1 [1]	0.47
$C_{2,q,C_F^2 C_A}^{(3)}$	5059	35	930	1.69267×10^6	30122.380	60 [290]	0.47780×10^6
$C_{2,q,C_F^2 C_A \zeta_3}^{(3)}$	284	8	64	34.00	33.400	7 [18]	28.53
$C_{2,q,C_F^2 C_A \zeta_4}^{(3)}$	48	3	11	0.32	0.643	1[1]	1.01
$C_{2,q,C_F^2 C_A \zeta_5}^{(3)}$	19	2	5	0.08	0.167	1 [1]	0.42
$C_{2,q,C_F C_A^2}^{(3)}$	4564	33	863	1.38918×10^6	24567.518	60 [258]	0.34941×10^6
$C_{2,q,C_F C_A^2 \zeta_3}^{(3)}$	284	8	63	26.83	29.918	7 [17]	30.46
$C_{2,q,C_F C_A^2 \zeta_4}^{(3)}$	48	3	11	0.32	0.643	1 [1]	1.01
$C_{2,q,C_F C_A^2 \zeta_5}^{(3)}$	19	2	5	0.08	0.175	1 [1]	0.42
$C_{2,q,C_F^2 N_F}^{(3)}$	1762	20	348	40237.45	2339.516	29 [107]	7548.56
$C_{2,q,C_F^2 N_F \zeta_3}^{(3)}$	87	4	21	1.94	2.354	3 [5]	2.83
$C_{2,q,C_F^2 N_F \zeta_4}^{(3)}$	15	2	3	0.07	0.101	1 [1]	0.34
$C_{2,q,C_F C_A N_F}^{(3)}$	1847	20	360	47661.64	2507.362	29 [111]	7525.89
$C_{2,q,C_F C_A N_F \zeta_3}^{(3)}$	89	4	24	2.47	2.935	3 [8]	3.19
$C_{2,q,C_F C_A N_F \zeta_4}^{(3)}$	15	2	3	0.06	0.101	1 [1]	0.34
$C_{2,q,C_F N_F^2}^{(3)}$	131	5	30	58.00	5.347	7 [22]	8.97
$C_{2,q,C_F N_F^2 \zeta_3}^{(3)}$	15	2	3	0.06	0.101	1 [1]	0.38
$C_{2,q,dabc}^{(3)}$	1199	14	242	6583.27	738.498	14 [62]	841.24
$C_{2,q,dabc\zeta_3}^{(3)}$	109	4	25	2.33	3.164	2[7]	2.40
$C_{2,q,dabc\zeta_5}^{(3)}$	8	1	2	0.03	0.041	0[0]	0.10

A complicated example

$$\underline{C_{2,q} \propto C_F^3} :$$

- 5114 moments needed. Use a clever way to calculate the input.
 - Largest moment: fraction: numerator 13388 digits; denominator 13381 digits.
 - CPU time to determine the recurrence: 20.7 days.
- modular prediction of the dimension: 4 h; modular LEQ's: 5.8 days; modular operator GCDs: 11 days; Chinese Remainder + Rat. Reconstruction: 3.8 days. 140 large primes needed.
- 31 MB recurrence is established; largest integer: 1227 digits; order: 35; degree: 938
- Solved by sigma within about one week.
 - 3 loop anomalous dimensions: much smaller recurrences & shorter computation times.
- ⇒ In practice no method does yet exists to calculate such a high number of moments.
- ⇒ Existence proof of a quite general and powerful automatic difference-equation solver, standing rather demanding tests.

Structure of the Results

- We carry out all algebraic reductions, J.B. 2003.
- Different color factor contributions lead to the same or nearly the same amount of sums at a given quantity.
- This points to the fact that the amount of harmonic sums is governed by topology rather than the fields and color.
- The linear harmonic sum representations by Vermaseren et al. 2004/05 require many more sums than our representation.
- There are reductions in the number of sums as $264 \longrightarrow 29$.
- Further use of structural relations will lead to maximally 35 sums for the 3-loop Wilson coefficients; J.B. arxiv: 0901.0837, arXiv:0901.3106.

$$\begin{aligned}
P_{qq}^0(n) &= C_F \left[4S_1 - \frac{3n^2 + 3n + 2}{n(n+1)} \right] \\
P_{qq}^{1,-}(n) &= C_F^2 \left[-\frac{3n^6 + 9n^5 + 9n^4 - 5n^3 - 24n^2 - 32n - 24}{2n^3(n+1)^3} - 16S_{-3} \right. \\
&\quad + S_{-2} \left(\frac{16}{n(n+1)} - 32S_1 \right) + S_1 \left(\frac{8(2n+1)}{n^2(n+1)^2} - 16S_2 \right) + \frac{4(3n^2 + 3n + 2)}{n(n+1)} S_2 \\
&\quad \left. - 16S_3 + 32S_{-2,1} + \frac{16(-1)^n}{(n+1)^3} \right] \\
&\quad + C_A C_F \left[-\frac{51n^5 + 102n^4 + 655n^3 + 484n^2 + 12n + 144}{18n^3(n+1)^2} + 8S_{-3} + \frac{268}{9} S_1 \right. \\
&\quad \left. + S_{-2} \left(16S_1 - \frac{8}{n(n+1)} \right) - \frac{44}{3} S_2 + 8S_3 - 16S_{-2,1} - \frac{8(-1)^n}{(n+1)^3} \right] \\
&\quad + C_F N_F \left[\frac{3n^4 + 6n^3 + 47n^2 + 20n - 12}{9n^2(n+1)^2} - \frac{40}{9} S_1 + \frac{8}{3} S_2 \right] \\
P_{qq}^{1,+}(n) &= C_F^2 \left[-\frac{3n^6 + 9n^5 + 9n^4 + 59n^3 + 40n^2 + 32n + 8}{2n^3(n+1)^3} - 16S_{-3} \right. \\
&\quad + S_{-2} \left(\frac{16}{n(n+1)} - 32S_1 \right) + S_1 \left(\frac{8(2n+1)}{n^2(n+1)^2} - 16S_2 \right) \\
&\quad \left. + \frac{4(3n^2 + 3n + 2)}{n(n+1)} S_2 - 16S_3 + 32S_{-2,1} + \frac{16(-1)^n}{(n+1)^3} \right] \\
&\quad + C_A C_F \left[-\frac{51n^5 + 153n^4 + 757n^3 + 851n^2 + 208n - 132}{18n^2(n+1)^3} + 8S_{-3} + \frac{268}{9} S_1 \right. \\
&\quad \left. + S_{-2} \left(16S_1 - \frac{8}{n(n+1)} \right) - \frac{44}{3} S_2 + 8S_3 - 16S_{-2,1} - \frac{8(-1)^n}{(n+1)^3} \right] \\
&\quad + C_F N_F \left[\frac{3n^4 + 6n^3 + 47n^2 + 20n - 12}{9n^2(n+1)^2} - \frac{40}{9} S_1 + \frac{8}{3} S_2 \right] \\
P_{qq}^{2,-}(n) &= C_F^3 \left\{ \left(\frac{64}{n(n+1)} - 128S_1 \right) S_{-2}^2 + \left(\frac{16(3n^6 + 9n^5 + 9n^4 + 17n^3 + 6n^2 + 8n + 2)}{n^3(n+1)^3} \right. \right. \\
&\quad \left. \left. + S_1 \left(\frac{64(3n^2 - n + 1)}{n^2(n+1)^2} - 1408S_2 \right) - \frac{64(3n^2 + 3n - 11)S_2}{n(n+1)} + 1536S_3 + 128S_{-2,1} \right. \right. \\
&\quad \left. \left. - 2304S_{2,1} \right) S_{-2} - \frac{16(3n^2 + 3n + 2)S_2^2}{n(n+1)} - \frac{P_1(n)}{2n^5(n+1)^5} - 576S_{-5} \right. \\
&\quad + S_{-4} \left(-\frac{16(9n^2 + 9n - 26)}{n(n+1)} - 832S_1 \right) \\
&\quad + S_{-3} \left(640S_1^2 - \frac{32(3n^2 + 3n + 20)S_1}{n(n+1)} + \frac{16(21n^2 + 17n + 20)}{n^2(n+1)^2} - 320S_{-2} - 2240S_2 \right) \\
&\quad + (-1)^n \left(-\frac{48(2n^2 - n + 1)}{(n+1)^5} + \frac{128S_{-2}}{(n+1)^3} + \frac{96(5n+3)S_1}{(n+1)^4} - \frac{64S_2}{(n+1)^3} \right) \\
&\quad + \frac{4(13n^4 + 26n^3 + 13n^2 - 16n - 20)S_3}{n^2(n+1)^2} - \frac{16(15n^2 + 15n + 2)S_4}{n(n+1)} - 192S_5 - 832S_{-4,1} \\
&\quad + \frac{896S_{-3,1}}{n(n+1)} + 1152S_{-3,2} + S_1^2 \left(-\frac{32(3n^2 + 3n + 1)}{n^3(n+1)^3} - 768S_{-2,1} \right) - \frac{32(15n^2 + 11n + 16)S_{-2,1}}{n^2(n+1)^2} \\
&\quad + S_2 \left(\frac{2(3n^6 + 9n^5 + 9n^4 + 19n^3 + 12n^2 - 4n - 16)}{n^3(n+1)^3} + 64S_3 + 2176S_{-2,1} \right) \\
&\quad + \frac{32(3n^2 + 3n - 26)S_{2,-2}}{n(n+1)} - 1472S_{3,-2} + \frac{64(3n^2 + 3n - 2)S_{3,1}}{n(n+1)} + 192S_{3,2} + 192S_{4,1} \\
&\quad + 2304S_{-3,1,1} + 512S_{-2,1,-2} + \frac{384(n^2 + n - 4)S_{-2,1,1}}{n(n+1)} + S_1 \left(64S_2^2 - \frac{64(2n+1)S_2}{n^2(n+1)^2} \right. \\
&\quad \left. + \frac{4(22n^6 + 186n^5 + 167n^4 - 40n^3 - 115n^2 - 120n - 44)}{n^4(n+1)^4} - 192S_3 + 64S_4 - 1792S_{-3,1} \right. \\
&\quad \left. - \frac{192(n^2 + n - 4)S_{-2,1}}{n(n+1)} + 1664S_{2,-2} + 256S_{3,1} + 3072S_{-2,1,1} \right) + 2304S_{-2,2,1} + 2304S_{2,1,-2} \\
&\quad - 384S_{3,1,1} - 4608S_{-2,1,1,1} \\
&\quad \left. + \left(C_F^3 - \frac{3}{2} C_F^2 C_A \right) C_3 \left[-\frac{24(5n^4 + 10n^3 + 9n^2 + 4n + 4)}{n^2(n+1)^2} - 192S_{-2} \right] \right\} \\
&\quad + C_A C_F^2 \left\{ \left(256S_1 - \frac{16(3n^2 + 3n + 8)}{n(n+1)} \right) S_{-2}^2 \right. \\
&\quad \left. + \left[-\frac{8(81n^6 + 243n^5 - 229n^4 - 389n^3 - 130n^2 + 228n + 72)}{9n^3(n+1)^3} + \frac{32(31n^2 + 31n - 81)S_2}{3n(n+1)} \right. \right. \\
&\quad \left. \left. + S_1 \left(1728S_2 - \frac{32(134n^4 + 268n^3 + 215n^2 + 45n + 54)}{9n^2(n+1)^2} \right) - 1792S_3 - 192S_{-2,1} + 2688S_{2,1} \right] S_{-2} \right. \\
&\quad \left. + \frac{176}{3} S_2^2 - \frac{P_2(n)}{36n^5(n+1)^5} + 672S_{-5} + S_{-4} \left(\frac{8(97n^2 + 97n - 210)}{3n(n+1)} + 1120S_1 \right) \right. \\
&\quad \left. + S_{-3} \left(-576S_1^2 + \frac{16(31n^2 + 31n + 108)S_1}{3n(n+1)} - \frac{8(268n^4 + 536n^3 + 811n^2 + 507n + 450)}{9n^2(n+1)^2} \right. \right. \\
&\quad \left. \left. + 480S_{-2} + 2656S_2 \right) + (-1)^n \left(\frac{8(382n^2 + 41n - 161)}{9(n+1)^5} - \frac{256S_{-2}}{(n+1)^3} - \frac{16(127n + 121)S_1}{3(n+1)^4} \right. \right. \\
&\quad \left. \left. + \frac{32S_2}{(n+1)^3} \right) - \frac{8(385n^4 + 770n^3 + 427n^2 + 6n - 126)S_3}{9n^2(n+1)^2} + \frac{8(151n^2 + 151n - 30)S_4}{3n(n+1)} \right. \\
&\quad \left. + 384S_5 + 864S_{-4,1} - \frac{960S_{-3,1}}{n(n+1)} - 1344S_{-3,2} \right\}
\end{aligned}$$

$$\begin{aligned}
& + S_2 \left(\frac{2(453n^5 + 906n^4 + 1325n^3 + 488n^2 - 120n + 144)}{9n^3(n+1)^2} - 32S_3 - 2624S_{-2,1} \right) \\
& + \frac{16(268n^4 + 536n^3 + 625n^2 + 321n + 414)S_{-2,1}}{9n^2(n+1)^2} + S_1^2(128S_3 + 896S_{-2,1}) \\
& - \frac{16(31n^2 + 31n - 174)S_{2,-2}}{3n(n+1)} + 1824S_{3,-2} - \frac{32(29n^2 + 29n - 24)S_{3,1}}{3n(n+1)} - 384S_{3,2} - 384S_{4,1} \\
& - 2688S_{-3,1,1} - 768S_{-2,1,-2} + S_1 \left(-\frac{8(135n^6 + 731n^5 + 245n^4 - 617n^3 - 395n^2 - 309n - 144)}{9n^4(n+1)^4} \right. \\
& - \frac{2144}{9}S_2 + \frac{32(31n^2 + 31n - 12)S_3}{3n(n+1)} + 160S_4 + 1920S_{-3,1} + \frac{32(31n^2 + 31n - 84)S_{-2,1}}{3n(n+1)} \\
& \left. - 1856S_{2,-2} - 512S_{3,1} - 3584S_{-2,1,1} \right) - \frac{64(31n^2 + 31n - 84)S_{-2,1,1}}{3n(n+1)} - 2688S_{-2,2,1} - 2688S_{2,1,-2} \\
& + 768S_{3,1,1} + 5376S_{-2,1,1,1} \Big\} \\
& + C_A^2 C_F \left[\left(\frac{24(n^2 + n + 2)}{n(n+1)} - 96S_1 \right) S_{-2}^2 + \left(\frac{8(27n^6 + 81n^5 - 155n^4 - 271n^3 - 92n^2 + 78n + 27)}{9n^3(n+1)^3} \right. \right. \\
& + S_1 \left(\frac{16(134n^4 + 268n^3 + 188n^2 + 54n + 45)}{9n^2(n+1)^2} - 512S_2 \right) - \frac{32(11n^2 + 11n - 24)S_2}{3n(n+1)} + 512S_3 \\
& + 64S_{-2,1} - 768S_{2,1} \Big) S_{-2} + \frac{P_3(n)}{108n^5(n+1)^5} - 192S_{-5} + S_{-4} \left(-\frac{8(35n^2 + 35n - 66)}{3n(n+1)} - 352S_1 \right) \\
& + (-1)^n \left(-\frac{16(82n^2 + 17n - 47)}{9(n+1)^5} + \frac{96S_{-2}}{(n+1)^3} + \frac{16(41n + 47)S_1}{3(n+1)^4} \right) \\
& + S_{-3} \left(128S_1^2 - \frac{16(11n^2 + 11n + 24)S_1}{3n(n+1)} + \frac{8(134n^4 + 268n^3 + 311n^2 + 177n + 135)}{9n^2(n+1)^2} \right. \\
& \left. - 160S_{-2} - 768S_2 \right) + \frac{4(389n^4 + 778n^3 + 398n^2 + 9n - 81)S_3}{9n^2(n+1)^2} - \frac{8(55n^2 + 55n - 24)S_4}{3n(n+1)} \\
& - 160S_5 - 224S_{-4,1} + \frac{256S_{-3,1}}{n(n+1)} + 384S_{-3,2} + S_1^2(-64S_3 - 256S_{-2,1}) \\
& - \frac{16(134n^4 + 268n^3 + 245n^2 + 111n + 135)S_{-2,1}}{9n^2(n+1)^2} + S_2 \left(768S_{-2,1} - \frac{4172}{27} \right) \\
& + \frac{16(11n^2 + 11n - 48)S_{2,-2}}{3n(n+1)} - 544S_{3,-2} + \frac{32(11n^2 + 11n - 12)S_{3,1}}{3n(n+1)} \\
& + 192S_{3,2} + 192S_{4,1} + 768S_{-3,1,1} + 256S_{-2,1,-2} + \frac{64(11n^2 + 11n - 24)S_{-2,1,1}}{3n(n+1)} \\
& + S_1 \left(\frac{2(245n^8 + 980n^7 + 1542n^6 + 1524n^5 + 851n^4 + 100n^3 + 36n^2 + 22n - 6)}{3n^4(n+1)^4} \right. \\
& \left. - \frac{8(11n^2 + 11n - 8)S_3}{n(n+1)} - 128S_4 - 512S_{-3,1} - \frac{32(11n^2 + 11n - 24)S_{-2,1}}{3n(n+1)} \right. \\
& \left. + 512S_{2,-2} + 256S_{3,1} + 1024S_{-2,1,1} \right) + 768S_{-2,2,1} + 768S_{2,1,-2} - 384S_{3,1,1}
\end{aligned}$$

$$\begin{aligned}
& - 1536S_{-2,1,1,1} \Big] \\
& + C_A^2 C_F \zeta_3 \left[-\frac{12(5n^4 + 10n^3 + 9n^2 - 4n - 4)}{n^2(n+1)^2} - 96S_{-2} \right] \\
& + C_F N_F^2 \left[\frac{51n^6 + 153n^5 + 57n^4 + 35n^3 + 96n^2 + 16n - 24}{27n^3(n+1)^3} - \frac{16}{27}S_1 - \frac{80}{27}S_2 + \frac{16}{9}S_3 \right] \\
& + C_F^2 N_F \left[-\frac{32}{3}S_2^2 - \frac{4(15n^4 + 30n^3 + 79n^2 + 16n - 24)S_2}{9n^2(n+1)^2} \right. \\
& \left. + \frac{207n^8 + 828n^7 + 1443n^6 + 1123n^5 - 38n^4 - 779n^3 - 632n^2 + 120}{9n^4(n+1)^4} - \frac{128}{3}S_{-4} \right. \\
& + S_{-3} \left(\frac{32(10n^2 + 10n + 3)}{9n(n+1)} - \frac{64}{3}S_1 \right) + (-1)^n \left(\frac{64S_1}{3(n+1)^3} - \frac{128(4n+1)}{9(n+1)^4} \right) \\
& + S_{-2} \left(-\frac{32(16n^2 + 10n - 3)}{9n^2(n+1)^2} + \frac{640}{9}S_1 - \frac{128}{3}S_2 \right) + \frac{16(29n^2 + 29n + 12)S_3}{9n(n+1)} - \frac{128}{3}S_4 \\
& + S_1 \left(-\frac{2(165n^5 + 330n^4 + 165n^3 + 160n^2 - 16n - 96)}{9n^3(n+1)^2} + \frac{320}{9}S_2 - \frac{128}{3}S_3 - \frac{128}{3}S_{-2,1} \right. \\
& \left. - \frac{64(10n^2 + 10n - 3)S_{-2,1}}{9n(n+1)} + \frac{64}{3}S_{2,-2} + \frac{64}{3}S_{3,1} + \frac{256}{3}S_{-2,1,1} \right) \\
& + (C_F^2 - C_F C_A) N_F \zeta_3 \left[32S_1 - \frac{8(3n^2 + 3n + 2)}{n(n+1)} \right] \\
& + C_A C_F N_F \left[-\frac{2(270n^7 + 810n^6 - 463n^5 - 1392n^4 - 211n^3 - 206n^2 - 156n + 144)}{27n^4(n+1)^3} \right. \\
& + \frac{64}{3}S_{-4} + S_{-3} \left(\frac{32}{3}S_1 - \frac{16(10n^2 + 10n + 3)}{9n(n+1)} \right) + (-1)^n \left(\frac{64(4n+1)}{9(n+1)^4} - \frac{32S_1}{3(n+1)^3} \right) \\
& + \frac{1336}{27}S_2 + S_{-2} \left(\frac{16(16n^2 + 10n - 3)}{9n^2(n+1)^2} - \frac{320}{9}S_1 + \frac{64}{3}S_2 \right) - \frac{8(14n^2 + 14n + 3)S_3}{3n(n+1)} + \frac{80}{3}S_4 \\
& + \frac{32(10n^2 + 10n - 3)S_{-2,1}}{9n(n+1)} + S_1 \left(-\frac{4(209n^6 + 627n^5 + 627n^4 + 281n^3 + 36n^2 + 36n + 18)}{27n^3(n+1)^3} \right. \\
& \left. + 16S_3 + \frac{64}{3}S_{-2,1} \right) - \frac{32}{3}S_{2,-2} - \frac{64}{3}S_{3,1} - \frac{128}{3}S_{-2,1,1} \Big] \\
P_{qq}^{2,+} & = C_F^3 \left[\left(\frac{64}{n(n+1)} - 128S_1 \right) S_{-2}^2 + \left(\frac{16(3n^6 + 9n^5 + 9n^4 + n^3 + 2n^2 + 4n + 2)}{n^3(n+1)^3} \right. \right. \\
& \left. + S_1 \left(-\frac{64(3n^2 + 7n + 5)}{n^2(n+1)^2} - 1408S_2 \right) - \frac{64(3n^2 + 3n - 11)S_2}{n(n+1)} + 1536S_3 + 128S_{-2,1} \right.
\end{aligned}$$

$$\begin{aligned}
& - 2304S_{2,1} \Big) S_{-2} - \frac{16(3n^2+3n+2)S_2^2}{n(n+1)} - \frac{P_4(n)}{2n^5(n+1)^5} - 576S_{-5} \\
& + S_{-4} \left(-\frac{16(9n^2+9n-26)}{n(n+1)} - 832S_1 \right) + S_{-3} \left(640S_1^2 - \frac{32(3n^2+3n+20)S_1}{n(n+1)} \right. \\
& + \frac{16(9n^2+5n+8)}{n^2(n+1)^2} - 320S_{-2} - 2240S_2 \Big) + (-1)^n \left(\frac{16(2n^2+11n+1)}{(n+1)^5} + \frac{128S_{-2}}{(n+1)^3} \right. \\
& + \frac{96(5n+3)S_1}{(n+1)^4} - \frac{64S_2}{(n+1)^3} \Big) + \frac{4(13n^4+26n^3+13n^2-16n-20)S_3}{n^2(n+1)^2} \\
& - \frac{16(15n^2+15n+2)S_4}{n(n+1)} - 192S_5 - 832S_{-4,1} + \frac{896S_{-3,1}}{n(n+1)} + 1152S_{-3,2} \\
& + S_1^2 \left(-\frac{32(3n^2+3n+1)}{n^3(n+1)^3} - 768S_{-2,1} \right) - \frac{32(3n^2-n+4)S_{-2,1}}{n^2(n+1)^2} \\
& + S_2 \left(\frac{2(3n^6+9n^5+9n^4+83n^3+76n^2+60n+16)}{n^3(n+1)^3} + 64S_3 + 2176S_{-2,1} \right) \\
& + \frac{32(3n^2+3n-26)S_{2,-2}}{n(n+1)} - 1472S_{3,-2} + \frac{64(3n^2+3n-2)S_{3,1}}{n(n+1)} + 192S_{3,2} + 192S_{4,1} \\
& + 2304S_{-3,1,1} + 512S_{-2,1,-2} + \frac{384(n^2+n-4)S_{-2,1,1}}{n(n+1)} + S_1 \left(64S_2^2 - \frac{64(2n+1)S_2}{n^2(n+1)^2} \right. \\
& + \frac{4(22n^6-54n^5+23n^4+88n^3+197n^2+160n+52)}{n^4(n+1)^4} - 192S_3 + 64S_4 - 1792S_{-3,1} \\
& - \left. \frac{192(n^2+n-4)S_{-2,1}}{n(n+1)} + 1664S_{2,-2} + 256S_{3,1} + 3072S_{-2,1,1} \right) + 2304S_{-2,2,1} \\
& + 2304S_{2,1,-2} - 384S_{3,1,1} - 4608S_{-2,1,1,1} \Big] \\
& + C_F^3 \zeta_3 \left[-\frac{24(5n^4+10n^3+n^2-4n-4)}{n^2(n+1)^2} - 192S_{-2} \right] \\
& + C_A C_F^2 \left\{ \left(256S_1 - \frac{16(3n^2+3n+8)}{n(n+1)} \right) S_{-2}^2 \right. \\
& + \left(-\frac{8(81n^5+243n^4-337n^3-1181n^2-526n-60)}{9n^2(n+1)^3} + \frac{32(31n^2+31n-81)S_2}{3n(n+1)} \right. \\
& + S_1 \left(1728S_2 - \frac{32(134n^4+268n^3+89n^2-81n-72)}{9n^2(n+1)^2} \right) - 1792S_3 - 192S_{-2,1} + 2688S_{2,1} \Big) S_{-2} \\
& + \frac{176}{3} S_2^2 - \frac{P_5(n)}{36n^4(n+1)^4} + 672S_{-5} + S_{-4} \left(\frac{8(97n^2+97n-210)}{3n(n+1)} + 1120S_1 \right) \\
& + S_{-3} \left(-576S_1^2 + \frac{16(31n^2+31n+108)S_1}{3n(n+1)} - \frac{8(268n^4+536n^3+487n^2+183n+126)}{9n^2(n+1)^2} \right.
\end{aligned}$$

$$\begin{aligned}
& + 480S_{-2} + 2656S_2 \Big) + (-1)^n \left(\frac{8(346n-125)}{9(n+1)^4} - \frac{256S_{-2}}{(n+1)^3} - \frac{16(103n+73)S_1}{3(n+1)^4} + \frac{32S_2}{(n+1)^3} \right) \\
& - \frac{8(385n^4+770n^3+427n^2+6n-126)S_3}{9n^2(n+1)^2} + \frac{8(151n^2+151n-30)S_4}{3n(n+1)} + 384S_5 \\
& + 864S_{-4,1} - \frac{960S_{-3,1}}{n(n+1)} - 1344S_{-3,2} + S_2 \left(\frac{2(453n^5+1359n^4+2231n^3+1525n^2+80n-264)}{9n^2(n+1)^3} \right. \\
& - 32S_3 - 2624S_{-2,1} \Big) + \frac{16(268n^4+536n^3+301n^2-3n+90)S_{-2,1}}{9n^2(n+1)^2} + S_1^2(128S_3 + 896S_{-2,1}) \\
& - \frac{16(31n^2+31n-174)S_{2,-2}}{3n(n+1)} + 1824S_{3,-2} - \frac{32(29n^2+29n-24)S_{3,1}}{3n(n+1)} - 384S_{3,2} - 384S_{4,1} \\
& - 2688S_{-3,1,1} - 768S_{-2,1,-2} + S_1 \left(-\frac{8(135n^6-649n^5-1039n^4-569n^3+487n^2+621n+216)}{9n^4(n+1)^4} \right. \\
& - \frac{2144}{9} S_2 + \frac{32(31n^2+31n-12)S_3}{3n(n+1)} + 160S_4 + 1920S_{-3,1} + \frac{32(31n^2+31n-84)S_{-2,1}}{3n(n+1)} \\
& - 1856S_{2,-2} - 512S_{3,1} - 3584S_{-2,1,1} \Big) - \frac{64(31n^2+31n-84)S_{-2,1,1,n}}{3n(n+1)} - 2688S_{-2,2,1} \\
& - 2688S_{2,1,-2} + 768S_{3,1,1} + 5376S_{-2,1,1,1} \Big] \\
& + C_A C_F^2 \zeta_3 \left[\frac{36(5n^4+10n^3+n^2-4n-4)}{n^2(n+1)^2} + 288S_{-2} \right] \\
& + C_A^2 C_F \left(\frac{24(n^2+n+2)}{n(n+1)} - 96S_1 \right) S_{-2}^2 + \left(\frac{8(27n^6+81n^5-209n^4-595n^3-272n^2-48n-9)}{9n^3(n+1)^3} \right. \\
& + S_1 \left(\frac{16(134n^4+268n^3+116n^2-18n-27)}{9n^2(n+1)^2} - 512S_2 \right) - \frac{32(11n^2+11n-24)S_2}{3n(n+1)} + 512S_3 \\
& + 64S_{-2,1} - 768S_{2,1} \Big) S_{-2} + \frac{P_6(N)}{108n^3(n+1)^5} - 192S_{-5} + S_{-4} \left(-\frac{8(35n^2+35n-66)}{3n(n+1)} - 352S_1 \right) \\
& + (-1)^n \left(-\frac{16(91n^2+80n-29)}{9(n+1)^5} + \frac{96S_{-2}}{(n+1)^3} + \frac{16(29n+23)S_1}{3(n+1)^4} \right) \\
& + S_{-3} \left(128S_1^2 - \frac{16(11n^2+11n+24)S_1}{3n(n+1)} + \frac{8(134n^4+268n^3+203n^2+69n+27)}{9n^2(n+1)^2} \right. \\
& - 160S_{-2} - 768S_2 \Big) + \frac{4(389n^4+778n^3+398n^2+9n-81)S_3}{9n^2(n+1)^2} - \frac{8(55n^2+55n-24)S_4}{3n(n+1)} \\
& - 160S_5 - 224S_{-4,1} + \frac{256S_{-3,1}}{n(n+1)} + 384S_{-3,2} + S_1^2(-64S_3 - 256S_{-2,1}) \\
& - \frac{16(134n^4+268n^3+137n^2+3n+27)S_{-2,1}}{9n^2(n+1)^2} + S_2 \left(768S_{-2,1} - \frac{4172}{27} \right) \\
& + \frac{16(11n^2+11n-48)S_{2,-2}}{3n(n+1)} - 544S_{3,-2} + \frac{32(11n^2+11n-12)S_{3,1}}{3n(n+1)} + 192S_{3,2}
\end{aligned}$$

$$\begin{aligned}
& + 192S_{4,1} + 768S_{-3,1,1} + 256S_{-2,1,-2} + \frac{64(11n^2 + 11n - 24)S_{-2,1,1}}{3n(n+1)} \\
& + S_1 \left(\frac{2(245n^8 + 980n^7 + 1542n^6 + 964n^5 + 211n^4 - 60n^3 + 156n^2 + 222n + 90)}{3n^4(n+1)^4} \right. \\
& - \frac{8(11n^2 + 11n - 8)S_3}{n(n+1)} - 128S_4 - 512S_{-3,1} \\
& - \left. \frac{32(11n^2 + 11n - 24)S_{-2,1}}{3n(n+1)} + 512S_{2,-2} + 256S_{3,1} + 1024S_{-2,1,1} \right) + 768S_{-2,2,1} \\
& + \left. 768S_{2,1,-2} - 384S_{3,1,1} - 1536S_{-2,1,1,1} \right\} \\
& + C_A^2 C_F \zeta_3 \left[-\frac{12(5n^4 + 10n^3 + n^2 - 4n - 4)}{n^2(n+1)^2} - 96S_{-2} \right] \\
& + C_F^2 N_F \left\{ -\frac{32}{3}S_2^2 - \frac{4(15n^4 + 30n^3 + 79n^2 + 16n - 24)S_2}{9n^2(n+1)^2} + \frac{P_7(n)}{9n^4(n+1)^4} - \frac{128}{3}S_{-4} \right. \\
& + S_{-3} \left(\frac{32(10n^2 + 10n + 3)}{9n(n+1)} - \frac{64}{3}S_1 \right) + (-1)^n \left(\frac{64S_1}{3(n+1)^3} - \frac{128(4n+1)}{9(n+1)^4} \right) \\
& + S_{-2} \left(-\frac{32(16n^2 + 10n - 3)}{9n^2(n+1)^2} + \frac{640}{9}S_1 - \frac{128}{3}S_2 \right) + \frac{16(29n^2 + 29n + 12)S_3}{9n(n+1)} - \frac{128}{3}S_4 \\
& + S_1 \left(-\frac{2(165n^5 + 495n^4 + 495n^3 + 517n^2 + 336n + 80)}{9n^2(n+1)^3} + \frac{320}{9}S_2 - \frac{128}{3}S_3 - \frac{128}{3}S_{-2,1} \right) \\
& - \left. \frac{64(10n^2 + 10n - 3)S_{-2,1}}{9n(n+1)} + \frac{64}{3}S_{2,-2} + \frac{64}{3}S_{3,1} + \frac{256}{3}S_{-2,1,1} \right\} \\
& + C_F^2 N_F \zeta_3 \left[32S_1 - \frac{8(3n^2 + 3n + 2)}{n(n+1)} \right] \\
& + C_F N_F^2 \left[\frac{51n^6 + 153n^5 + 57n^4 + 35n^3 + 96n^2 + 16n - 24}{27n^3(n+1)^3} - \frac{16}{27}S_1 - \frac{80}{27}S_2 + \frac{16}{9}S_3 \right] \\
& + C_A C_F N_F \left[-\frac{2(270n^7 + 1080n^6 + 383n^5 - 979n^4 - 571n^3 + 507n^2 + 106n - 132)}{27n^3(n+1)^4} \right. \\
& + \frac{64}{3}S_{-4} + S_{-3} \left(\frac{32}{3}S_1 - \frac{16(10n^2 + 10n + 3)}{9n(n+1)} \right) + (-1)^n \left(\frac{64(4n+1)}{9(n+1)^4} - \frac{32S_1}{3(n+1)^3} \right) \\
& + \frac{1336}{27}S_2 + S_{-2} \left(\frac{16(16n^2 + 10n - 3)}{9n^2(n+1)^2} - \frac{320}{9}S_1 + \frac{64}{3}S_2 \right) - \frac{8(14n^2 + 14n + 3)S_3}{3n(n+1)} + \frac{80}{3}S_4 \\
& + \left. \frac{32(10n^2 + 10n - 3)S_{-2,1}}{9n(n+1)} + S_1 \left(-\frac{4(209n^6 + 627n^5 + 627n^4 + 137n^3 - 108n^2 - 108n - 54)}{27n^3(n+1)^3} \right. \right. \\
& + \left. \left. 16S_3 + \frac{64}{3}S_{-2,1} \right) - \frac{32}{3}S_{2,-2} - \frac{64}{3}S_{3,1} - \frac{128}{3}S_{-2,1,1} \right] \\
& + C_A C_F N_F \zeta_3 \left[\frac{8(3n^2 + 3n + 2)}{n(n+1)} - 32S_1 \right]
\end{aligned}$$

$$\begin{aligned}
P_{qq}^{2,-,dabc} & = \frac{d_{abc}d^{abc}}{N_c} N_F \left[-\frac{P_8(n)}{3n^5(n+1)^5(n+2)^3} + \frac{4(n^2 + n + 2)S_{-3}}{n^2(n+1)^2} - \frac{P_9(n)S_1}{3n^4(n+1)^4(n+2)^3} \right. \\
& + S_{-2} \left(-\frac{8S_1(n^2 + n + 2)^2}{(n-1)n^2(n+1)^2(n+2)} - \frac{4(n^6 + 3n^5 - 8n^4 - 21n^3 - 23n^2 - 12n - 4)}{(n-1)n^3(n+1)^3(n+2)} \right) \\
& + (-1)^n \left(\frac{16(5n^6 + 29n^5 + 78n^4 + 118n^3 + 114n^2 + 72n + 16)S_1}{3(n-1)n^2(n+1)^3(n+2)^3} \right. \\
& - \left. \frac{4(13n^8 + 74n^7 + 179n^6 + 314n^5 + 644n^4 + 1000n^3 + 816n^2 + 352n + 64)}{3(n-1)n^3(n+1)^4(n+2)^3} \right) \\
& - \left. \frac{2(n^2 + n + 2)S_3}{n^2(n+1)^2} - \frac{8(n^2 + n + 2)S_{-2,1}}{n^2(n+1)^2} \right]
\end{aligned}$$

12. Conclusions - B

- We established a general algorithm to calculate the **exact expression** for **single scale** quantities from a **finite** (suitably large) number of moments (zero scale quantities).
- The latter ones are much more easily calculable.
- We applied the method to the **anomalous dimensions** and **Wilson coefficients** up to **3-loop order**.
- To solve 3-loop problems this way is not possible at present, since the number of required moments is too large for the methods available.
- We attempted to solve the quantities for all **color projections** at once. This problem is too voluminous.
- Yet we showed that giant **difference equations** [order 35; degree ~ 1000] can be reliably and fast **established** and **solved unconditionally** for advanced problems in Quantum Field Theory.