$O(\alpha_s^3)$ Heavy Flavor Wilson Coefficients in DIS @ $Q^2 \gg m^2$

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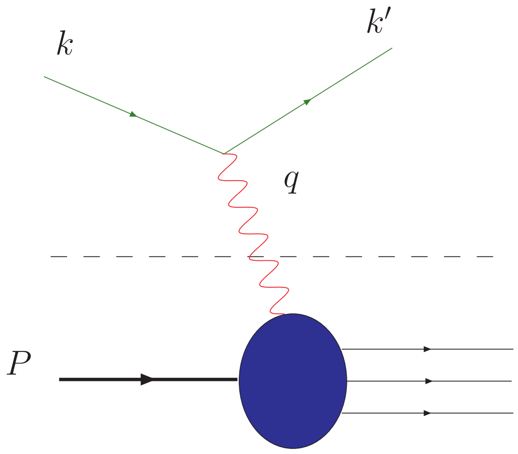
- Introduction
- Theory Status
- The Method
- Asymptotic 2 Loop Results (all N)
- Asymptotic 3 Loop Results (fixed moments)
- Conclusions

1. Introduction

Deep–Inelastic Scattering (DIS):



$\longrightarrow \quad L_{\mu\nu}$

$\longrightarrow \quad W_{\mu\nu}$

$$Q^2 := -q^2, \quad x := \frac{Q^2}{2pq}$$ Bjorken–x

$$\nu := \frac{Pq}{M}\,,$$

$$\frac{d\sigma}{dQ^2\,dx} \sim W_{\mu\nu}L^{\mu\nu}$$

$$W_{\mu\nu}(q,P,s) = \frac{1}{4\pi}\int d^4\xi \exp(iq\xi)\langle P,s\mid [J_\mu^{em}(\xi), J_\nu^{em}(0)]\mid P,s\rangle$$

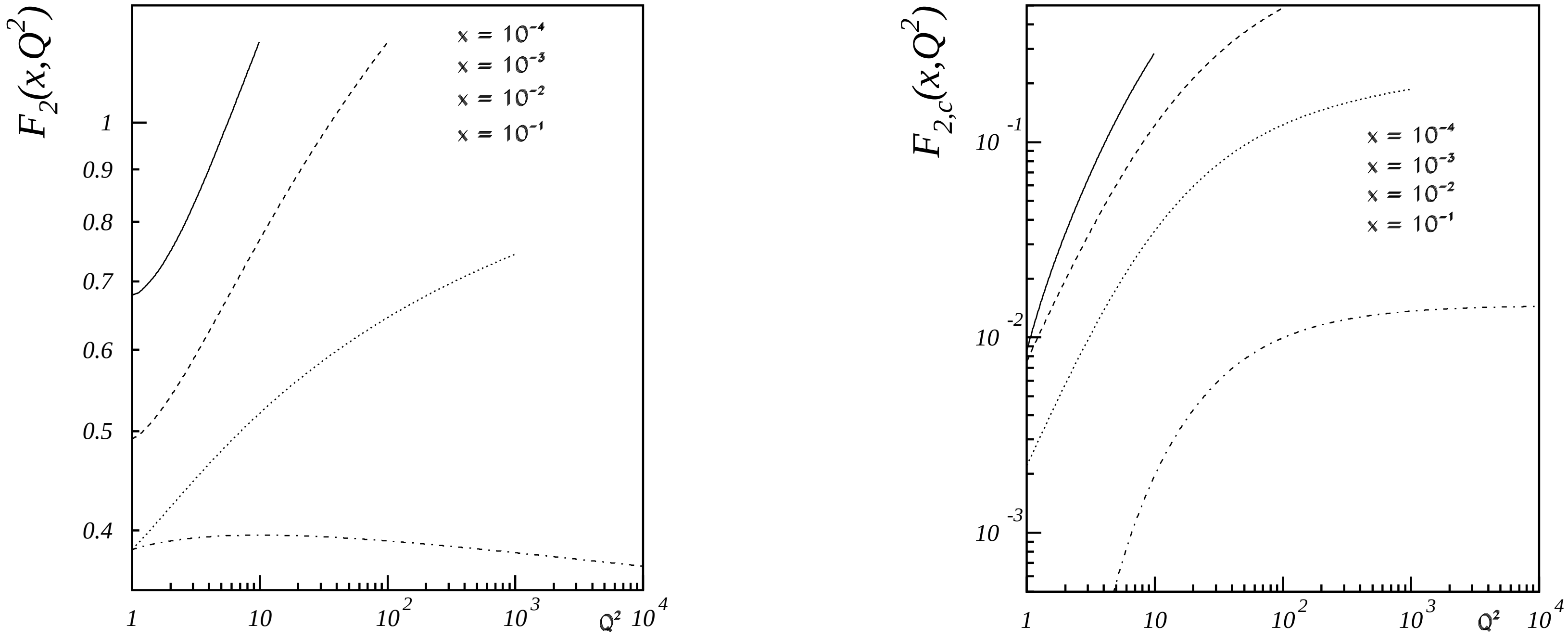
$$\text{unpol.}\left\{ \; = \frac{1}{2x}\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) F_L(x,Q^2) + \frac{2x}{Q^2}\left(P_\mu P_\nu + \frac{q_\mu P_\nu + q_\nu P_\mu}{2x} - \frac{Q^2}{4x^2}g_{\mu\nu}\right) F_2(x,Q^2)\right.$$

$$\text{pol.}\left\{ \; -\frac{M}{2Pq}\varepsilon_{\mu\nu\alpha\beta}q^\alpha\left[s^\beta g_1(x,Q^2) + \left(s^\beta - \frac{sq}{Pq}p^\beta\right) g_2(x,Q^2)\right]\right. .$$

Structure Functions: $F_{2,L}, \quad g_{1,2}$

contain light and heavy quark contributions

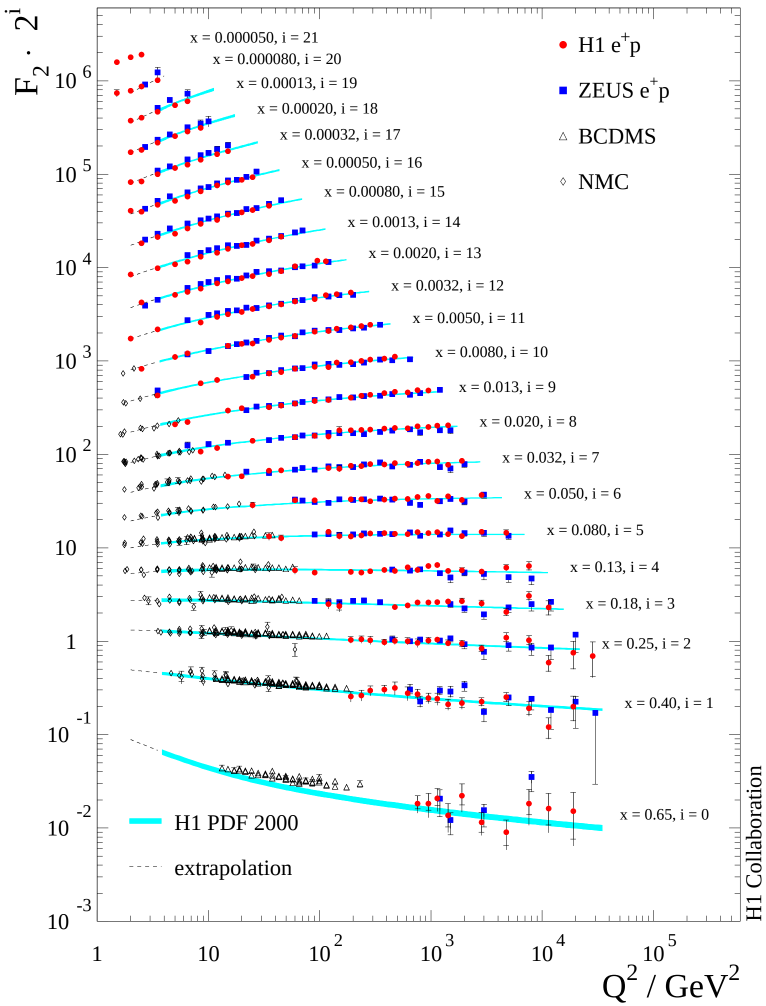
Let us compare the scaling violations in the massless and massive case.

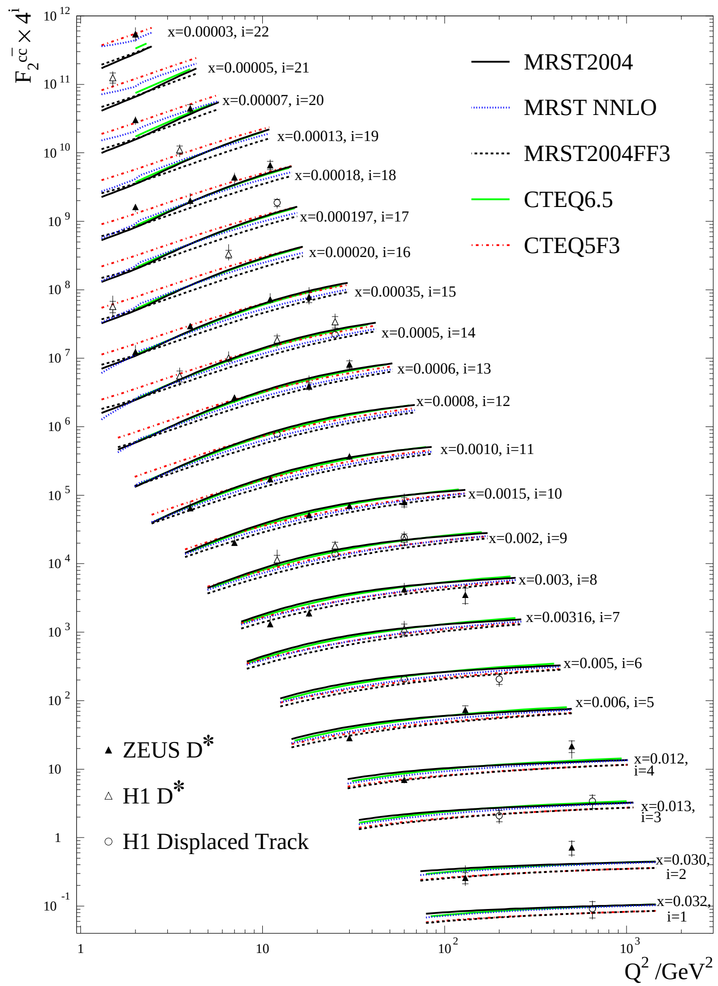


LO charm contributions : PDFs from [Alekhin, Melnikov, Petriello, 2006.]

$\Longrightarrow$ **different scaling violations!**

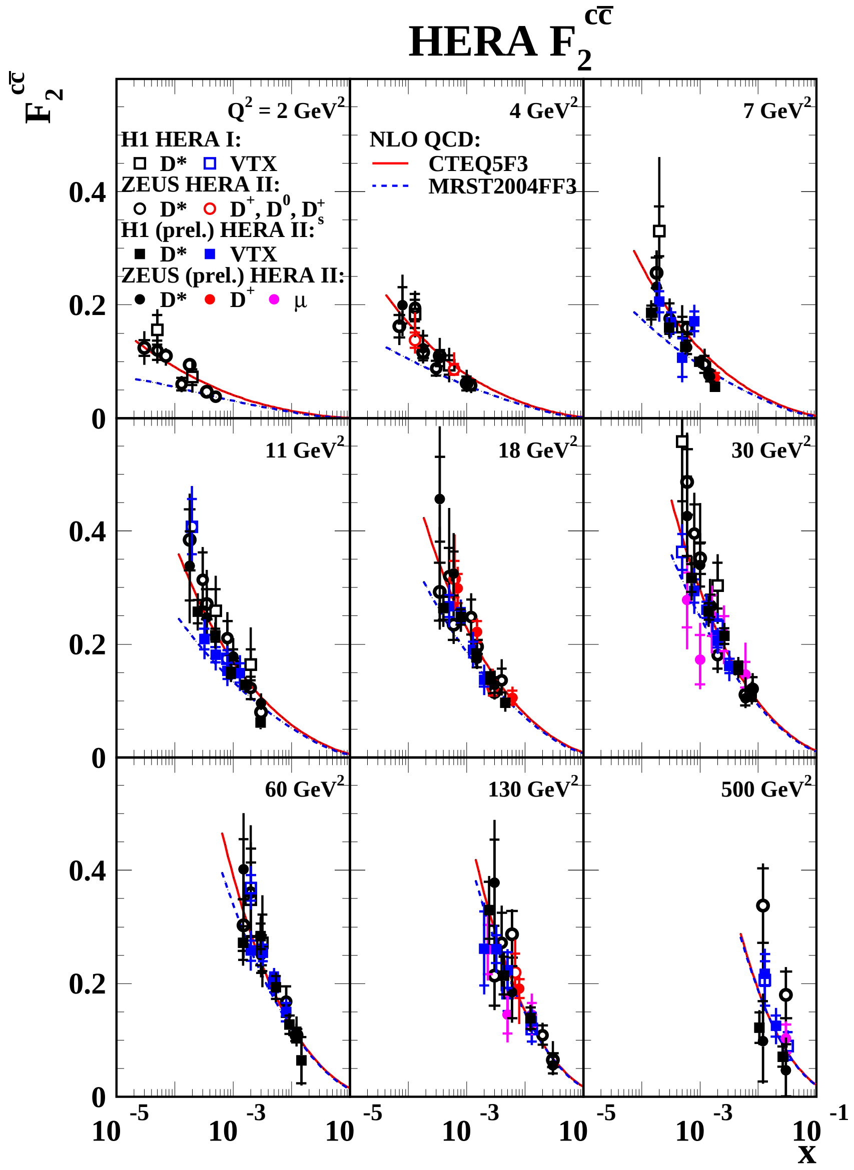
Light and Heavy Quark Contributions



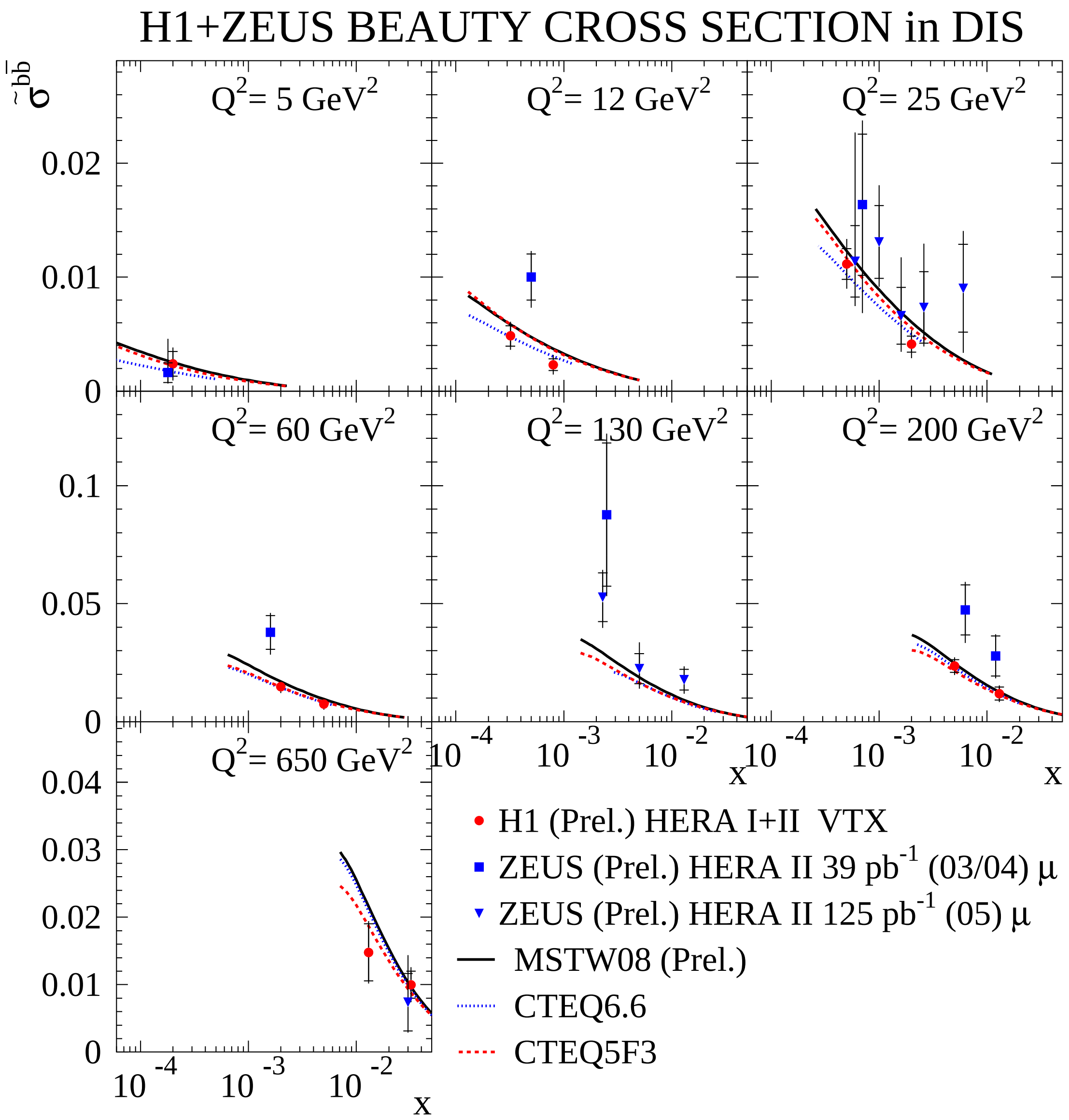


[Thompson, 2007]

High statistics for F_2 and $F_2^{c\bar{c}}$ $\Longrightarrow$ Accuracy will increase in the future.



[Krüger (H1 and Z. Coll.), 2008.]



[Krüger (H1 and Z. Coll.), 2008.]

Evolution of Light Quark Distributions

- The scaling violations are described by the splitting functions $P_{ij}(x, a_s)$.
- They describe the probability to find parton i radiated from parton j and carrying its momentum fraction x.
- They are related to the anomalous dimensions via a Mellin–Transform:

$$\mathbf{M}[f](N) := \int_0^1 dz z^{N-1} f(z) \ , \ \gamma_{ij}(N, a_s) := -\mathbf{M}[P_{ij}](N, a_s) \ .$$

- The splitting functions govern the scale–evolution of the parton densities.

$$\frac{d}{d \ln Q^2} \begin{pmatrix} \Sigma(N, Q^2) \\ G(N, Q^2) \end{pmatrix} = - \begin{pmatrix} \gamma_{qq} & \gamma_{qg} \\ \gamma_{gq} & \gamma_{gg} \end{pmatrix} \otimes \begin{pmatrix} \Sigma(N, Q^2) \\ G(N, Q^2) \end{pmatrix} ,$$

$$\frac{d}{d \ln Q^2} q_{NS}(N, Q^2) = -\gamma_{qq,NS} \otimes q_{NS} \ .$$

- The singlet light flavor density is defined by

$$\Sigma(n_f, \mu^2) = \sum_{i=1}^{n_f} (f_i(n_f, \mu^2) + \bar{f}_i(n_f, \mu^2)) \ .$$

- The anomalous dimensions are presently known at NNLO [Moch, Vermaseren, Vogt, 2004] and N^3LO for NS^+, N=2 [Baikov & Chetyrkin, 2006]

2. Theory Status

Leading Order : $F_{2,L}(x, Q^2)$ [Witten, 1976; Babcock, Sivers, 1978; Shifman, Vainshtein, Zakharov, 1978; Leveille, Weiler, 1979; Glück, Reya, 1979; Glück, Hoffmann, Reya, 1982.]

Leading Order : $g_1(x, Q^2)$ [Watson, 1982; Glück, Reya, Vogelsang, 1991; Vogelsang, 1991]

Leading Order : $g_2(x, Q^2)$ [J.B., Ravindran, van Neerven, 2003] $\Longrightarrow$ holds to all orders Wandzura-Wilczek relation.

Soft resummation: $F_{2,L}(x, Q^2)$ [Laenen & Moch, 1998; Alekhin & Moch, 2008]

Next-to-Leading Order : $F_{2,L}(x, Q^2)$ [Laenen, Riemersma, Smith, van Neerven, 1993, 1995] asymptotic: [Buza, Matiounine, Smith, Migneron, van Neerven, 1996; Bierenbaum, J.B., Klein, 2007]

Next-to-Leading Order : $g_1(x, Q^2)$ asymptotic: [Buza, Matiounine, Smith, Migneron, van Neerven, 1997; Bierenbaum, J.B., Klein, 2009]

- Fast semi-analytic representation in Mellin space: [Alekhin & J.B., 2003] This includes power corrections.

$\Longrightarrow$ 3-Loop corrections needed to line up with the accuracy reached for light flavor contributions.

3. The Method

- massless RGE and light–cone expansion in Bjorken–limit $\{Q^2,\ \nu\} \ \rightarrow\ \infty,\ x$ fixed:

$$\lim_{\xi^2 \to 0} \Big[J(\xi), J(0) \Big] \propto \sum_{i,N,\tau} c_{i,\tau}^{N}(\xi^2, \mu^2) \xi_{\mu_1} ... \xi_{\mu_N} O_{i,\tau}^{\mu_1 \cdots \mu_m}(0, \mu^2) \ .$$

- Operators: flavor non-singlet (≤ 3), pure–singlet and gluon; consider leading twist.
- RGE for collinear singularities: mass factorization of the structure functions into Wilson coefficients and parton densities:

$$F_i(x, Q^2) = \sum_j \underbrace{C_i^j\left(x, \frac{Q^2}{\mu^2}\right)}_{\text{perturbative}} \otimes \underbrace{f_j(x, \mu^2)}_{\text{non-perturbative}}$$

- Light-flavor Wilson coefficients: process dependent ($O(a_s^3)$: [Moch, Vermaseren, Vogt, 2005.])

$$C_{(2,L);i}^{\text{fl}}\left(\frac{Q^2}{\mu^2}\right) = \delta_{i,q} + \sum_{l=1}^{\infty} a_s^l C_{(2,L),i}^{\text{fl},(l)}, \qquad i = q, g$$

- Heavy quark contributions given by heavy quark Wilson coefficients, $H_{(2,L),i}^{\text{S,NS}}\left(\frac{Q^2}{\mu^2}, \frac{m^2}{Q^2}\right)$.

- In the limit $Q^2 \gg m_h^2$ [$Q^2 \approx 10\ m^2$ for F_2, g_1]:
  massive RGE, derivative $m^2\partial/\partial m^2$ acts on Wilson coefficients only: all terms but power corrections calculable through partonic operator matrix elements, $\langle i|A_l|j\rangle$, which are process independent objects!

$$H_{(2,L),i}^{\mathrm{S,NS}}\left(\frac{Q^2}{\mu^2},\frac{m^2}{\mu^2}\right) = \underbrace{A_{k,i}^{\mathrm{S,NS}}\left(\frac{m^2}{\mu^2}\right)}_{\text{massive OMEs}} \otimes \underbrace{C_{(2,L),k}^{\mathrm{S,NS}}\left(\frac{Q^2}{\mu^2}\right)}_{\text{light–parton–Wilson coefficients}}.$$

- holds for polarized and unpolarized case. OMEs obey expansion

$$A_{k,i}^{\mathrm{S,NS}}\left(\frac{m^2}{\mu^2}\right) = \langle i|O_k^{\mathrm{S,NS}}|i\rangle = \delta_{k,i} + \sum_{l=1}^{\infty} a_s^l A_{k,i}^{\mathrm{S,NS},(l)}\left(\frac{m^2}{\mu^2}\right), \qquad i = q, g$$

  [Buza, Matiounine, Migneron, Smith, van Neerven, 1996; Buza, Matiounine, Smith, van Neerven, 1997.]

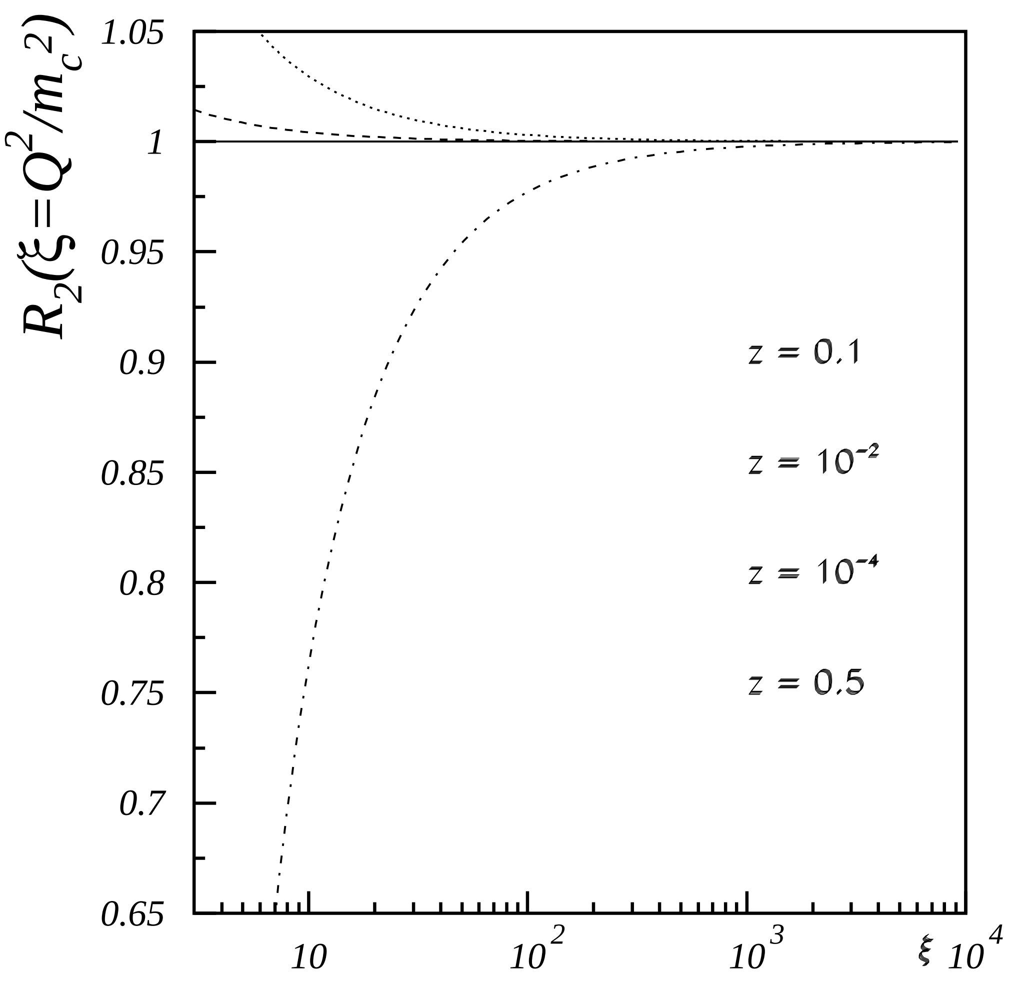
- Heavy OMEs occur as well as transition functions to define the VFNS starting from a fixed flavor number scheme(FFNS).
  [Buza, Matiounine, Smith, van Neerven, 1998; Chuvakin, Smith, van Neerven, 1998.]

- Comparison for LO:

$$R_2\left(\xi \equiv \frac{Q^2}{m^2}\right) \quad \equiv \quad \frac{H^{(1)}_{2,g}}{H^{(1)}_{2,g,(asym)}} \; .$$

- Comparison to exact order $O(a_s^2)$ result: asymptotic formulae valid for $Q^2 \gtrsim 20\ (\mathrm{GeV}/c)^2$ in case of $F_2^{c\bar{c}}(x,Q^2)$ and $Q^2 \gtrsim 800\ (\mathrm{GeV}/c)^2$ for $F_L^{c\bar{c}}(x,Q^2)$

- Drawbacks:
  - Power corrections $(m^2/Q^2)^k$ cannot be calculated using this method.
  - The case of two different heavy quark masses is still too complicated $\Longrightarrow$ 2 scale problem to be treated semi-analytically.
  - Only inclusive quantities can be calculated $\Longrightarrow$ structure functions.



Renormalization

$$\hat{\hat{A}}_{ij} = \delta_{ij} + \sum_{k=0}^{\infty} \hat{a}_s^k \hat{\hat{A}}_{ij}^{(k)}$$

- Mass renormalization (on–mass shell scheme)
- Charge renormalization

MOM scheme → $\overline{\rm MS}$ scheme ($D = 4 + \varepsilon$):→ decoupling formalism [Ovrut, Schnitzer, 1981; Bernreuther, Wetzel, 1982.].

- Renormalization of ultraviolet singularities
  $\Longrightarrow$ are absorbed into Z–factors given in terms of anomalous dimensions γ_{ij}.
- Factorization of collinear singularities
  $\Longrightarrow$ are factored into Γ–factors Γ_{NS}, $\Gamma_{ij,S}$ and $\Gamma_{qq,PS}$.
  For massless quarks it would hold: $\Gamma = Z^{-1}$.
  Here: Γ–matrices apply to parts of the diagrams with massless lines only .

Generic formula for operator renormalization and mass factorization:

$$A_{ij} = Z_{il}^{-1} \hat{A}_{lm} \Gamma_{mj}^{-1}$$

$\Longrightarrow$ $O(\varepsilon)$–terms of the 2–loop OMEs are needed for renormalization at 3–loops.

4. $O(a_s^2)$ Contributions to $O(\varepsilon)$

Refs.: [I. Bierenbaum, J.B., S. Klein, Phys. Lett. B672 (2009) 401; B648 (2007) 195; Nucl. Phys. B780 (2007) 1; I. Bierenbaum, J.B., S. Klein, C. Schneider, Nucl. Phys. B803 (2008) 1]

- Use of generalized hypergeometric functions for general analytic results

  $\Longrightarrow$ allows feasible computation of higher orders in ε & automated check for fixed values of N.

- use of Mellin-Barnes integrals for numerical checks (MB, [Czakon, 2006.])

- Summation of lots of new infinite one-parameter sums into harmonic sums. E.g.:

$$N \sum_{i,j=1}^{\infty} \frac{S_1(i)S_1(i+j+N)}{i(i+j)(j+N)} = 4S_{2,1,1} - 2S_{3,1} + S_1\left(-3S_{2,1} + \frac{4S_3}{3}\right) - \frac{S_4}{2}$$
$$- S_2^2 + S_1^2 S_2 + \frac{S_1^4}{6} + 6S_1\zeta_3 + \zeta_2\left(2S_1^2 + S_2\right) .$$

  use of integral techniques and the Mathematica package SIGMA [Schneider, 2007.], [Bierenbaum, J.B., Klein, Schneider, 2007, 2008.]

- Partial checks for fixed values of N using SUMMER, [Vermaseren, 1999.]

- Computation within general R_ξ gauge.

We calculated all 2–loop $O(\varepsilon)$–terms in the unpolarized case

and several 2–loop $O(\varepsilon)$–terms in the polarized case:

$$\overline{a}_{Qg}^{(2)}, \quad \overline{a}_{Qq}^{(2),\mathbf{PS}}, \quad \overline{a}_{gg,Q}^{(2)}, \quad \overline{a}_{gq,Q}^{(2)}, \quad \overline{a}_{qq,Q}^{(2),\mathbf{NS}}.$$

$$\Delta\overline{a}_{Qg}^{(2)}, \quad \Delta\overline{a}_{Qq}^{(2),\mathbf{PS}}, \quad \Delta\overline{a}_{qq,Q}^{(2),\mathbf{NS}}.$$

We verified all corresponding 2–loop $O(\varepsilon^0)$–results by van Neerven et. al.

- A remark on the appearing functions:

  van Neerven et al. to O(1): unpolarized: 48 basic functions; polarized: 24 basic functions.

  O(1): $\{S_1, \ S_2, \ S_3, \ S_{-2}, S_{-3}\}, \qquad S_{-2,1} \Longrightarrow$ 2 basic objects.

  O(ε): $\{S_1, \ S_2, \ S_3, \ S_4, \ S_{-2}, S_{-3}, S_{-4}\}, \quad S_{2,1}, \quad S_{-2,1}, \quad S_{-3,1}, \quad S_{2,1,1}, \quad S_{-2,1,1}$
  $\Longrightarrow$ 6 basic objects

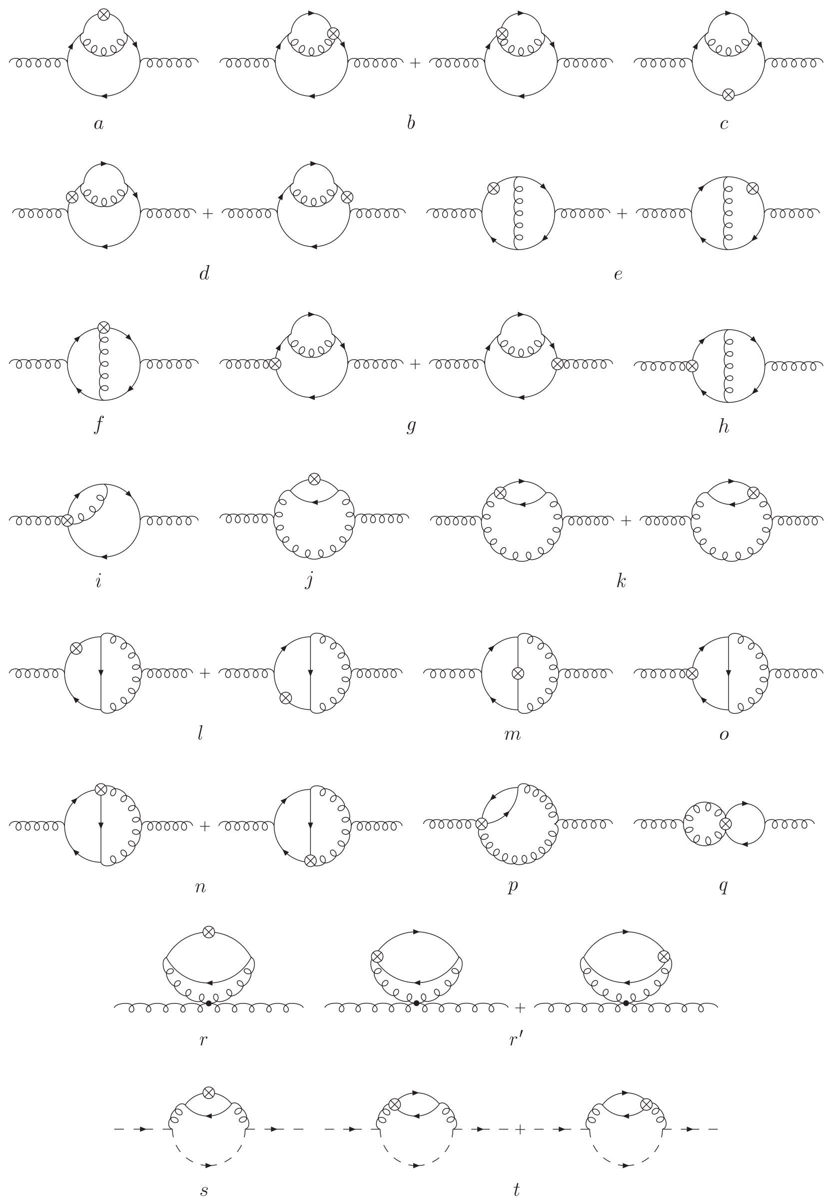
- harmonic sums with index $\{-1\}$ cancel (holds even for each diagram)

  J.B., 2004; J.B., Ravindran, 2005,2006; J.B., Klein, 2007; J.B., Moch in preparation;
  J.B. 0901.3106 [hep-ph]; 0901.0837 [math-ph]]

- Expectation for 3–loops: weight 5 (6) harmonic sums

- Diagrams contain two scales: the mass m and the Mellin–parameter N.
- 2–point functions with on–shell external momentum, $p^2 = 0$.
  $\longrightarrow$ reduce for $N = 0$ to massive tadpoles.
- E.g. diagrams contributing to the gluonic OME

$$\hat{A}_{Qg}^{(2)} \qquad \Longrightarrow$$



Example: Unpolarized case, Singlet, O(ε)

$$
\begin{aligned}
\overline{a}_{Qg}^{(2)} = T_F C_F \Bigg\{ & \frac{2}{3}\frac{(N^2+N+2)(3N^2+3N+2)}{N^2(N+1)^2(N+2)}\zeta_3 + \frac{P_1}{N^3(N+1)^3(N+2)}S_2 + \frac{N^4-5N^3-32N^2-18N-4}{N^2(N+1)^2(N+2)}S_1^2 \\
& + \frac{N^2+N+2}{N(N+1)(N+2)}\Big(16S_{2,1,1} - 8S_{3,1} - 8S_{2,1}S_1 + 3S_4 - \frac{4}{3}S_3S_1 - \frac{1}{2}S_2^2 - S_2S_1^2 - \frac{1}{6}S_1^4 + 2\zeta_2S_2 - 2\zeta_2S_1^2 - \frac{8}{3}\zeta_3S_1\Big) \\
& - 8\frac{N^2-3N-2}{N^2(N+1)(N+2)}S_{2,1} + \frac{2}{3}\frac{3N+2}{N^2(N+2)}S_1^3 + \frac{2}{3}\frac{3N^4+48N^3+43N^2-22N-8}{N^2(N+1)^2(N+2)}S_3 + 2\frac{3N+2}{N^2(N+2)}S_2S_1 + 4\frac{S_1}{N^2}\zeta_2 \\
& + \frac{N^5+N^4-8N^3-5N^2-3N-2}{N^3(N+1)^3}\zeta_2 - 2\frac{2N^5-2N^4-11N^3-19N^2-44N-12}{N^2(N+1)^3(N+2)}S_1 + \frac{P_2}{N^5(N+1)^5(N+2)}\Bigg\} \\
+ T_F C_A \Bigg\{ & \frac{N^2+N+2}{N(N+1)(N+2)}\Big(16S_{-2,1,1} - 4S_{2,1,1} - 8S_{-3,1} - 8S_{-2,2} - 4S_{3,1} - \frac{2}{3}\beta''' + 9S_4 - 16S_{-2,1}S_1 \\
& + \frac{40}{3}S_1S_3 + 4\beta''S_1 - 8\beta'S_2 + \frac{1}{2}S_2^2 - 8\beta'S_1^2 + 5S_1^2S_2 + \frac{1}{6}S_1^4 - \frac{10}{3}S_1\zeta_3 - 2S_2\zeta_2 - 2S_1^2\zeta_2 - 4\beta'\zeta_2 - \frac{17}{5}\zeta_2^2\Big) \\
& + \frac{4(N^2-N-4)}{(N+1)^2(N+2)^2}\Big(-4S_{-2,1} + \beta'' - 4\beta'S_1\Big) - \frac{2}{3}\frac{N^3+8N^2+11N+2}{N(N+1)^2(N+2)^2}S_1^3 + 8\frac{N^4+2N^3+7N^2+22N+20}{(N+1)^3(N+2)^3}\beta' \\
& + 2\frac{3N^3-12N^2-27N-2}{N(N+1)^2(N+2)^2}S_2S_1 - \frac{16}{3}\frac{N^5+10N^4+9N^3+3N^2+7N+6}{(N-1)N^2(N+1)^2(N+2)^2}S_3 - 8\frac{N^2+N-1}{(N+1)^2(N+2)^2}\zeta_2S_1 \\
& - \frac{2}{3}\frac{9N^5-10N^4-11N^3+68N^2+24N+16}{(N-1)N^2(N+1)^2(N+2)^2}\zeta_3 - \frac{P_3}{(N-1)N^3(N+1)^3(N+2)^3}S_2 - \frac{2P_4}{(N-1)N^3(N+1)^3(N+2)^2}\zeta_2 \\
& - \frac{P_5}{N(N+1)^3(N+2)^3}S_1^2 + \frac{2P_6}{N(N+1)^4(N+2)^4}S_1 - \frac{2P_7}{(N-1)N^5(N+1)^5(N+2)^5}\Bigg\}.
\end{aligned}
$$

5. Fixed moments at 3–Loop: $F_2^{Q\overline{Q}}$

$F_L^{Q\overline{Q}}(x,Q^2)$ @ 3-loop: [J.B., De Freitas, Klein, van Neerven, Nucl. Phys. B755 (2006) 272.]

Refs.: [I. Bierenbaum, J.B., S. Klein, Nucl.Phys.Proc.Suppl. 183 (2008) 162; arXiv:0812.2427; PoS]

Contributing OMEs:

Singlet	A_{Qg}	$A_{qg,Q}$	$A_{gg,Q}$	$A_{gq,Q}$	} mixing
Pure–Singlet		$A_{Qq}^{\rm PS}$	$A_{qq,Q}^{\rm PS}$		} mixing
Non–Singlet	$A_{qq,Q}^{\rm NS,+}$				

- All 2–loop $O(\varepsilon)$–terms in the unpolarized case are known.
- Unpolarized anomalous dimensions are known up to $O(a_s^3)$ [Moch, Vermaseren, Vogt, 2004.]
  $\Longrightarrow$ All terms needed for the renormalization of unpolarized 3–Loop heavy OMEs are present.
  $\Longrightarrow$ The calculation will provide first independent checks on $\gamma_{qg}^{(2)}$, $\gamma_{qq}^{(2),\rm PS}$ and on respective color projections of $\gamma_{qq}^{(2),\rm NS+}$, $\gamma_{gg}^{(2)}$ and $\gamma_{gq}^{(2)}$.
- The calculation proceeds in the same way in the polarized case.
- Independent checks provided by pole terms (anomalous dimensions, lower order OMEs) and sum rules for $N=2$.

Fixed moments using MATAD

- three–loop "self-energy" type diagrams with an operator insertion
- Extension: additional scale compared to massive propagators: Mellin variable N
- Genuine tensor integrals due to

$$\Delta^{\mu_1}...\Delta^{\mu_n} \langle p|O_{\mu_1...\mu_n}|p\rangle = \Delta^{\mu_1}...\Delta^{\mu_n} \langle p|S\, \bar{\Psi}\gamma_{\mu_1} D_{\mu_2}...D_{\mu_n}\Psi|p\rangle = A(N)\cdot(\Delta p)^N$$

$$D_\mu = \partial_\mu - igt_a A^a_\mu \quad , \qquad \Delta^2 = 0.$$

- Construction of a projector to obtain the desired moment in N [undo Δ-contraction]
- 3–loop OMEs are generated with QGRAF [Nogueira, 1993.]
- Color factors are calculated with [van Ritbergen, Schellekens, Vermaseren, 1998.]
- Translation to suitable input for MATAD [Steinhauser, 2001.]

Tests performed:

- Various 2–loop calculations for $N = 2, 4, 6, ...$ were repeated
  → agreement with our previous calculation.
- Several non–trivial scalar 3–loop diagrams were calculated using Feynman–parameters for all N
  → agreement with MATAD.

General structure of the result: the PS –case

$$
\begin{aligned}
A_{Qq}^{(3),\mathrm{PS}} + A_{qq,Q}^{(3),\mathrm{PS}} = & \frac{\hat{\gamma}_{qg}^{(0)}\gamma_{gq}^{(0)}}{48}\Biggl\{\gamma_{gg}^{(0)} - \gamma_{qq}^{(0)} + 4(n_f+1)\beta_{0,Q} + 6\beta_0\Biggr\}\ln^3\left(\frac{m^2}{\mu^2}\right) \\
& +\Biggl\{\frac{\hat{\gamma}_{qq}^{(1),\mathrm{PS}}}{2}\Bigl((n_f+1)\beta_{0,Q} - \beta_0\Bigr) + \frac{\hat{\gamma}_{qg}^{(0)}}{8}\Bigl((n_f+1)\hat{\gamma}_{gq}^{(1)} - \gamma_{gq}^{(1)}\Bigr) - \frac{\gamma_{gq}^{(0)}\hat{\gamma}_{qg}^{(1)}}{8}\Biggr\}\ln^2\left(\frac{m^2}{\mu^2}\right) \\
& +\Biggl\{\frac{\hat{\gamma}_{qq}^{(2),\mathrm{PS}}}{2} - \zeta_2\frac{\hat{\gamma}_{qg}^{(0)}\gamma_{gq}^{(0)}}{16}\Bigl(\gamma_{gg}^{(0)} - \gamma_{qq}^{(0)} + 4(n_f+1)\beta_{0,Q} + 6\beta_0\Bigr) - 2a_{Qq}^{(2),\mathrm{PS}}\beta_0 \\
& +\frac{n_f+1}{2}\hat{\gamma}_{qg}^{(0)}a_{gq,Q}^{(2)} - \frac{\gamma_{gq}^{(0)}}{2}a_{Qg}^{(2)}\Biggr\}\ln\left(\frac{m^2}{\mu^2}\right) + \zeta_3\frac{\gamma_{gq}^{(0)}\hat{\gamma}_{qg}^{(0)}}{48}\Bigl(\gamma_{gg}^{(0)} - \gamma_{qq}^{(0)} + 4n_f\beta_{0,Q} + 6\beta_0\Bigr) \\
& +\frac{\zeta_2}{16}\Bigl(-4n_f\beta_{0,Q}\hat{\gamma}_{qq}^{(1),\mathrm{PS}} + \hat{\gamma}_{qg}^{(0)}\gamma_{gq}^{(1)}\Bigr) + 4(\beta_0+\beta_{0,Q})\overline{a}_{Qq}^{(2),\mathrm{PS}} + \gamma_{gq}^{(0)}\overline{a}_{Qg}^{(2)} - (n_f+1)\hat{\gamma}_{qg}^{(0)}\overline{a}_{gq,Q}^{(2)} \\
& +C_F\Bigl(-(4+\frac{3}{4}\zeta_2)\hat{\gamma}_{qg}^{(0)}\gamma_{gq}^{(0)} - 4\hat{\gamma}_{qq}^{(1),\mathrm{PS}} + 12a_{Qq}^{(2),\mathrm{PS}}\Bigr) + a_{Qq}^{(3),\mathrm{PS}} + a_{qq,Q}^{(3),\mathrm{PS}} \ .
\end{aligned}
$$

- n_f–dependence non–trivial. Take all quantities at n_f flavors and adopt notation

$$
\hat{\gamma}_{ij} \equiv \gamma_{ij}(n_f+1) - \gamma_{ij}(n_f) \ , \quad \beta_{0,Q} \equiv \beta_0(n_f+1) - \beta_0(n_f) \ .
$$

- There are similar formulas for the remaining OMEs.

- Number of Diagrams to be calculated:
$A_{Q(q)q}^{(3),\mathsf{PS}}$: 132, $A_{qq}^{(3),\mathsf{NS}}$: 128, $A_{gq}^{(3)}$: 89, $A_{Qg}^{(3)}$: 1498, $A_{gg,Q}^{(3)}$: 865. $\sim$ 250 days CT
- We calculated the terms

$$A_{Qq}^{(3),\mathsf{PS}} + A_{qq,Q}^{(3),\mathsf{PS}}\,, \quad A_{qq,Q}^{(3),\mathsf{NS}}\,, \quad A_{gq,Q}^{(3)}: \quad (2-12); \quad A_{Qg}^{(3)},\quad A_{gg,Q}^{(3)},\quad A_{gg,Q}^{(3)}: \quad (2-10);$$

for $N = 2, 4, 6, 8, 10, (12)$ using MATAD and find agreement of the pole terms with the prediction obtained from renormalization.

- An additional check is provided by the sum rule

$$A_{qq,Q}^{(3),\mathsf{NS}}\Big|_{N=2} + A_{qq,Q}^{(3),\mathsf{PS}}\Big|_{N=2} + A_{Qq}^{(3),\mathsf{PS}}\Big|_{N=2} + A_{gq,Q}^{(3)}\Big|_{N=2} = 0\,,$$

which is fulfilled by our result.

- All terms proportional to ζ_2 cancel in the renormalized result for $A_{Q(q)q}^{(3),\mathrm{NS\,PS}}$, $A_{gq,Q}^{(3)}$
- We observe the number

$$\mathsf{B4} = -4\zeta_2 \ln^2 2 + \frac{2}{3}\ln^4 2 - \frac{13}{2}\zeta_4 + 16\mathsf{Li}_4\left(\frac{1}{2}\right) = -8\sigma_{-3,-1} + \frac{11}{2}\zeta_4\,.$$

- The term B4 appears as

$$\left(C_F - \frac{C_A}{2}\right)\mathsf{B4}\,.$$

Result for the renormalized PS–term for $N = 4$.

$$A_{Qq}^{(3),\mathsf{PS}} + A_{qq,Q}^{(3),\mathsf{PS}}\Big|_{N=4} = \Big\{ -\frac{484}{2025} C_F {T_F}^2 (2n_f + 1) + \frac{4598}{3375} C_F C_A T_F - \frac{18997}{40500} {C_F}^2 T_F \Big\} \ln^3\Big(\frac{m^2}{\mu^2}\Big)$$
$$+ \Big\{ -\frac{16}{125} C_F {T_F}^2 + \frac{36751}{202500} C_F C_A T_F - \frac{697631}{405000} {C_F}^2 T_F \Big\} \ln^2\Big(\frac{m^2}{\mu^2}\Big) + \Big\{ -\frac{2131169}{303750} C_F {T_F}^2 n_f$$
$$- \frac{427141}{121500} C_F {T_F}^2 + \Big(-\frac{484}{75}\zeta_3 + \frac{24888821}{2700000} \Big) C_F C_A T_F + \Big(\frac{484}{75}\zeta_3 + \frac{63582197}{16200000} \Big) {C_F}^2 T_F \Big\} \ln\Big(\frac{m^2}{\mu^2}\Big)$$
$$+ \Big(\frac{7744}{2025}\zeta_3 - \frac{143929913}{27337500} \Big) C_F {T_F}^2 n_f + \Big(-\frac{13552}{2025}\zeta_3 + \frac{218235943}{54675000} \Big) C_F {T_F}^2 + \Big(\frac{242}{225}\mathsf{B4} - \frac{242}{25}\zeta_4$$
$$+ \frac{86833}{13500}\zeta_3 + \frac{4628174}{1265625} \Big) C_F C_A T_F + \Big(-\frac{484}{225}\mathsf{B4} + \frac{242}{25}\zeta_4 + \frac{298363}{20250}\zeta_3 - \frac{57518389433}{2187000000} \Big) {C_F}^2 T_F \;.$$

The constant terms: N = 12 $\hat{\hat{a}}_{gq,Q}^{(3)}$:

$$\hat{\hat{a}}_{gq,Q}^{(3)}\Big|_{N=12} = T_F\left(\frac{m^2}{\mu^2}\right)^{3\varepsilon/2}\Biggl\{C_FT_Fn_f\left(-\frac{8848}{3861}\zeta_3-\frac{14257247}{57972915}\zeta_2+\frac{9228836319135394697}{2258497163057335200}\right)$$
$$+C_FC_A\Biggl[-\frac{1264}{1287}\mathsf{B}_4+\frac{2528}{715}\zeta_2^2-\frac{999900989}{173918745}\zeta_3-\frac{693594486209}{3798385390800}\zeta_2$$
$$-\frac{1515875996003174876943331}{147976734123516602304000}\Biggr]+C_F^2\Biggl[\frac{2528}{1287}\mathsf{B}_4-\frac{2528}{715}\zeta_2^2+\frac{43693776149}{2260943685}\zeta_3$$
$$-\frac{2486481253717}{1671289571952}\zeta_2-\frac{48679935129017185612582919}{4069360188396706563360000}\Biggr]+C_FT_F\Biggl[\frac{20224}{3861}\zeta_3$$
$$-\frac{28514494}{57972915}\zeta_2-\frac{2105210836073143063}{1129248581528667600}\Biggr]\Biggr\}.$$

The constant terms: N = 12 $\hat{\hat{a}}_{qq,Q}^{(3),\mathrm{NS}}$:

$$
\begin{aligned}
\hat{\hat{a}}_{qq,Q}^{(3),\mathsf{NS}}\Big|_{N=12} = \; & T_F \left(\frac{m^2}{\mu^2}\right)^{3\varepsilon/2} \Bigg\{ C_F T_F n_f \left(\frac{6774784}{173745}\zeta_3 - \frac{90143221429}{7304587290}\zeta_2 - \frac{1201733391177720469772303}{711426606363060588 0000} \right) \\
& + C_F C_A \Bigg[\frac{3387392}{135135}\mathsf{B}_4 - \frac{6774784}{75075}\zeta_2^2 + \frac{51577729507}{158107950}\zeta_3 + \frac{2401246832561}{243486243000}\zeta_2 \\
& - \frac{1262073436041562279 42043}{2463815086971638400000} \Bigg] + C_F^2 \Bigg[-\frac{6774784}{135135}\mathsf{B}_4 + \frac{6774784}{75075}\zeta_2^2 \\
& - \frac{79117185295}{243486243}\zeta_3 + \frac{108605787257580461}{1096783781593500}\zeta_2 + \frac{68296027149155250557867961293}{12208080565190119690080 0000} \Bigg] \\
& + C_F T_F \Bigg[-\frac{108396544}{1216215}\zeta_3 - \frac{90143221429}{3652293645}\zeta_2 - \frac{189306988923316881320303}{35571330318153029400 00} \Bigg] \Bigg\} .
\end{aligned}
$$

The constant terms: N = 12 $\hat{\hat{a}}_{Qq}^{(3),\mathrm{PS}} + \hat{\hat{a}}_{qq,Q}^{(3),\mathrm{PS}}$:

$$
\begin{aligned}
\hat{\hat{a}}_{Qq}^{(3),\mathsf{PS}}\Big|_{N=12} = \; & T_F \left(\frac{m^2}{\mu^2}\right)^{3\varepsilon/2} \Biggl\{ C_F T_F n_f \left(\frac{24964}{150579}\zeta_3 - \frac{1291174013}{63306423180}\zeta_2 - \frac{662155770929305616017 7}{1233139451029305019200 0} \right) \\
& + C_F C_A \left[\frac{12482}{117117}\mathsf{B}_4 - \frac{24964}{65065}\zeta_2^2 - \frac{64839185833913}{16206444334080}\zeta_3 + \frac{489403711559293}{1382612282251200}\zeta_2 \right. \\
& \left. + \frac{9683070501568269053982065 47}{1077270624419200864773120 00} \right] + C_F^2 \left[-\frac{24964}{117117}\mathsf{B}_4 + \frac{24964}{65065}\zeta_2^2 \right. \\
& + \frac{418408135384633}{8103222167040}\zeta_3 - \frac{72904483229177}{152087351047632 00}\zeta_2 \\
& \left. - \frac{19021129843983468515905514828 9}{29624942171528023781260800 00} \right] + C_F T_F \left[-\frac{798848}{1054053}\zeta_3 \right. \\
& \left. + \frac{11471393}{347837490}\zeta_2 + \frac{1727596215111011341}{1355098297834401120 0} \right] \Biggr\} . \\
\hat{\hat{a}}_{qq,Q}^{(3),\mathsf{PS}}\Big|_{N=12} = \; & \left(\frac{m^2}{\mu^2}\right)^{3\varepsilon/2} C_F T_F^2 n_f \left\{ \frac{24964}{150579}\zeta_3 + \frac{583767694}{15826605795}\zeta_2 - \frac{6724380801633998071}{385356078446657818 50} \right\} .
\end{aligned}
$$

The constant terms: N = 10 $\hat{\hat{a}}_{gg,Q}^{(3)}$:

$$
\begin{aligned}
\hat{\hat{a}}_{gg,Q}^{(3)}\Big|_{N=10} = \; & T_F \left(\frac{m^2}{\mu^2}\right)^{3\varepsilon/2} \Bigg\{ n_f T_F \Bigg(C_A \left[\frac{215128}{4455}\zeta_3 - \frac{81362132}{5145525}\zeta_2 - \frac{6069333056458984}{26476427525625}\right] \\
& + C_F \left[-\frac{351232}{735075}\zeta_3 - \frac{799867252}{121287375}\zeta_2 - \frac{1006983638998844296}{4368610541728125}\right]\Bigg) + C_A^2 \left[\frac{17788828}{571725}\mathsf{B}_4\right. \\
& - \frac{17746492}{63525}\zeta_4 + \frac{2690944765495521109}{519314374656000}\zeta_3 + \frac{1444408720649}{55468759500}\zeta_2 \\
& \left. - \frac{15434483462331661005275759}{327337774462347264000000}\right] + C_A C_F \left[-\frac{35662328}{571725}\mathsf{B}_4 + \frac{17831164}{63525}\zeta_4\right. \\
& \left. - \frac{3288460968359099}{37093883904000}\zeta_3 + \frac{6078270984602}{46695639375}\zeta_2 + \frac{207095356146239371087405921}{771581896946961408000000}\right] \\
& + C_F^2 \left[\frac{896}{3025}\mathsf{B}_4 - \frac{4032}{3025}\zeta_4 + \frac{7140954579599}{198717235200}\zeta_3 - \frac{282148432}{4002483375}\zeta_2\right. \\
& \left. + \frac{55377792586772052149323}{206673722396507520000000}\right] + T_F C_A \left[-\frac{85188238297}{729907200}\zeta_3 - \frac{33330316}{735075}\zeta_2\right. \\
& \left. - \frac{630598434818955502807}{4337897885798400000}\right] + T_F C_F \left[-\frac{71350574183}{12043468800}\zeta_3 - \frac{3517889264}{121287375}\zeta_2\right. \\
& \left. - \frac{655690580559958774157}{357876575578368000000}\right] + \frac{64}{27}\zeta_3 {T_F}^2 \Bigg\} \,.
\end{aligned}
$$

The constant terms: N = 10 $\quad \hat{\hat{a}}_{Qg}^{(3)} + \hat{\hat{a}}_{qg,Q}^{(3)}$:

$$
\begin{aligned}
\hat{\hat{a}}_{Qg}^{(3)}\Big|_{N=10} = \; & T_F \left(\frac{m^2}{\mu^2}\right)^{3\varepsilon/2} \Biggl\{ n_f T_F \Biggl(C_A \Bigl[-\frac{1505896}{245025}\zeta_3 + \frac{189965849}{188669250}\zeta_2 + \frac{297277185134077151}{15532837481700000} \Bigr] \\
& + C_F \Bigl[\frac{62292104}{13476375}\zeta_3 - \frac{49652772817}{93391278750}\zeta_2 - \frac{1178560772273339822317}{107642563748181000000} \Bigr] \Biggr) + C_A^2 \Bigl[-\frac{563692}{81675}\mathrm{B}_4 \\
& + \frac{483988}{9075}\zeta_4 - \frac{103652031822049723}{415451499724800}\zeta_3 - \frac{20114890664357}{581101290000}\zeta_2 \\
& + \frac{68303634635669246922536599}{685850575063965696000000} \Bigr] + C_A C_F \Bigl[\frac{1286792}{81675}\mathrm{B}_4 - \frac{643396}{9075}\zeta_4 \\
& - \frac{761897167477437907}{33236119977984000}\zeta_3 + \frac{15455008277}{660342375}\zeta_2 + \frac{872201479486471797889957487}{29928025093700321280000000} \Bigr] \\
& + C_F^2 \Bigl[-\frac{11808}{3025}\mathrm{B}_4 + \frac{53136}{3025}\zeta_4 + \frac{9636017147214304991}{7122025709568000}\zeta_3 + \frac{14699237127551}{15689734830000}\zeta_2 \\
& - \frac{247930147349635960148869654541}{148143724213816590336000000} \Bigr] + T_F C_A \Bigl[\frac{4206955789}{377338500}\zeta_2 + \frac{123553074914173}{5755172290560}\zeta_3 \\
& + \frac{2323118975810619964522 9}{63339735648043008000 0} \Bigr] + T_F C_F \Bigl[-\frac{502987059528463}{113048027136000}\zeta_3 + \frac{24683221051}{46695639375}\zeta_2 \\
& - \frac{1831993118263044461191214 9}{1410892611560158003200000} \Bigr] - \frac{896}{1485} T_F^2 \zeta_3 \Biggr\} . \\
\hat{\hat{a}}_{qg,Q}^{(3)}\Big|_{N=10} = \; & n_f T_F^2 \left(\frac{m^2}{\mu^2}\right)^{3\varepsilon/2} \Biggl\{ C_A \Bigl[-\frac{1505896}{245025}\zeta_3 + \frac{1109186999}{377338500}\zeta_2 + \frac{6542127929072987}{191763425700000} \Bigr] \\
& + C_F \Bigl[\frac{62292104}{13476375}\zeta_3 - \frac{83961181063}{93391278750}\zeta_2 - \frac{353813854966442889041}{21528512749636200000} \Bigr] \Biggr) \Biggr\}
\end{aligned}
$$

We obtain for the moments of the PS, and $\hat{\gamma}_{gq}^{(2)}$ anomalous dimensions

N	$\hat{\gamma}_{qq}^{(2),\mathsf{PS}}/T_F/C_F$
2	$-\frac{5024}{243}T_F(1+2n_f)+\frac{256}{3}\Big(C_F-C_A\Big)\zeta_3+\frac{10136}{243}C_A-\frac{14728}{243}C_F$
4	$-\frac{618673}{151875}T_F(1+2n_f)+\frac{968}{75}\Big(C_F-C_A\Big)\zeta_3+\frac{2485097}{506250}C_A-\frac{2217031}{675000}C_F$
6	$-\frac{126223052}{72930375}T_F(1+2n_f)+\frac{3872}{735}\Big(C_F-C_A\Big)\zeta_3+\frac{1988624681}{4084101000}C_A+\frac{11602048711}{10210252500}C_F$
8	$-\frac{13131081443}{13502538000}T_F(1+2n_f)+\frac{2738}{945}\Big(C_F-C_A\Big)\zeta_3-\frac{343248329803}{648121824000}C_A+\frac{39929737384469}{22684263840000}C_F$
10	$-\frac{265847305072}{420260754375}T_F(1+2n_f)+\frac{50176}{27225}\Big(C_F-C_A\Big)\zeta_3-\frac{1028766412107043}{1294403123475000}C_A+\frac{839864254987192}{485401171303125}C_F$
12	$-\frac{2566080055386457}{5703275664286200}T_F(1+2n_f)+\frac{49928}{39039}\Big(C_F-C_A\Big)\zeta_3-\frac{6969748954384649691}{8303969367200707200}C_A+\frac{8603325540244325619 7}{548061978235246675 20}C_F$

N	$\hat{\gamma}_{gq}^{(2)}/T_F/C_F$
2	$\frac{2272}{81}T_F(1+2n_f)+\frac{512}{3}\Big(C_A-C_F\Big)\zeta_3+\frac{88}{9}C_A+\frac{28376}{243}C_F$
4	$\frac{109462}{10125}T_F(1+2n_f)+\frac{704}{15}\Big(C_A-C_F\Big)\zeta_3-\frac{799}{12150}C_A+\frac{14606684}{759375}C_F$
6	$\frac{22667672}{3472875}T_F(1+2n_f)+\frac{2816}{105}\Big(C_A-C_F\Big)\zeta_3-\frac{253841107}{145860750}C_A+\frac{20157323311}{2552563125}C_F$
8	$\frac{339184373}{75014100}T_F(1+2n_f)+\frac{1184}{63}\Big(C_A-C_F\Big)\zeta_3-\frac{3105820553}{1687817250}C_A+\frac{8498139408671}{2268426384000}C_F$
10	$\frac{1218139408}{363862125}T_F(1+2n_f)+\frac{7168}{495}\Big(C_A-C_F\Big)\zeta_3-\frac{18846629176433}{11767301122500}\Big)C_A+\frac{529979902254031}{323600780868750}C_F$
12	$\frac{13454024393417}{5222779912350}T_F(1+2n_f)+\frac{5056}{429}\Big(C_A-C_F\Big)\zeta_3-\frac{6419049307813978 9}{48885219979596000}C_A+\frac{1401404001326440151}{3495293228541114000}C_F$

The moments of the $\hat{\gamma}_{gg}^{(2)}$ and $\hat{\gamma}_{qg}^{(2)}$ anomalous dimensions

N	$\hat{\gamma}_{qg}^{(2)}/T_F$
2	$(1+2n_f)T_F\left(\frac{8464}{243}C_A-\frac{1384}{243}C_F\right)+\frac{\zeta_3}{3}\left(-416C_AC_F+288C_A^2+128C_F^2\right)-\frac{7178}{81}C_A^2+\frac{556}{9}C_AC_F-\frac{8620}{243}C_F^2$
4	$(1+2n_f)T_F\left(\frac{4481539}{303750}C_A+\frac{9613841}{3037500}C_F\right)+\frac{\zeta_3}{25}\left(2832C_A^2-3876C_AC_F+1044C_F^2\right)$ $-\frac{295110931}{3037500}C_A^2+\frac{278546497}{2025000}C_AC_F-\frac{757117001}{12150000}C_F^2$
6	$(1+2n_f)T_F\left(\frac{86617163}{11668860}C_A+\frac{1539874183}{340341750}C_F\right)+\frac{\zeta_3}{735}\left(69864C_A^2-94664C_AC_F+24800C_F^2\right)$ $-\frac{58595443051}{653456160}C_A^2+\frac{1199181909343}{8168202000}C_AC_F-\frac{2933980223981}{40841010000}C_F^2$
8	$(1+2n_f)T_F\left(\frac{10379424541}{2755620000}C_A+\frac{7903297846481}{1620304560000}C_F\right)+\zeta_3\left(\frac{128042}{1575}C_A^2-\frac{515201}{4725}C_AC_F+\frac{749}{27}C_F^2\right)$ $-\frac{24648658224523}{289340100000}C_A^2+\frac{4896295442015177}{32406091200000}C_AC_F-\frac{4374484944665803}{56710659600000}C_F^2$
10	$(1+2n_f)T_F\left(\frac{1669885489}{988267500}C_A+\frac{1584713325754369}{323600780868750}C_F\right)+\zeta_3\left(\frac{1935952}{27225}C_A^2-\frac{2573584}{27225}C_AC_F+\frac{70848}{3025}C_F^2\right)$ $-\frac{21025430857658971}{255684567600000}C_A^2+\frac{926990216580622991}{6040547909550000}C_AC_F-\frac{1091980048536213833}{13591232796487500}C_F^2$

N	$\hat{\gamma}_{gg}^{(2)}/T_F$
2	$(1+2n_f)T_F\left(-\frac{8464}{243}C_A+\frac{1384}{243}C_F\right)+\frac{\zeta_3}{3}\left(-288C_A^2+416C_AC_F-128C_F^2\right)+\frac{7178}{81}C_A^2-\frac{556}{9}C_AC_F+\frac{8620}{243}C_F^2$
4	$(1+2n_f)T_F\left(-\frac{757861}{30375}C_A-\frac{979774}{151875}C_F\right)+\frac{\zeta_3}{25}\left(-6264C_A^2+6528C_AC_F-264C_F^2\right)$ $+\frac{53797499}{607500}C_A^2-\frac{235535117}{1012500}C_AC_F+\frac{2557151}{759375}C_F^2$
6	$(1+2n_f)T_F\left(-\frac{52781896}{2083725}C_A-\frac{560828662}{72930375}C_F\right)+\zeta_3\left(-\frac{75168}{245}C_A^2+\frac{229024}{735}C_AC_F-\frac{704}{147}C_F^2\right)$ $+\frac{9763460989}{116688600}C_A^2-\frac{9691228129}{32672808}C_AC_F-\frac{11024749151}{10210252500}C_F^2$
8	$(1+2n_f)T_F\left(-\frac{420970849}{16074450}C_A-\frac{6990254812}{843908625}C_F\right)+\zeta_3\left(-\frac{325174}{945}C_A^2+\frac{327764}{945}C_AC_F-\frac{74}{27}C_F^2\right)$ $+\frac{2080130771161}{25719120000}C_A^2-\frac{220111823810087}{648121824000}C_AC_F-\frac{14058417959723}{5671065960000}C_F^2$
10	$(1+2n_f)T_F\left(-\frac{2752314359}{101881395}C_A-\frac{3631303571944}{420260754375}C_F\right)+\zeta_3\left(-\frac{70985968}{190575}C_A^2+\frac{71324656}{190575}C_AC_F-\frac{5376}{3025}C_F^2\right)$ $+\frac{43228502203851731}{549140719050000}C_A^2-\frac{3374081335517123191}{9060821864325000}C_FC_A-\frac{3009386129483453}{970802342606250}C_F^2$

The moments of the NS^+ $\hat{\gamma}_{qq}^{(2)}$ anomalous dimensions

N	$\hat{\gamma}_{qq}^{(2),\mathsf{NS},+}/T_F/C_F$
2	$-\frac{1792}{243}T_F(1+2n_f)+\frac{256}{3}\Big(C_F-C_A\Big)\zeta_3-\frac{12512}{243}C_A-\frac{13648}{243}C_F$
4	$-\frac{384277}{30375}T_F(1+2n_f)+\frac{2512}{15}\Big(C_F-C_A\Big)\zeta_3-\frac{8802581}{121500}C_A-\frac{165237563}{1215000}C_F$
6	$-\frac{160695142}{10418625}T_F(1+2n_f)+\frac{22688}{105}\Big(C_F-C_A\Big)\zeta_3-\frac{13978373}{171500}C_A-\frac{44644018231}{243101250}C_F$
8	$-\frac{38920977797}{2250423000}T_F(1+2n_f)+\frac{79064}{315}\Big(C_F-C_A\Big)\zeta_3-\frac{1578915745223}{18003384000}C_A-\frac{91675209372043}{420078960000}C_F$
10	$-\frac{27995901056887}{1497656506500}T_F(1+2n_f)+\frac{192880}{693}\Big(C_F-C_A\Big)\zeta_3-\frac{9007773127403}{97250422500}C_A-\frac{75522073210471127}{307518802668000}C_F$
12	$-\frac{65155853387858071}{3290351344780500}T_F(1+2n_f)+\frac{13549568}{45045}\Big(C_F-C_A\Big)\zeta_3-\frac{25478252190337435009}{263228107582440000}C_A$ $-\frac{35346062280941906036867}{131745667845011220000}C_F$

$\Longrightarrow$ Agreement for the terms $\propto\ T_F$ with

[Larin, Nogueira, Ritbergen, Vermaseren, 1997; Moch, Vermaseren, Vogt, 2004.]

- How far can we go ?

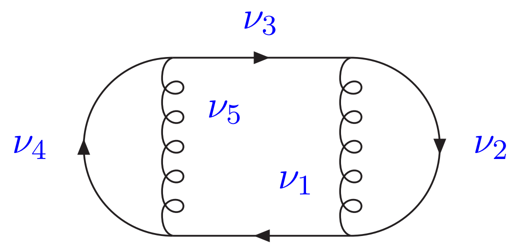
$N = 14$ in some cases; generally: $N = 10$ (achieved). $\Longrightarrow$ Phenomenology

CPU time: various months; up to 64 Gb processors needed.

Unfortunately not enough to perform the automatic
fixed moments $\rightarrow$ all moments turn. [J.B., Kauers, Klein, Schneider, 0902.4091 [hep-ph]].

Fixed moments using Feynman–parameters

Consider e.g the 3–loop tadpole diagram



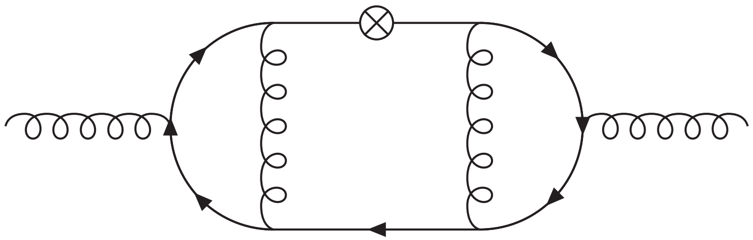
Using Feynman–parameters, one obtains a representation in terms of a double sum

$$I = C\Gamma\left[\begin{matrix} 2+\varepsilon/2-\nu_1, 2+\varepsilon/2-\nu_5, \nu_{12}-2-\varepsilon/2, \nu_{45}-2-\varepsilon/2, \nu_{1345}-4-\varepsilon, \nu_{12345}-6-3/2\varepsilon \\ \nu_1, \nu_2, \nu_4, 2+\varepsilon/2, \nu_{345}-2-\varepsilon/2, \nu_{12345}-4-\varepsilon \end{matrix}\right]$$
$$\sum_{m,n=0}^{\infty} \frac{(\nu_{345}-2-\varepsilon/2)_{n+m}(\nu_{12345}-6-3/2\varepsilon)_m(2+\varepsilon/2-\nu_1)_m(2+\varepsilon/2-\nu_5)_n(\nu_{45}-2-\varepsilon/2)_n}{m!n!(\nu_{12345}-4-\varepsilon)_{n+m}(\nu_{345}-2-\varepsilon/2)_m(\nu_{345}-2-\varepsilon/2)_n},$$

which derives from a Appell–function of the first kind, F_1 .

First all–N results

As in the 2–loop case, for any diagram deriving from the tadpole–ladder topology, one obtains for fixed values of N a finite sum over double sums of the same type. Consider e.g. the scalar diagram



For the above diagram, we obtained a result for arbitrary N using similar techniques as in the 2–loop case and the package SIGMA.

$$I_1 = -\frac{4(N+1)S_1+4}{(N+1)^2(N+2)}\zeta_3 + \frac{2S_{2,1,1}}{(N+2)(N+3)} + \frac{1}{(N+1)(N+2)(N+3)}\Bigg\{-2(3N+5)S_{3,1} - \frac{{S_1}^4}{4}$$
$$+\frac{4(N+1)S_1-4N}{N+1}S_{2,1} + 2\Big((2N+3)S_1 + \frac{5N+6}{N+1}\Big)S_3 + \frac{9+4N}{4}{S_2}^2 + \Big(\frac{5N}{N+1}S_1 + 2\frac{7N+11}{(N+1)(N+2)}$$
$$-\frac{5}{2}{S_1}^2\Big)S_2 + \frac{N}{N+1}{S_1}^3 + \frac{2(3N+5){S_1}^2}{(N+1)(N+2)} + \frac{4(2N+3)S_1}{(N+1)^2(N+2)} - \frac{(2N+3)S_4}{2} + 8\frac{2N+3}{(N+1)^3(N+2)}\Bigg\} + O(\varepsilon)\ .$$

For fixed N, this formula agrees with the result we obtain using MATAD.

6. Conclusions

- The heavy flavor contributions to the structure function F_2 are rather large in the region of lower values of x.
- Competitive QCD precision analyzes therefore require the description of the heavy quark contributions to 3–loop order.
- The inclusive heavy flavor contributions to $F_{2,L}(x, Q^2)$ are known to NLO (in semi-analytic form with fast numeric implementations.)
- Complete analytic results are known in the region $Q^2 \gg m^2$ at NLO for $F_{2,L}(x, Q^2), g_{1,2}(x, Q^2)$. They are expressed in terms of massive operator matrix elements and the corresponding massless Wilson coefficients.
  Threshold resummations were performed.
- $F_L(x, Q^2)$ is known to NNLO for $Q^2 \gg m^2$.
- The calculation of fixed moments of the massive operator matrix elements at $O(a_s^3)$ has been finished (Monday) for $N = 10, 12$, which will be followed by some first phenomenological parameterization.
- We also calculate the matrix elements necessary to transform from the FFNS to the VFNS.
- First steps towards the calculation of the massive operator matrix elements at $O(a_s^3)$ for general values of N were performed.