



Harmonic Sums and Polylogarithms Generated by Cyclotomic Polynomials

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Introduction

Harmonic sums: (Vermaseren; JB, Kurth, 1998)

$$S_{b, \vec{a}}(N) = \sum_{k=1}^N \frac{(\text{sign}(b))^k}{k^{|b|}} S_{\vec{a}}(k), \quad S_{\emptyset}(N) = 1, \quad b, a_i \in \mathbb{Z} \setminus \{0\}.$$

The Mellin transform mapping between N and x space:

$$\mathbf{M}[f(x)](N) = \int_0^1 dx x^N f(x).$$

Generalized harmonic sums: (Moch, Uwer, Weinzierl, 2001)

$$S_{b, \vec{a}}(\zeta, \vec{\xi}; N) = \sum_{k=1}^N \frac{\zeta^k}{k^b} S_{\vec{a}}(\vec{\xi}; k), \quad b, a_i \in \mathbb{N}_+; \zeta, \xi_i \in \mathbb{R}^* = \mathbb{R} \setminus \{0\}$$

Known examples are related to the second index set $\xi_i \in \{1, -1, 1/2, -1/2, 2, -2\}$.

Other Summands occurring:

$$\frac{(\pm 1)^k}{(l \cdot k + m)^n}, \quad \text{with } l, m, n \in \mathbb{N}_+.$$

New Mellin variable:

$$N \rightarrow k \cdot N,$$



Basic Formalism

$$S_{\{a_1, b_1, c_1\}, \dots, \{a_l, b_l, c_l\}}(s_1, \dots, s_l; N) = \sum_{k_1=1}^N \frac{s_1^{k_1}}{(a_1 k_1 + b_1)^{c_1}} S_{\{a_2, b_2, c_2\}; \dots; \{a_l, b_l, c_l\}}(s_2, \dots, s_l; k_1),$$

$$S_{\emptyset} = 1, \quad a_i, c_i \in \mathbb{N}_+, b_i \in \mathbb{N}, \quad s_i = \pm 1, a_i > b_i$$

Summand representation:

$$\frac{(\pm 1)^k}{ak + b} = \int_0^1 dx x^{ak+b-1} (\pm 1)^k$$

$$\frac{(\pm 1)^k}{(ak + b)^c} = \int_0^1 \frac{dx_1}{x_1} \int_0^{x_1} \frac{dx_2}{x_2} \dots \int_0^{x_{c-2}} \frac{dx_{c-1}}{x_{c-1}} \int_0^{x_{c-1}} dx_c x_c^{ak+b-1} (\pm 1)^k,$$

$$\sum_{k=1}^l (\pm 1)^k x^{ak+b-1} = x^{a+b-1} \frac{(\pm x^a)^{l+1} - 1}{(\pm x^a) - 1}.$$





Let us illustrate the principle steps in case of the following example :

$$S_{\{3,2,2\},\{2,1,1\}}(1, -1; N) = \sum_{k=1}^N \frac{1}{(3k+2)^2} \sum_{l=1}^k \frac{(-1)^l}{(2l+1)} .$$

$$S_{\{3,2,2\},\{2,1,1\}}(1, -1; N) = \sum_{k=1}^N \int_0^1 dx \frac{x^2}{x^2+1} \frac{(-x^2)^k + 1}{(3k+2)^2} .$$

Setting $x = y^3$ one obtains

$$\begin{aligned} S_{\{3,2,2\},\{2,1,1\}}(1, -1; N) &= 12 \int_0^1 dy \frac{y^8}{y^6+1} \sum_{k=1}^N \frac{(-y^6)^k - 1}{(6k+4)^2} \\ &= 12 \int_0^1 dy \frac{y^4}{y^6+1} \left\{ \int_0^y \frac{dz}{z} \int_0^z dt t^9 \frac{(-t^6)^N - 1}{t^6+1} \right. \\ &\quad \left. - y^4 \int_0^1 \frac{dz}{z} \int_0^z dt t^9 \frac{t^{6N} - 1}{t^6 - 1} \right\} \\ &= 12 \int_0^1 dy \frac{y^4}{y^6+1} \int_0^y \frac{dz}{z} \int_0^z dt t^9 \frac{(-t^6)^N - 1}{t^6+1} \\ &\quad - (4 - \pi) \int_0^1 \frac{dz}{z} \int_0^z dt t^9 \frac{t^{6N} - 1}{t^6 - 1} \Bigg\} . \end{aligned}$$



In general, the polynomials

$$x^a - 1$$

decompose in a product of cyclotomic polynomials, except for $a = 1$ for which the expression is $\Phi_1(x)$. Moreover, the polynomials

$$x^a + 1 = \frac{x^{2a} - 1}{x^a - 1}$$

are either cyclotomic for $a = 2^n, n \in \mathbb{N}$ or decompose into products of cyclotomic polynomials in other cases. All factors divide $(x^a)^l - 1$, resp. $(-x^a)^l - 1$.

$$\Phi_1(x) = x - 1$$

$$\Phi_2(x) = x + 1$$

$$\Phi_3(x) = x^2 + x + 1$$

$$\Phi_4(x) = x^2 + 1$$

$$\Phi_5(x) = x^4 + x^3 + x^2 + x + 1$$

$$\Phi_6(x) = x^2 - x + 1$$

$$\Phi_7(x) = x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$$

$$\Phi_8(x) = x^4 + 1$$

$$\Phi_9(x) = x^6 + x^3 + 1$$

$$\Phi_{10}(x) = x^4 - x^3 + x^2 - x + 1$$

$$\Phi_{11}(x) = x^{10} + x^9 + x^8 + x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$$

$$\Phi_{12}(x) = x^4 - x^2 + 1, \text{ etc.}$$



Cyclotomic Harmonic Polylogarithms

To account for the newly emerging sums in perturbative calculations in Quantum Field Theory we introduce Poincaré-iterated integrals over the alphabet \mathfrak{A}

$$\mathfrak{A} = \left\{ \frac{1}{x} \right\} \cup \left\{ \frac{x^l}{\Phi_k(x)} \mid k \in \mathbb{N}_+, 0 \leq l < \varphi(k) \right\},$$

where $\Phi_k(x)$ denotes the k th cyclotomic polynomial and $\varphi(k)$ denotes Euler's totient function.

$$\Phi_n(x) = \frac{x^n - 1}{\prod_{d|n, d < n} \Phi_d(x)}, \quad d, n \in \mathbb{N}_+,$$

The alphabet \mathfrak{A} is an extension of the alphabet

$$\mathfrak{A}_H = \left\{ \frac{1}{x}, \frac{1}{\Phi_1(x)}, \frac{1}{\Phi_2(x)} \right\},$$

generating the usual harmonic polylogarithms (Remiddi, Vermaseren, 1999)



A shorthand notation for the letters of \mathfrak{Q} :

$$f_0^0(x) = \frac{1}{x}$$

$$f_k^l(x) = \frac{x^l}{\Phi_k(x)}, \quad k \in \mathbb{N}_+, l \in \mathbb{N}, l < \varphi(k).$$

Examples: (N. Nielsen, 1906)

$$\frac{1}{2}\beta\left(\frac{x}{2}\right) = \int_0^1 dt \frac{t^{x-1}}{t^2+1}$$

$$\beta\left(\frac{x}{3}\right) = \beta(x) - \int_0^1 dt t^{x-1} \frac{t-2}{t^2-t+1},$$

$$\beta(x) = \frac{1}{2} \left[\Psi\left(\frac{x+1}{2}\right) - \Psi\left(\frac{x}{2}\right) \right].$$



We form the Poincaré iterated integrals

$$C_{k_1, \dots, k_m}^{l_1, \dots, l_m}(z) = \frac{1}{m!} \ln^m(x) \quad \text{if } (l_1, \dots, l_m) = (0, \dots, 0),$$

$$(k_1, \dots, k_m) = (0, \dots, 0),$$

$$C_{k_m}^{l_m}(z) = \int_0^z dx f_{k_m}^{l_m}(x) \quad \text{if } k_m \neq 0,$$

$$C_{k_1, \dots, k_m}^{l_1, \dots, l_m}(z) = \int_0^z dx f_{k_1}^{l_1}(x) C_{k_2, \dots, k_m}^{l_2, \dots, l_m}(x) \quad \text{if } (k_1, \dots, k_m) \neq (0, \dots, 0),$$

and $C_{\vec{a}}^{\vec{l}}(z)$ denotes cyclotomic harmonic polylogarithms of weight $w = m$. They form a shuffle algebra by multiplication

$$C_{\vec{a}_1}^{\vec{a}_2}(z) \cdot C_{\vec{b}_1}^{\vec{b}_2}(z) = C_{\vec{a}_1}^{\vec{a}_2}(z) \sqcup C_{\vec{b}_1}^{\vec{b}_2}(z) = \sum_{\left[\begin{smallmatrix} \vec{c}_2 \\ \vec{c}_1 \end{smallmatrix} \right] \in \left[\begin{smallmatrix} \vec{a}_2 \\ \vec{a}_1 \end{smallmatrix} \right] \sqcup \left[\begin{smallmatrix} \vec{b}_2 \\ \vec{b}_1 \end{smallmatrix} \right]} C_{\vec{c}_1}^{\vec{c}_2}(z)$$

of M^w elements at weight w , where M denotes the number of chosen letters from \mathfrak{A} .

$$N^{\text{basic}}(w) = \frac{1}{w} \sum_{d|w} \mu\left(\frac{w}{d}\right) M^d, w \geq 1$$



weight	letters						
	2	3	4	5	6	7	8
1	2	3	4	5	6	7	8
2	1	3	6	10	15	21	28
3	2	8	20	40	70	112	168
4	3	18	60	150	315	588	1008
5	6	48	204	624	1554	3360	6552
6	9	116	670	2580	7735	19544	43596
7	18	312	2340	11160	39990	117648	299592
8	30	810	8160	48750	209790	729300	2096640

Table: Number of basic cyclotomic harmonic polylogarithms in dependence of the number of letters and weight.

$f_{i_3}^{j_b}$ representation:

$$\frac{y^4}{y^6 + 1} = \frac{1}{3} [f_4^0(y) - f_{12}^0(y) + 2f_{12}^2(y)] .$$



With integration by parts one obtains the following Mellin transforms of argument $6N$ of cyclotomic harmonic polylogarithms $C_{k_1, \dots, k_m}^{l_1, \dots, l_m}(x)$ weighted by the letters $f_l^k(x)$ of the alphabet \mathfrak{A} :

$$\begin{aligned}
 S_{\{3,2,2\}, \{2,1,1\}}(1, -1; N) = & \\
 \frac{1}{6}(4 - \pi) \int_0^1 dx x^3 (x^{6N} - 1) [6 + f_1^0(x) - f_2^0(x) - 2f_3^0(x) & \\
 - f_3^1(x) - 2f_6^0(x) + f_6^1(x)] C_0^0(x) & \\
 - 2 \int_0^1 dx x^3 [(-1)^N x^{6N} - 1] [3 - f_4^0(x) - 2f_{12}^0(x) + 2f_{12}^2(x)] C_0^0(x) & \\
 - \frac{4}{3} [C_{0,4}^{0,0}(1) - C_{0,12}^{0,0}(1) + 2C_{0,12}^{0,2}(1)] \int_0^1 dx x^3 [(-1)^N x^{6N} - 1] & \\
 \times [3 - f_4^0(x) - 2f_{12}^0(x) + 2f_{12}^2(x)] & \\
 + \frac{4}{3} \int_0^1 dx x^3 [(-1)^N x^{6N} - 1] [C_{0,4}^{0,0}(x) - C_{0,12}^{0,0}(x) + 2C_{0,12}^{0,2}(x)] & \\
 \times [3 - f_4^0(x) - 2f_{12}^0(x) + 2f_{12}^2(x)] . &
 \end{aligned}$$

$$C_{0,4}^{0,0} = -C, \quad C_{0,12}^{0,0(2)} = \alpha_1 \psi'(1/12) + \alpha_2 \psi'(5/12)$$



Cyclotomic Harmonic Sums

(i) The Single Sums:

$$S_{l,m,n}(N) = \sum_{k=0}^N \frac{(\pm 1)^k}{(l \cdot k + m)^n}; \quad \phi_k(l, N) = \int_0^1 dx x^N f_k'(x)$$

The summation package Sigma reduces to :

$$S_{-1}(N), S_{-1}(3N), S_{-1}(5N), S_1(N), S_1(2N), S_1(3N), S_1(4N), S_1(5N), S_1(6N), \\ \phi_3(3N)_+, \phi_3(6N)_+, \phi_4(2N), \phi_4(4N), \phi_4(6N), \phi_5(5N)_+, \phi_5(1, 5N)_+, \\ \phi_5(2, 5N)_+, \phi_6(3N), \phi_8(4N), \phi_8(2, 4N), \phi_{10}(5N), \phi_{10}(1, 5N), \phi_{10}(2, 5N), \phi_{12}(6N).$$

and the sums at $N \rightarrow \infty$

$$\sigma_{\{1,0,-1\}}, \sigma_{\{2,1,-1\}}, \sigma_{\{3,1,-1\}}, \sigma_{\{4,1,-1\}}, \sigma_{\{4,3,-1\}}, \sigma_{\{5,1,-1\}}, \sigma_{\{5,2,-1\}}, \sigma_{\{5,3,-1\}}, \sigma_{\{6,1,-1\}}$$

Asymptotic representations $|N| \rightarrow \infty$ are derived easily.

Needed within the summation calculus.





(ii) Relations between sums:

- Differentiation w.r.t. N , (D)
- Stuffle Algebra (A)
- Synchronization (M)
- Duplication Relations: (H_1, H_2)

(iii) Nested Sums: Consider:

$$\frac{1}{k^1}, \quad \frac{(-1)^k}{k^2}, \quad \frac{1}{(2k+1)^3}, \quad \frac{(-1)^k}{(2k+1)^4} \quad (*)$$

w	N_S	H_1	H_1, H_2	H_1, M	H_1, H_2, M	D	H_1, H_2, M, D	A	H_1, H_2, M, A	A, D	all
1	4	3	3	2	2	4	2	4	2	4	2
2	20	18	17	16	15	16	13	10	8	6	6
3	100	96	93	92	89	80	74	40	35	30	27
4	500	492	485	484	477	400	388	150	142	110	107
5	2500	2484	2469	2468	2453	2000	1976	624	607	474	465

• Reduction of the number of cyclotomic harmonic sums N_S over the elements at given weight w by applying the three multiple argument relations (H_1, H_2, M), differentiation w.r.t. to the external sum index N , (D), and the algebraic relations (A). A sequence of symbols corresponds to the combination of these relations.





Explicit counting relations for the number of basis elements given in the above Table can be derived :

$$N_A(w) = \frac{1}{w} \sum_{d|w} \mu\left(\frac{w}{d}\right) 5^d$$

$$N_D(w) = N_S(w) - N_S(w-1) = 16 \cdot 5^{w-2}$$

$$N_{H_1}(w) = N_S(w) - 2^{w-1} = 4 \cdot 5^{w-1} - 2^{w-1}$$

$$N_{H_1 H_2}(w) = N_S(w) - (2 \cdot 2^{w-1} - 1) = 4 \cdot 5^{w-1} - (2 \cdot 2^{w-1} - 1)$$

$$N_{H_1 M}(w) = N_S(w) - 2 \cdot 2^{w-1} = 4 \cdot 5^{w-1} - 2 \cdot 2^{w-1}$$

$$N_{H_1 H_2 M}(w) = N_S(w) - (3 \cdot 2^{w-1} - 1) = 4 \cdot 5^{w-1} - (3 \cdot 2^{w-1} - 1)$$

$$N_{AD}(w) = N_A(w) - N_A(w-1)$$

$$N_{AH_1 H_2 M}(w) = \frac{1}{w} \sum_{d|w} \mu\left(\frac{w}{d}\right) 5^d - \left(3 \cdot \frac{1}{w} \sum_{d|w} \mu\left(\frac{w}{d}\right) 2^d - 1\right)$$

$$N_{DH_1 H_2 M}(w) = N_{H_1 H_2 M}(w) - N_{H_1 H_2 M}(w-1) = 16 \cdot 5^{w-2} - 3 \cdot 2^{w-2}$$

$$N_{ADH_1 H_2 M}(w) = N_{AH_1 H_2 M}(w) - N_{AH_1 H_2 M}(w-1)$$

Here μ denotes the Möbius function.



Special Values: Single Sums

$$w = 1$$

$$\psi\left(\frac{p}{q}\right) = -\gamma_E - \ln(2q) - \frac{\pi}{2} \cot\left(\frac{p\pi}{q}\right) + 2 \sum_{k=1}^{[(q-1)/2]} \cos\left(\frac{2\pi kp}{q}\right) \ln\left[\sin\left(\frac{\pi k}{q}\right)\right]$$

$$\psi\left(\frac{1}{n}\right) = -n(\gamma_E + \ln(n)) - \sum_{k=2}^n \psi\left(\frac{k}{n}\right).$$

Further simplification if the regular q -polygon is constructible

$$q \in \{2, 3, 4, 5, 6, 8, 10, 12, 15, 16, 17, 20, 24, 30, 32, 34, 40, 48, 51, 60, 64, \dots\}$$

Consider the above alphabet (*).

The basis elements are:

$$\sigma_0, \quad \ln(2), \quad \pi$$

$$w \geq 2, l \leq 6:$$

$$\zeta_{2k+1}, \psi^{(2k+1)}\left(\frac{1}{3}\right), \text{Ti}_{2k}(1), \psi^{(k)}\left(\frac{1}{5}\right), \psi^{(2k+1)}\left(\frac{2}{5}\right), \psi^{(k)}\left(\frac{1}{8}\right), \psi^{(2k)}\left(\frac{1}{12}\right)$$





$$\text{Ti}_l(1) = \beta_D(l) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^l}$$

$$\text{Ti}_2(1) = \mathbf{C}$$

Some new numbers occurring at $w = 1$:

$$\ln(3), \ln(\sqrt{2} - 1), \ln(\sqrt{3} - 1), \ln(\sqrt{5} - 1), \dots$$

+ algebraic (irrational) numbers



$w > 1$, higher depth, alphabet (*)

weight	N_S	A	SH	A + SH	A + sh + H_1	A + SH + H_1 + H_2	A + SH + H_1 + H_2 + M
1	4	4	4	4	4	3	3
2	20	10	13	3	3	2	1
3	100	40	46	6	6	5	3
4	500	150	163	10	10	9	6
5	2500	624	650	21	21	19	13
6	12500	2580	2635	36	36	34	25

Some basis elements:

$$\begin{aligned} \sigma_{\{2,1,-2\}} &= -1 + \mathbf{C} \\ \sigma_{\{1,0,-2\},\{2,1,-1\}} &= \frac{\pi^2}{12} - \frac{\pi^3}{48} + \frac{1}{2} \int_0^1 dx \frac{\sqrt{x}}{x+1} \text{Li}_2(x) \\ \sigma_{\{2,1,-2\},\{1,0,-1\}} &= -\mathbf{C} \ln(2) + \int_0^1 dx \frac{1}{1+x} \frac{\chi_2(\sqrt{x})}{\sqrt{x}} \\ \chi_\nu(x) &= \frac{1}{2} [\text{Li}_\nu(x) - \text{Li}_\nu(-x)] \end{aligned}$$





Infinite Sums with more Cyclotomic Letters

$$w = 1$$

$$\frac{(\pm 1)^k}{(lk + m)^n}, \quad 1 \leq n \leq 2, 1 \leq l \leq 20, m < l.$$

l	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
sums	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30	32	34	36	38	40
basis	2	3	4	5	6	6	8	9	8	10	12	10	14	14	11	17	18	14	20	18
new basis sums	2	1	2	2	4	1	6	4	4	3	10	2	12	5	3	8	16	4	18	6

The number of the $w = 1$ cyclotomic harmonic sums (1) up to $l = 20$, the basis elements at fixed value of l , and the new basis elements in ascending sequence.

$$\sigma_{\{1,0,1\}}, \sigma_{\{1,0,-1\}}, \sigma_{\{2,1,-1\}}, \sigma_{\{3,1,1\}}, \sigma_{\{3,1,-1\}}, \sigma_{\{4,1,-1\}}, \sigma_{\{4,3,-1\}}, \\ \sigma_{\{5,1,1\}}, \sigma_{\{5,1,-1\}}, \sigma_{\{5,2,-1\}}, \sigma_{\{5,3,-1\}}, \sigma_{\{6,1,-1\}}, \dots$$



$$w = 2$$

l	N_S	SH	A	A + SH	A + SH + H_1	A + SH + H_1 + H_2	A + SH + H_1 + H_2 + M
1	6	4	3	1	1	1	1
2	20	13	10	3	3	2	1
3	42	27	21	7	6	6	5
4	72	46	36	12	11	10	3
5	110	70	55	19	17	17	16
6	156	99	78	27	25	24	5
7	210	133	105	37	34	34	33
8	272	172	136	48	45	44	12
9	342	216	171	61	57	57	52
10	420	265	210	75	71	70	22
11	506	319	253	91	86	86	85
12	600	378	300	108	103	102	21
13	702	442	351	127	121	121	120
14	812	511	406	147	141	140	49
15	930	585	465	169	162	162	145
16	1056	664	528	192	185	184	50
17	1190	748	595	217	209	209	208
18	1332	837	666	243	235	234	63
19	1482	931	741	271	262	262	261
20	1640	1030	820	300	291	290	74

Number of basis elements of the $w = 2$ cyclotomic harmonic sums up to cyclotomy $l = 20$ after applying the quasi-shuffle algebra of the sums (A), the shuffle algebra of the cyclotomic harmonic polylogarithms (SH), and the three multiple argument relations (H_1 , H_2 , M) of the sums.





Counting Relations:

$$N_S(l) = 2l(2l + 1)$$

$$N_A(l) = l(2l + 1)$$

$$N_{SH}(l) = \frac{(5l + 3)l}{2}$$

$$N_{A,SH}(l) = \frac{6l^2 + 1 - (-1)^l}{8}$$

$$N_{A,SH,H_1}(l) = \frac{6l^2 - 4l + 7 - (-1)^l}{8}$$

$$N_{A,SH,H_1,H_2}(l) = \frac{6l^2 - 4l + 3(1 - (-1)^l)}{8}$$

$$N_{A,SH,H_1,H_2}(l) = \frac{3}{4}l^2 - 12l + if(\text{mod}p(1, 2) = 0, 1, 3/4) \quad (\text{Broadhurst})$$



Generalized Harmonic Sums at Roots of Unity

$$\lim_{N \rightarrow \infty} S_{k_1, \dots, k_m}(x_1, \dots, x_m; N) = \sigma_{k_1, \dots, k_m}(x_1, \dots, x_m), x_j \in \mathcal{C}_n, n \geq 1, k_1 \neq 1 \text{ for } x_1 = 1$$

$$\mathcal{C}_n \in \{e_n | e_n^n = 1, e_n \in \mathbb{C}\}$$

$$\sigma_w(x) = \text{Li}_w(x), \quad w \in \mathbb{N}, w \geq 1$$

$$\sigma_1(x) = \text{Li}_1(x) = -\ln(1-x)$$

$$\sigma_{1,1}(x, y) = \text{Li}_2(x) + \frac{1}{2} \ln^2(1-x) + \text{Li}_2\left(-\frac{x(1-y)}{1-x}\right)$$

$$\sigma_{1,1}(x, x^*) = \text{Li}_2(x) + \frac{1}{2} \ln^2(1-x) + \zeta_2$$

$$\text{Li}_w(x) = \text{Li}_w^*(x^*)$$

$$\sigma_{1,1}(x, y) + \sigma_{1,1}(y, x) = \ln(1-x)\ln(1-y) + \text{Li}_2(xy).$$

Root structure $l = 12$:

$$\left\{ e_{12}^k \Big|_{k=1}^{12} \right\} \equiv \left\{ e_{12}, e_6, e_4, e_3, e_{12}^5, e_2, e_{12}^{5*}, e_3^*, e_4^*, e_6^*, e_{12}^*, e_1 \right\}.$$

$$\text{Im} \left[\text{Li}_n(e_l^k) \right] = r_{n,l,k} \pi^n \text{ for } n \text{ odd, } \text{Re} \left[\text{Li}_n(e_l^k) \right] = r_{n,l,k} \pi^n \text{ for } n \text{ even, } r_{n,l,k} \in \mathbb{Q}$$



$$w = 1$$

l	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
basis	0	1	2	2	3	3	4	3	4	4	6	4	7	5	6	5	9	5	10	6
Racinet '01 new elements	0	1	2	0	2	0	3	1	2	0	5	1	6	0	2	2	8	0	9	2

The number of the basis elements spanning the $w=1$ cyclotomic harmonic polylogarithms at l th roots of unity up to 20.

Basis Elements:

$$\begin{aligned} & \ln(2); \ln(3), \pi; \operatorname{Re}(\operatorname{Li}_1(e_5)), \operatorname{Re}(\operatorname{Li}_1(e_5^2)); \operatorname{Re}(\operatorname{Li}_1(e_7^k)) \Big|_{k=1}^3; \operatorname{Re}(\operatorname{Li}_1(e_8)); \\ & \operatorname{Re}(\operatorname{Li}_1(e_9)), \operatorname{Re}(\operatorname{Li}_1(e_9^2)); \operatorname{Re}(\operatorname{Li}_1(e_{11}^k)) \Big|_{k=1}^5; \operatorname{Re}(\operatorname{Li}_1(e_{12})); \operatorname{Re}(\operatorname{Li}_1(e_{13})) \Big|_{k=1}^6; \\ & \operatorname{Re}(\operatorname{Li}_1(e_{15})), \operatorname{Re}(\operatorname{Li}_1(e_{15}^2)); \operatorname{Re}(\operatorname{Li}_1(e_{16})), \operatorname{Re}(\operatorname{Li}_1(e_{16}^3)); \operatorname{Re}(\operatorname{Li}_1(e_{17})) \Big|_{k=1}^8; \\ & \operatorname{Re}(\operatorname{Li}_1(e_{19})) \Big|_{k=1}^9; \operatorname{Re}(\operatorname{Li}_1(e_{20})), \operatorname{Re}(\operatorname{Li}_1(e_{20}^3)); \dots \end{aligned}$$



$$w = 2$$

l	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
Li_2 basis	1	1	2	2	3	2	4	3	4	3	6	3	7	4	5	5	9	4	10	5
Li_2 new basis	1	0	1	1	2	0	3	1	2	0	5	0	6	0	1	2	8	0	9	1
Racinnet '01 new elements	1	1	1	1	2	2	4	4	7	6	10									
	1	0	1	1	2	1	4	3	5	4	10	5	14	8	12	12	24	11	30	16

The number of the basis elements spanning the dilogarithms resp. $w = 2$ cyclotomic harmonic sums at l th roots of unity up to 20.

Basis Elements:

$$\begin{aligned} & \pi; \text{Im}(\text{Li}_2(e_3)); \mathbf{C}; \text{Im}(\text{Li}_2(e_5)), \text{Im}(\text{Li}_2(e_5^2)); \text{Li}_4(1/2); \text{Im}(\text{Li}_2(e_7^k)) \Big|_{k=1}^3, \sigma_{1,1}(e_7, e_7^2); \\ & \text{Im}(\text{Li}_2(e_8)), \sigma_{1,1}(e_8, e_4), \sigma_{1,1}(e_8, e_8^3); \text{Im}(\text{Li}_2(e_9)), \text{Im}(\text{Li}_2(e_9^2)), \sigma_{1,1}(e_9, e_9^2), \\ & \sigma_{1,1}(e_9, e_3), \sigma_{1,1}(e_9^2, e_3); \sigma_{1,1}(e_5, e_2), \sigma_{1,1}(e_5^2, e_2), \sigma_{1,1}(e_{10}, e_5), \sigma_{1,1}(e_{10}, e_{10}^3); \\ & \text{Im}(\text{Li}_2(e_{11}^k)) \Big|_{k=1}^5, \sigma_{1,1}(e_{11}, e_{11}^k) \Big|_{k=2}^4, \sigma_{1,1}(e_{11}^2, e_{11}^k) \Big|_{k=3}^4, \dots \end{aligned}$$



