Analytic evaluation

of Feynman graph integrals

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- description of the differential equation approach to the anal. ev. of F. graphs. Back to Euler's variation of constants formula (+ FORM) for solving the eq.s, writing the solution and defining the new functions which appear
- elementary example: 1 loop self-mass
- (recent) application: a 3 loop sunrise

  (P. Mastrolia & ER)

Prolegomena

zu einer jeden künftigen Rechnung, die als Analytische wird auftreten können

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- regularization in d continuous dimensions

- consider the whole set of *all* the integrals associated with a given Feynman graph;

- write the ibp (and Lorentz) identities for them;

- (obtain a *modulus of functions on the ring of the polynomials* in d and the scalar (Mandelstam) variables) ; [something like an algebra]

- find the master integrals (M.I.) of the set;

  - finite difference equations [Laporta]

- act with $p^i_\mu \dfrac{\partial}{\partial p^j_\mu}$ on the M.I. and express the result in terms of the M.I. and scalar derivatives;

[• numerical sol. of diff. eqs: M. Caffo, H. Czyz & ER]

- write the system of linear, inhomogeneous diff. eq.s for the M.I. (in any of the variables)

- transform the system in a single equation of higher order for one of the M.I.;

- expand the equation in $(d-4)$; obtain a system of chained diff. eq.s for the coefficients of the expansion of the M.I. in $(d-4)$;

  - *all* the eq.s have the same homogeneous *"kernel"*;

  - coeff.s of lower order in $(d-4)$ become inhomogeneous terms in the eq.s for the coeff.s of higher order;

- solve the *kernel* equation !

- solve recursively the system of chained eq.s by Euler's variation of constants method, starting from the lowest coeff. in the $(d-4)$ expansion;

- exploit known (qualitative) information on analytic behaviour to get quantitative information for fixing the integration constants;

- Euler's formula is nothing but the very *analytic definition* of the result ! ($i.e.$ it gives the result in *closed analytic form*).

- (generalized) harmonic polylog.s (and further generalizations) arise in that way.

No general algorithm known for solving differential eq.s – even when homogeneous and with rational coefficients.

- in most cases, the diff. eq.s were trivial (applies to 2loop off-massshell 3j ampl's, T. Gehrmann & ER)

- integral representations are suggested by hypergeometric function theory, or immaginary parts of F. graphs (or direct calculation!)

- the equation itself can be used for the definition and the study of the properties of the solutions

- the number of different scales seems to give more problems than the number of loops

Features of an analytic calculation

- using known symbols and greek letters is not the main point!

- no hidden zeroes!
  use a notation convention in which two expression, when equal, are identical
  ex: in trigonometry, use $\cos^2 x = 1 - \sin^2 x$, so that $(\cos^2 x + \sin^2 x - 1)^2 = 0$ (and is not expanded); same for $Li_2(1-x) + Li_2(x) + \ln(1-x)\ln x \ldots$

- the knowledge of analyticity* properties (in particular the relevant behaviours and expansions at all the potentially singular points) so that a numerical (FORTRAN)* routine for the *fast & accurate* evaluation can be written.

* recent papers by T. Gehrmann & ER

(2 dimensional) Harmonic PolyLog's

$$g(0;y) = \frac{1}{y} \qquad G(0;y) = \lg y$$

$$g(1;y) = \frac{1}{y-1} \qquad G(1;y) = \lg(1-y)$$

$$\left[g(-1;y) = \frac{1}{y+1} \qquad G(-1;y) = \lg(1+y) \right]$$

$$g(-z;y) = \frac{1}{y+z} \qquad G(-z;y) = \lg\left(1+\frac{y}{z}\right)$$

$$g(1-z;y) = \frac{1}{y+z-1} \qquad G(1-z;y) = \lg\left(1-\frac{y}{1-z}\right)$$

in general $\quad G(a,\vec{b};y) = \int_0^y dy'\, g(a;y')\, G(\vec{b};y')$

Ex:

$$G(0,-1,1;x) = \int_0^x dx' \frac{1}{x'} \int_0^{x'} dx'' \frac{1}{x''+1} \int_0^{x''} dx''' \frac{1}{x'''-1}$$

weight : number of repeated integrations

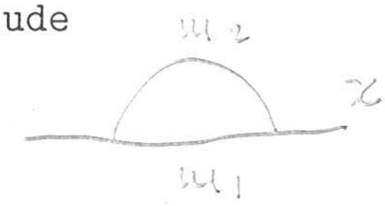
Linearly independent!

Algebra - ex:

$$G(a;x)\, G(b,c;x) = G(a,b,c;x) + G(b,a,c;x) + G(b,c,a;x) \quad \text{etc.}$$

Real parts of 2 loop QED vac. pol + Vertex f.f. can be expressed in terms of Harmonic Polylogarithms (P. Mastrolia & ER) diff. eq. for the 1-loop self mass amplitude

$$\frac{\partial}{\partial x} S(d,x) =$$

$S(d,x) =$ 

$$+ \quad \frac{1}{2}\left(\frac{1}{x+(m_1+m_2)^2} + \frac{1}{x+(m_1-m_2)^2} - \frac{2}{x}\right) S(d,x)$$
$$+ \quad \frac{d-4}{2}\left(\frac{1}{x+(m_1+m_2)^2} + \frac{1}{x+(m_1-m_2)^2} - \frac{1}{x}\right) S(d,x)$$
$$+ \quad \frac{d-2}{4}\frac{m_1}{m_1^2-m_2^2}\left(\frac{m_1+m_2}{x+(m_1+m_2)^2} + \frac{m_1-m_2}{x+(m_1-m_2)^2} - \frac{2}{x}\right) T(d,m_1^2)$$
$$+ \quad \frac{d-2}{4}\frac{m_2}{m_2^2-m_1^2}\left(\frac{m_1+m_2}{x+(m_1+m_2)^2} + \frac{m_2-m_1}{x+(m_1-m_2)^2} - \frac{2}{x}\right) T(d,m_2^2)$$

with

$$T(d,m^2) = \frac{m^{d-2}}{(d-2)(d-4)} \quad =$$

expansion in $(d-4)$

$$S(d,x) = \frac{S^{(-1)}(x)}{d-4} + S^{(0)}(x) + (d-4)S^{(1)}(x) + \ldots.$$

eq.s for the coeff.s of the $(d-4)$-expansion

$$\frac{\partial}{\partial x} S^{(-1)}(x) =$$
$$+ \quad \frac{1}{2}\left(\frac{1}{x+(m_1+m_2)^2} + \frac{1}{x+(m_1-m_2)^2} - \frac{2}{x}\right) S^{(0)}(x)$$
$$+ \quad \frac{1}{4}\left(\frac{1}{x+(m_1+m_2)^2} + \frac{1}{x+(m_1-m_2)^2} - \frac{2}{x}\right)$$

$$
\begin{aligned}
\frac{\partial}{\partial x} S^{(k)}(x) = & \\
+ & \frac{1}{2}\left(\frac{1}{x+(m_1+m_2)^2}+\frac{1}{x+(m_1-m_2)^2}-\frac{2}{x}\right) S^{(k)}(x) \\
+ & N^{(k)}(x)
\end{aligned}
$$

homogeneous equation (kernel)

$$\frac{\partial}{\partial x} K(x) = \frac{1}{2}\left(\frac{1}{x+(m_1+m_2)^2}+\frac{1}{x+(m_1-m_2)^2}-\frac{2}{x}\right) K(x)$$

solution

$$K(x) = \frac{1}{x}\sqrt{(x+(m_1+m_2)^2)(x+(m_1-m_2)^2)}$$

Euler's formula

$$S^{(k)}(x) = K(x)\left(C_k + \int^x \frac{dy}{K(y)}\right) N^{(k)}(y)$$

C_k fixed by the regularity at $x=0$

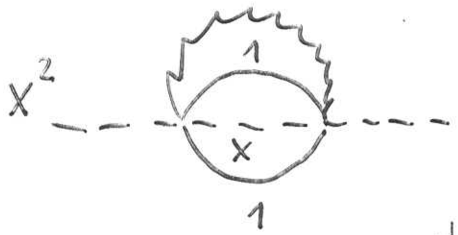
with the change of variable

$$u = \frac{\sqrt{y+(m_1+m_2)^2}-\sqrt{y+(m_1-m_2)^2}}{\sqrt{y+(m_1+m_2)^2}+\sqrt{y+(m_1-m_2)^2}}$$

harmonic polylogarithms are recovered

only $\frac{1}{u}$, $\frac{1}{u-1}$, $\frac{1}{u+1}$ appear in the eq.

3rd order Differential Equation exact in d



$$x^2(1-x)(1+x)\frac{d^3}{dx^3}\Phi(d,x)$$

$$+\left\{\,[2-(d-4)]x+[2+5(d-4)]x^3\,\right\}\frac{d^2}{dx^2}\Phi(d,x)$$

$$+\left\{\,[-6-8(d-4)-2(d-4)^2]+[2-4(d-4)-6(d-4)^2]x^2\,\right\}\frac{d}{dx}\Phi(d,x)$$

$$-[8+14(d-4)+6(d-4)^2]x\;\Phi(d,x)\;=\;\frac{1}{(d-4)^3}\frac{x^{(d-2)}}{x}$$

$x \to 0^+$ behaviour:

$\boxed{x \to 1 : \quad \Phi(d,x) \text{ regular}}$

$$\Phi(d,x)=\sum_i x^{\alpha_i}\;\psi_i(x^2)\;,\qquad \psi_i(x^2)\text{ regular @ } x=0$$

$\underline{\alpha_1=0}$, $\quad \underset{\text{NO}}{\alpha_2=-(d-2)}$, $\quad \underset{\text{NO}}{\alpha_3=(2d-5)}$, $\quad \underline{\alpha_4=(d-2)}$.

when $x \to 0$, $\Phi(d,x)$ finite if $2 < d < \frac{5}{2}$

qualitative information

3rd order Differential Equation

expanded in $d-4$

$$\left[\frac{d^3}{dx^3}+2\left(\frac{1}{x}+\frac{1}{1-x}-\frac{1}{1+x}\right)\frac{d^2}{dx^2}-2\left(\frac{3}{x^2}+\frac{1}{1-x}+\frac{1}{1+x}\right)\frac{d}{dx}-4\left(\frac{2}{x}+\frac{1}{1-x}-\frac{1}{1+x}\right)\right]\Phi^{(n)}(x)$$

$$= \left(\frac{1}{x}-\frac{2}{1-x}+\frac{2}{1+x}\right)\frac{d^2}{dx^2}\Phi^{(n-1)}(x)$$

$$+2\left(\frac{4}{x^2}+\frac{3}{1-x}+\frac{3}{1+x}\right)\frac{d}{dx}\Phi^{(n-1)}(x)+7\left(\frac{2}{x}+\frac{1}{1-x}-\frac{1}{1+x}\right)\Phi^{(n-1)}(x)$$

$$+2\left(\frac{1}{x^2}+\frac{2}{1-x}+\frac{2}{1+x}\right)\frac{d}{dx}\Phi^{(n-2)}(x)+3\left(\frac{2}{x}+\frac{1}{1-x}-\frac{1}{1+x}\right)\Phi^{(n-2)}(x)$$

$$+\frac{1}{2}\left(\frac{2}{x}+\frac{1}{1-x}-\frac{1}{1+x}\right)\frac{1}{(n+3)!}\ \ln^{(n+3)}(x) \quad ,$$

$$\Phi^{(n)}(x)=0 \ , n<-3 .$$

Solutions of the Homogeneous Eq.

$$\begin{cases} \phi_1(x) = (1-x^2) & \\ \phi_2(x) = \phi_1(x)u_1(x) & \Leftarrow \quad u_1'(x) = (1-x^2)^2/x^3 \\ \phi_3(x) = \phi_1(x)u_2(x) & \Leftarrow \quad u_2'(x) = u_1'(x)v(x) \quad \Leftarrow \quad v'(x) = -x^4/(1-x^2)^5 \end{cases}$$

$$\begin{aligned}
\phi_1(x) &= (1-x^2)\ , \\
\phi_2(x) &= -\frac{1}{2}\frac{(1-x^2)(1-x^4)}{x^2} - 2(1-x^2)H(0;x)\ , \\
\phi_3(x) &= +\frac{3}{512}\frac{(1-x^2)(1-x^4)}{x^2}\left[H(-1;x)+H(1;x)\right] \\
&\quad +\frac{3}{128}(1-x^2)\left[H(0,-1;x)+H(0,1;x)\right] \\
&\quad -\frac{1}{256}\frac{(x^2+3)(3x^2+1)}{x}\ ,
\end{aligned}$$

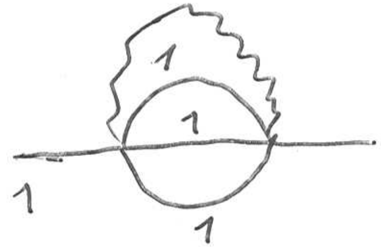
$$W(x) = -\frac{(1-x)^2(1+x)^2}{x^2}\ .$$

Euler's formula

$$
\begin{aligned}
\Phi^{(n)}(x) \quad = \quad & +\phi_1(x)\left[\Phi_1^{(n)} + \int^x dy \frac{\phi_2(y)\phi_3'(y) - \phi_2'(y)\phi_3(y)}{W(y)}\; K^{(n)}(y)\right] \\
& +\phi_2(x)\left[\Phi_2^{(n)} - \int^x dy \frac{\phi_1(y)\phi_3'(y) - \phi_1'(y)\phi_3(y)}{W(y)}\; K^{(n)}(y)\right] \\
& +\phi_3(x)\left[\Phi_3^{(n)} + \int^x dy \frac{\phi_1(y)\phi_2'(y) - \phi_1'(y)\phi_2(y)}{W(y)}\; K^{(n)}(y)\right] ,
\end{aligned}
$$

$\phi_i^{(n)}$: fixed by the allowed behaviour at $x=0$, $x=1$

$$
\begin{aligned}
\Phi(x=1) \quad = \quad & -\frac{1}{8}\frac{1}{(d-4)^3}+\frac{7}{32}\frac{1}{(d-4)^2}-\frac{253}{1152}\frac{1}{(d-4)}+\frac{2501}{13824} \\
& +(d-4)\left(-\frac{59437}{165888}+\frac{1}{18}\pi^2\right) \\
& +(d-4)^2\left(+\frac{2831381}{1990656}-\frac{71}{216}\pi^2+\frac{1}{3}\pi^2\ln 2-\frac{7}{9}\zeta(3)\right) \\
& +(d-4)^3\left(-\frac{117529021}{23887872}+\frac{3115}{2592}\pi^2-\frac{71}{36}\pi^2\ln 2+\frac{497}{108}\zeta(3)+\frac{7}{9}\pi^2\ln^2 2-\frac{43}{1080}\pi^4+\frac{2}{9}\ln^4 2+\frac{16}{3}a_4\right) \\
& +(d-4)^4\left(+\frac{4081770917}{286654464}-\frac{109403}{31104}\pi^2+\frac{3115}{432}\pi^2\ln 2-\frac{21805}{1296}\zeta(3)-\frac{497}{108}\pi^2\ln^2 2+\frac{3053}{12960}\pi^4\right. \\
& \qquad \left.-\frac{71}{54}\ln^4 2-\frac{284}{9}a_4-\frac{43}{180}\pi^4\ln 2+\frac{14}{9}\pi^2\ln^3 2+\frac{7}{36}\pi^2\zeta(3)+\frac{4}{15}\ln^5 2+\frac{341}{12}\zeta(5)-32a_5\right) \\
& +(d-4)^5\left(-\frac{125873914573}{3439853568}+\frac{3386467}{373248}\pi^2-\frac{109403}{5184}\pi^2\ln 2+\frac{765821}{15552}\zeta(3)+\frac{21805}{1296}\pi^2\ln^2 2\right. \\
& \qquad -\frac{26789}{31104}\pi^4+\frac{3115}{648}\ln^4 2+\frac{3115}{27}a_4-\frac{497}{54}\pi^2\ln^3 2-\frac{497}{432}\pi^2\zeta(3)+\frac{3053}{2160}\pi^4\ln 2 \\
& \qquad -\frac{24211}{144}\zeta(5)+\frac{568}{3}a_5-\frac{71}{45}\ln^5 2+\frac{41}{810}\pi^6+\frac{10}{9}\pi^2\ln^4 2-\frac{151}{180}\pi^4\ln^2 2+\frac{88}{3}\zeta(3)\ln^3 2 \\
& \qquad \left.-\frac{27}{2}\pi^2\zeta(3)\ln 2+176\zeta(5)\ln 2-176a_5\ln 2+\ln^6 2+\frac{2723}{36}\zeta^2(3)-160\ a_6+176\ b_6\right) \\
& +\mathcal{O}(\ (d-4)^6\ )\ ,
\end{aligned}
$$



$$b_6 = H(0,0,0,0,1,1;1/2) = S_{4,2}(1/2).$$

Solutions

Harmonic polylog's only

$$\Phi^{(-3)}(x) = -\frac{1}{12}x^2 - \frac{1}{24} \; ;$$

$$\Phi^{(-2)}(x) = +\frac{5}{32}x^2 - \frac{1}{96}x^4 - \frac{1}{8}x^2\, H(0;x) + \frac{7}{96} \; ;$$

$$\Phi^{(-1)}(x) = -\frac{71}{384}x^2 + \frac{35}{1152}x^4 - \frac{1}{16}x^2\, H(0;x)H(0;x) + \frac{1}{4}x^2\left(1 - \frac{1}{8}x^2\right) H(0;x) - \frac{25}{384} \; ;$$

$$\cdots$$

$$\Phi^{(3)}(x) = (\text{ HPL's up to } w = 6 \text{ })$$

Conclusion

- Differential equations do provide with a powerful tool for the analytic evaluation of Feynman graph integrals
- Euler's variation of constants method gives the result and defines the functions which appear in the result
- the functions which appear in the result are suitable generalizations of Euler's dilogarithm.