RadCor 2002 / Loops 'n Legs 9/12/02

DELOCALIZED OPERATOR EXPANSION

André H. Hoang
MPI Munich

in collaboration with Ralf Hofmann
(hep-ph/0206201)

WILSON'S (LOCAL) OPE

"practical version" only !

* countless applications: – current correlators, sum rules (SVZ)
  – heavy quarks, HQET, NRQCD
  – etc.

* Separation of short-distance and long-distance contributions

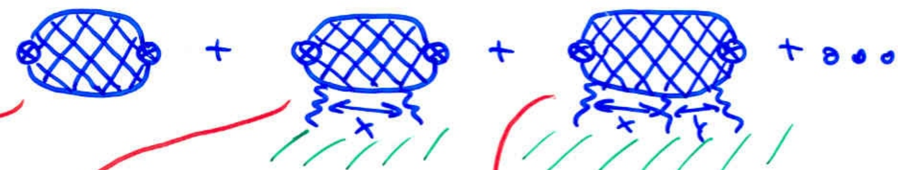
short-distance	long-distance
$x \ll \Lambda_{QCD}^{-1}$	$x \sim \Lambda_{QCD}^{-1}$
perturb. theory	non-perturb. effects
$\Downarrow$	$\Downarrow$
process-dependent	universal <u>local</u> matrix elements

Example: heavy quark current correlator

$\rightarrow$ neglect anom. dims.

$Q^2 = -q^2 \rightarrow \infty$

$\int d^4x \, e^{iqx} \langle T j(x) j(0) \rangle =$



$$= 1 + \frac{\alpha_s}{\pi} \# + \ldots + \int d^4x \, f^Q(x) \langle G(0) G(x) \rangle + \int d^4x \, d^4y \, f^Q(x,y) \langle G(0) G(x) G(y) \rangle + \ldots$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$= 1 + \frac{\alpha_s}{\pi} \# + \ldots + \sum C_n \langle G(0) D^n G(0) \rangle + \sum C'_{nm} \langle G(0) D^n G(0) D^m G(0) \rangle + \ldots$$

$\oplus$ OPE provides expansion in $\frac{\Lambda_{QCD}}{Q} \ll 1$

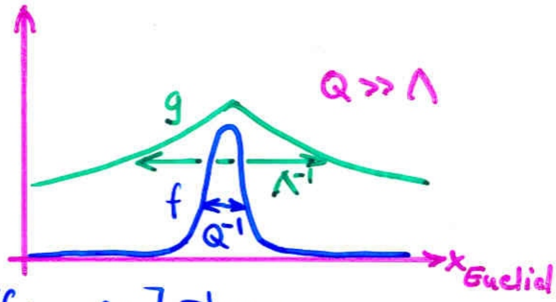
$\oplus$ basis of modern EFT's (HQET, NRQCD)

$\ominus$ asymptotics of condensate series
truncation of "practical" OPE $\Big\} \Rightarrow$ duality violation ($q^2 > 0$)

$\ominus$ breakdown of OPE for $\frac{\Lambda_{QCD}}{Q} \rightarrow \mathcal{O}(1)$

$\Rightarrow$ DOE: Approach that keeps the $\oplus$'s and provides an improved handling of the $\ominus$'s.

One-Dimensional Case:



$$\int dx\, f(x)\, g(x) =: (f,g)$$

Local Expansion: $f(x) = \left[\int dx\, f(x)\right] \delta(x) - \left[\int dx\, x f(x)\right] \delta'(x) + \dots$

multipole expansion

f, g lie in a dual space with the basis

$$\boxed{e_n(x) = \frac{(-1)^n}{n!}\,\delta^{(n)}(x) \;, \quad \tilde{e}_n(x) = x^n \quad ; \; n = 0,1,\dots,\infty}$$

$$\Rightarrow \int dx\, f(x)\, g(x) = \sum_{n=0}^{\infty} (f, \tilde{e}_n) \times (e_n, g) = \sum_{n=0}^{\infty} f_n \times g_n$$

Delocalized Expansion: expand $f(x)$ in fct's with finite width Ω^{-1}

(Ω free parameter, "practical" choice: $\underline{\Omega = Q}$)

e.g.
$$\boxed{e_n^{\Omega}(x) = \frac{\Omega^{n+1}}{\sqrt{\pi}\, n!} H_n(\Omega x)\, e^{-\Omega^2 x^2} \;, \quad \tilde{e}_n^{\Omega}(x) = \frac{H_n(\Omega x)}{(2\Omega)^n} \; ; \; n = 0,\dots,\infty}$$

$$\Rightarrow \int dx\, f(x)\, g(x) = \sum_{n=0}^{\infty} (f, \tilde{e}_n^{\Omega}) \times (e_n^{\Omega}, g) = \sum_{n=0}^{\infty} f_n(\Omega) \times g_n(\Omega)$$

Properties:

$$e_n^{\Omega} \xrightarrow{\Omega \to \infty} e_n$$

$$\tilde{e}_n^{\Omega} \xrightarrow{\Omega \to \infty} \tilde{e}_n$$

$$\tilde{e}_n^{\Omega} = \sum_{m=0}^{\infty} a_{nm}(\Omega, \Omega')\, \tilde{e}_m^{\Omega'}$$

$$e_n^{\Omega} = \sum_{m=0}^{\infty} a_{mn}(\Omega', \Omega)\, e_m^{\Omega'}$$

$$a_{nm}(\Omega, \Omega') = \begin{cases} \dfrac{n!}{m!\left(\frac{n-m}{2}\right)!} \left(\dfrac{\Omega^2 - \Omega'^2}{4\Omega^2 \Omega'^2}\right)^{\left(\frac{n-m}{2}\right)} & n \geq m,\ n-m \text{ even} \\ 0, & \text{otherwise} \end{cases}$$

"orthogonal"

Properties (cont'd):

$$f_n(\Omega) = \sum_{m=0}^{\infty} a_{nm}(\Omega,\Omega')\, f_m(\Omega')$$

$$g_n(\Omega) = \sum_{m=0}^{\infty} a_{mn}(\Omega',\Omega)\, g_m(\Omega')$$

DOE (Ω) $\Longleftrightarrow$ OPE ($\Omega' = \infty$)

$f_n(\infty) = f_n \,,\; g_n(\infty) = g_n$

$$\left.\begin{aligned}
f_0(\Omega) &= f_0 \\
f_2(\Omega) &= f_2 - \frac{1}{2\Omega^2} f_0 \\
&\;\vdots \\
f_n(\Omega) &= \sum_{i=0}^{[\frac{n}{2}]} \frac{n!}{(n-2i)!\,i!}\left(-\frac{1}{4\Omega^2}\right)^i f_{n-2i} \\
g_n(\Omega) &= \sum_{i=0}^{\infty} \frac{(n+2i)!}{n!\,i!}\left(\frac{1}{4\Omega^2}\right)^i g_{n+2i}
\end{aligned}\right\}$$

Definition of the DoE

→ applicable for non-zero anomalous dimensions

* DOE (Ω) is a generalization of the OPE ($\Omega = \infty$)
* $C_n(\Omega,\mu) \sim \sum_{i=0}^{n} \Omega^{n-i} C_i^{\text{local}}(\mu)$ $\Rightarrow$ $C^{\langle g^2G^2\rangle}$ Ω-independent
* $\langle O\rangle(\Omega,\mu) \sim \sum_{i=0}^{\infty} \frac{\langle D^i O\rangle(\mu)}{\Omega^i}$ $\Rightarrow$ $\langle g^2G^2\rangle(\Omega) = \langle g^2G^2\rangle + \frac{1}{4\Omega^2}\langle g^2G\partial^2G\rangle + \dots$
* DoE does NOT INTERFERE with the regularization scheme

Evolution Equation

$$\frac{d}{d\Omega} g_n(\Omega) = -\frac{(n+1)(n+2)}{2\Omega^3}\, g_{n+2}(\Omega) \quad \Rightarrow \quad \frac{d}{d\Omega}\langle g^2G^2\rangle(\Omega) = -\frac{\langle g^2G\partial^2G\rangle(\Omega)}{2\Omega^3}$$

Power counting

→ parametric counting unchanged

$$\left.\begin{aligned}
g_n \sim \Lambda^n &\;\Rightarrow\; g_n(\Omega) \sim \Lambda^n \sum \left(\frac{\Lambda}{\Omega}\right)^i \sim \Lambda^n \\
f_n \sim Q^{-n} &\;\Rightarrow\; f_n(\Omega) \sim Q^{-n} \sum \left(\frac{Q}{\Omega}\right)^i \sim Q^{-n}
\end{aligned}\right\} \rightarrow \boxed{\Omega \gtrsim Q}$$

→ practical counting: $f_n \times g_n \sim \left(\frac{\Lambda}{Q}\right)^n$

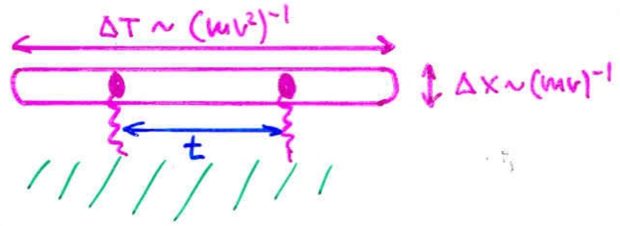
$$\Rightarrow f_n(\Omega) \times g_n(\Omega) \sim \left(\frac{\Lambda}{aQ}\right)^n \,,\; a > 1$$

Heavy Quarkonium Ground State Level

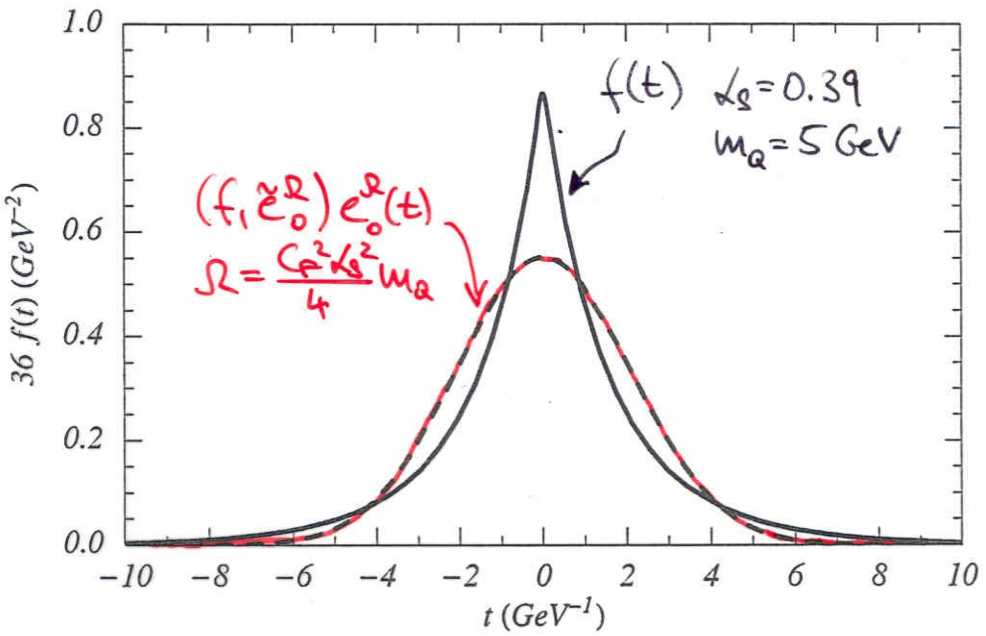
Voloshin
Leutwyler

→ leading order OPE in $\frac{p}{m_Q}$, $p \sim m_Q v$

→ DOE in $\frac{\Lambda_{QCD}}{E}$, $E \sim m_Q v^2$



$$\Delta E_{\text{nonpert}} = \int dt\, f(t)\, g(t)$$



$$f(t) \sim \int \frac{dq_0}{(2\pi)} e^{-q_0 t} \int d^3\vec{x}\, d^3\vec{y}\; \varphi(x)\, x^i\, G_{\text{Octett}}(\vec{x},\vec{y},E)\, y^i\, \varphi(y)$$

$$g(t) \sim e^{-t/\lambda_a} \quad (t \text{ large})$$

$$\lambda_a^{-1} \approx 0.7\ \text{GeV} \qquad (\text{D'Elia, Di Giacomo, Meggiolaro})$$

$\lambda_a^{-1} = 0.7\ \text{GeV}$

$Q \sim \frac{C_F^2 \alpha_s^2}{4} m$ $\quad \Delta E_{\text{nonpert}}$ $\quad \Omega = \infty$ $\quad \Omega = \frac{C_F^2 \alpha_s^2}{4} m$

					$\Omega = \infty$		$\Omega = k^2/m$	
m (GeV)	α_s	k^2/m (MeV)	E^{np} (MeV)	n	$f_n g_n$ (MeV)	$\sum_{i=0}^{n} f_i g_i$ (MeV)	$f_n g_n$ (MeV)	$\sum_{i=0}^{n} f_i g_i$ (MeV)
5	0.39	0.338	24.8	0	38.6	38.6	24.2	24.2
				2	−65.7	−27.2	−3.9	20.3
				4	832.7	805.5	12.1	32.4
				6	−35048.0	−34242.4	−43.1	−10.8
25	0.23	0.588	12.6	0	16.0	16.0	12.8	12.8
				2	−9.0	7.0	−1.2	11.5
				4	37.8	44.8	2.6	14.1
				6	−526.6	−481.8	−6.7	7.4
45	0.19	0.722	4.9	0	5.9	5.9	5.0	5.0
				2	−2.2	3.7	−0.4	4.6
				4	6.1	9.8	0.7	5.3
				6	−56.4	−46.6	−1.4	3.8
90	0.17	1.156	1.05	0	1.15	1.15	1.07	1.07
				2	−0.17	0.98	−0.04	1.02
				4	0.18	1.16	0.04	1.07
				6	−0.65	0.51	−0.07	1.00
175	0.15	1.750	0.245	0	0.258	0.258	0.249	0.249
				2	−0.016	0.242	−0.005	0.244
				4	0.008	0.250	0.003	0.247
				6	−0.012	0.237	−0.003	0.244

* Series $\sum f_n(\Omega) g_n(\Omega)$ asymptotic for any Ω
* terms for $\Omega = \frac{C_F^2 \alpha_s^2}{4} m$ by 2^{-n} smaller than for $\Omega = \infty$
* leading order OPE : overestimate by 1–60 %

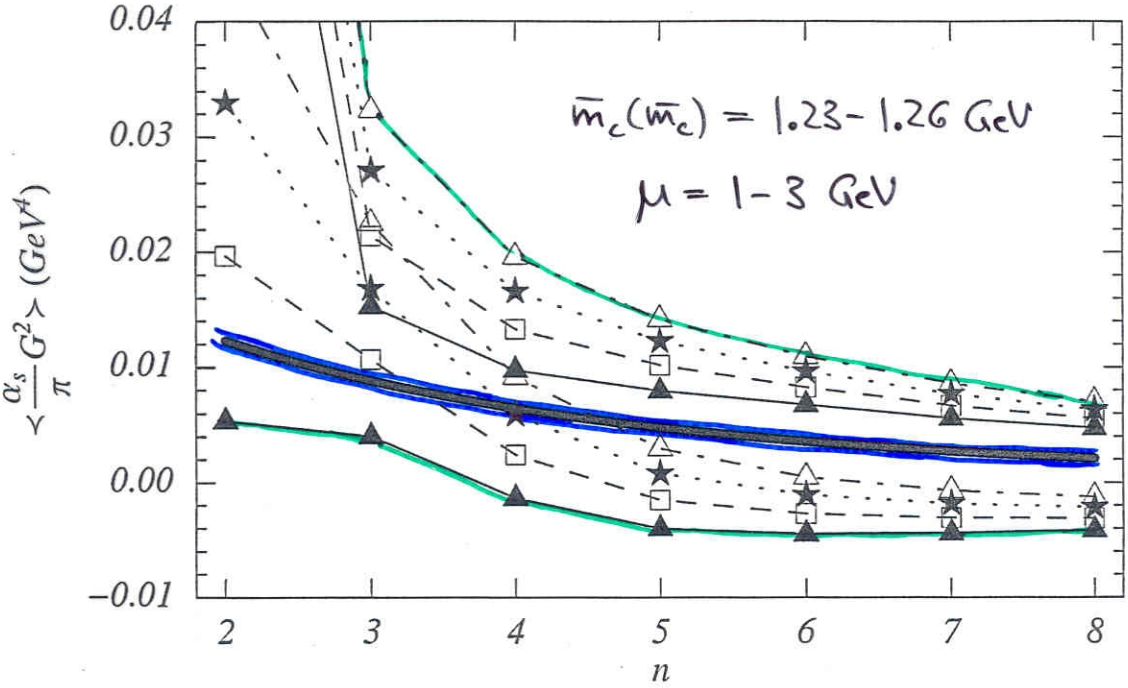
  leading order $\Omega = \frac{C_F^2 \alpha_s^2}{4} m$: agreement within 2‰ – 2%

Charmonium Sum Rules

Novikov, SVZ

$$M_n \sim \left(\frac{d}{dQ}\right)^n \Pi^c(Q^2)\Big|_{Q^2=0} \sim \int \frac{ds}{s^{n+1}} R_{c\bar c}(s)$$

$$r_n = \frac{M_n}{M_{n-1}} \quad \Rightarrow \text{extraction of } \langle \tfrac{\alpha_s}{\pi} G^2 \rangle$$



$$M_n^{th} = M_n^{(0)} \left\{ 1 + \frac{\alpha_s}{\pi} + \ldots + \delta_n^{(4)} \langle g^2 G^2 \rangle + \ldots \right\} \qquad \text{OPE}$$

$$= M_n^{(0)} \left\{ 1 + \frac{\alpha_s}{\pi} + \ldots + \delta_n^{(4)} \langle g^2 G^2 \rangle(\Omega) + \ldots \right\} \qquad \text{DOE}$$

$*$ $\Omega \sim \frac{2m_c}{n}$: $\delta_n^{(4)}\langle g^2G^2\rangle(\Omega) = \delta_n^{(4)}\langle g^2G^2\rangle + \underbrace{\frac{\delta_n^{(4)}}{4\Omega^2}\langle g^2 G D^2 G\rangle}$

equal sign & app. size of $\delta_{n,G}^{(6)}\langle g^3 f G^3\rangle + \delta_{n,j}^{(6)}\langle g^4 j^2\rangle$

$*$ for $2 \le n \le 8$: $\mathcal{O}(\alpha_s^2) \le \frac{1}{2}\mathcal{O}(\alpha_s)$

$\Rightarrow$ $\langle \frac{\alpha_s}{\pi} G^2 \rangle(\Omega)$ monotonic increasing

consistent with lattice results

$$\langle g^2 G(0) S(0,x) G(x) S(x,0) \rangle_{NP} \sim e^{-|x|/\lambda_a}$$

$\lambda_a^{-1} = 0.7$ GeV (D'Elia et. al.)

V+A Adler Function

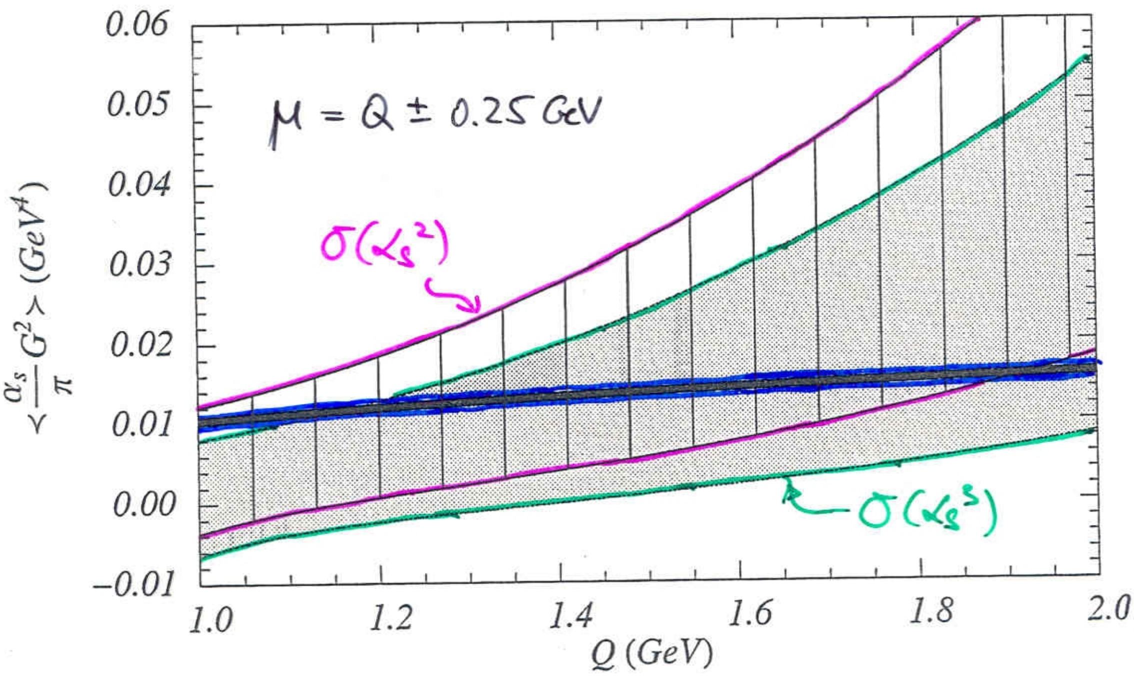
$$D(Q^2) = -Q^2 \frac{d}{dQ^2} \Pi^{V+A}(Q^2)$$

$\Rightarrow$ extraction of $\langle \frac{\alpha_s}{\pi} G^2 \rangle (\Omega)$

$$\sim \int \frac{ds}{(s+Q^2)^2} \rho_{V+A}(s)$$

$\Uparrow$

ALEPH, OPAL (τ decays)



* $\Omega \sim Q$
* perturbative series well behaved for $Q = 1-2$ GeV
* dimension 6 terms (OPE) dominated by $\langle g^2 G D^2 G \rangle$
* chiral limit

$\Rightarrow$ $\langle \frac{\alpha_s}{\pi} G^2 \rangle (\Omega)$ monotonic increasing
consistent to lattice results
uncertainties large

$\hookrightarrow$ need good data for $s > 2\,\mathrm{GeV}^2$

MESON DECAY CONSTANT IN HQET

$$\langle 0|\bar{q}\gamma^\mu\gamma^5 Q|P(v)\rangle = i f_P \sqrt{M_P}\, v^\mu$$

$$\left(\frac{f_D}{f_B}\right)_{\text{lattice}} = 1.0 - 1.3$$

HQET: expansion in $\frac{\Lambda_{QCD}}{m_Q}$ $\Rightarrow$ here: leading order

Local Expansion: $\bar{q}\gamma^\mu\gamma^5 Q \Rightarrow \underbrace{C_0(m_Q,\mu)}_{\left(\frac{\alpha_s(\mu)}{\alpha_s(m_Q)}\right)^{2/\beta_0}} \bar{q}\gamma^\mu\gamma^5 h_Q + \ldots$

Shifman, Voloshin
Politzer, Wise

$$\frac{f_D}{f_B} = \sqrt{\frac{M_B}{M_D}}\left(\frac{\alpha_s(m_b)}{\alpha_s(m_c)}\right)^{2/\beta_0} \times 1 \approx 1.5$$

$$\frac{\langle 0|\bar{q}\gamma^\mu\gamma^5 h_c|D\rangle(\mu)}{\langle 0|\bar{q}\gamma^\mu\gamma^5 h_b|B\rangle(\mu)} = 1$$

* $\frac{1}{m_b}, \frac{1}{m_c}$ corrections uncontrollable (Neubert)

Delocalized Expansion:

* DOE does not interfere with regularization scheme !

$$\frac{f_D}{f_B} = \sqrt{\frac{M_B}{M_D}}\left(\frac{\alpha_s(m_b)}{\alpha_s(m_c)}\right)^{2/\beta_0} \underbrace{\frac{\langle 0|\bar{q}\gamma_\mu\gamma_5 h_c|D\rangle(\Omega=m_c,\mu)}{\langle 0|\bar{q}\gamma_\mu\gamma_5 h_b|B\rangle(\Omega=m_b,\mu)}}_{=:\,\rho}$$

* $\frac{1}{m_Q}$ corrections likely to be large too $\rightarrow$ investigation planned
* much better LO approximation $\Rightarrow \rho_{\text{lattice}} = 0.6 - 0.8$
* lattice (DeGrand et al.) $\rightarrow \rho_D(x) \approx \rho_B(x) \simeq e^{-x/\lambda}$

$$\rho \simeq \left(\frac{1 + 1/\lambda^2 m_b^2}{1 + 1/\lambda^2 m_c^2}\right)^{3/2} \Rightarrow \lambda^{-1} \simeq 0.3 - 0.7\ \text{GeV}$$

consistent with potential models
(e.g. Huang, Kim)

Summary

* Delocalized Operator Expansion (DOE) is a generalization of Wilson's (local) OPE
  - → DOE based on delocalized multipole expansion of s.-d. process
  - → resolution scale Ω: $\Omega = \infty \to$ OPE
    "optimal" choice: $\Omega \sim Q$

* OPE : $\sum C_n(\mu) \langle O_n \rangle(\mu)$
  DOE : $\sum \underbrace{C_n(\Omega,\mu)}_{\sim \frac{C_0(\mu)}{\Omega^n} + \dots + C_n(\mu)} \underbrace{\langle O_n \rangle(\Omega,\mu)}_{\to \sum_{i=0}^{\infty} \frac{\langle D^i O_n \rangle}{\Omega^i}}$

* DOE: improved convergence behavior $\left(\frac{\Lambda}{Q} \to 1\right)$

* charmonium sum rules & V+A → "running gluon condensate"
  consistent to lattice results

Outlook

* Can the DOE do better than the OPE in phenomenology?
  → Reanalysis of many processes (sum rules, heavy quarks, etc.)

* Quantitativ understanding of Duality Violation?
  ($\leftrightarrow$ truncated OPE series)

* subtle relation : $\langle O \rangle_a^{\text{lattice}} \leftrightarrow \langle O \rangle(\Omega \sim \frac{1}{a})$

⇒ DOE appears promising for future investigations

Spires: MPI-PhT 2003-25