SIX SCANDALS IN THE RELATION BETWEEN NUMBER THEORY & Q.F.T.

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1) What is  in D dimensions?

2) What comes after polylogs of roots of unity in massive bubbles?  $\sim \frac{1+\sqrt{-3}}{2}$  $\sim$?

3) Why does QCD hate $\zeta(4) = \pi^4/90$?

4) Why would it take weeks of FORM on MZV's to prove what PSLQ finds in seconds?

5) How to solve a non-trivial Dyson-Schwinger equation in ϕ^3_6 as done for Yukawa at $D=4$?

6) Where do the tedious products of zetas come from?
"Pseudo-exponentiation"

Warning: No use of POWERPOINT
Moreover we (all?) lack the POWER
and maybe there is no POINT

SCANDALS REVISITED

1)  IS STILL A SCANDAL

2) LAPORTA HAS FOUND, IN 4-LOOP MASSIVE BUBBLES, GOLDEN NORMS OF ELLIPTIC INTEGRALS

3) QCD IS A BIT LIKE QED
BUT QUENCHED QED IS
PROBABLY ZETA-FREE

4) PSLQ >> FORM BECAUSE
QFT EXPT >> BEST MATH IN THE WORLD

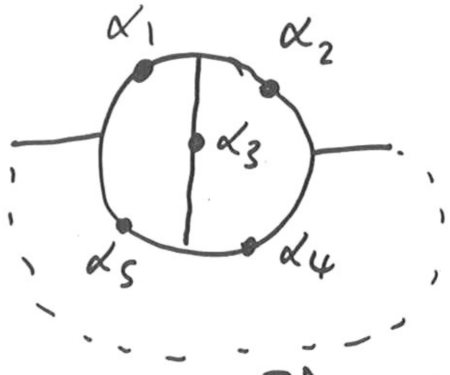
5) erfc in YUKAWA is the first
non-trivial DYSON-SCHWINGER
resummation; OTHERS may follow

6) KREIMER PSEUDOEXPONENTIATION
IS A WEIRD PROCEDURE EVERY
STEP OF WHICH WAS DETERMINED
BY QFT DATA WITH STILL ONLY
$O(1/(12\times 4^9))$ CHANCE OF
SUCCESS. BUT IT WORKED!

OUT OF FIELD THEORY ALWAYS
SOMETHING NEW

EX QFT SEMPER ALIQUID NOVA

6) KREIMER'S PSEUDOEXPONENTIATION IN DRESSED 2-LOOP 2-POINT

$I_6 =$ 

$D = 4 - 2\varepsilon$

$\alpha_i = 1 + n_i \varepsilon$

$$\sum_{i=1}^{6} n_i = -3$$

$$\alpha_6 = \frac{3D}{2} - \alpha_1 - \alpha_2 - \alpha_3 - \alpha_4 - \alpha_5$$

DJB This has a symmetry group of order 1440 $Z_2 \times S_6$ (mind boggling)

GPXT : 4-foldsum

$$\sum_{n, m_1, m_2, m_3} (\Gamma\text{'s})(\text{rational functions with denoms that are of factors of the form } n + \sum_i n_i \varepsilon + \sum_j m_j)$$

products

USELESS!

gives 2-foldsum for

$I_4 =$  $\alpha_4 = \alpha_5 = 1$

whereas inspired soln of recurrence relations gives $\;{}_3F_2 = \sum_{n \geq 0} \Gamma\text{'s}(\alpha_i, n)$ DJB

CAN I_6 BE THE SOURCE OF NON-MZV'S at WEIGHT=10 ie $O(\varepsilon^7)$

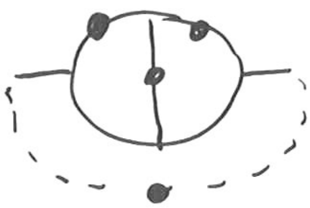
NO! I_4 to $O(\varepsilon^9)$

$\Longrightarrow$ I_6 to $O(\varepsilon^9)$ (weight = 12)

PUSHDOWN

Form of ${}_3F_2$ Solution

$$I_4 = \Gamma S + \bar{\Gamma} \bar{S}$$



$$S = \frac{\mathrm{Col}}{H} + \sum_{m>1} \text{Pochhammers}$$

$$H(a,b,c,d) = \frac{\Gamma(1+a)\Gamma(1+b)\Gamma(1+c)\Gamma(1+d)\Gamma(1+a+b+c+d)}{\Gamma(1+a+c)\Gamma(1+a+d)\Gamma(1+b+c)\Gamma(1+b+d)}$$

Kreimer: Maybe

$S =$ pseudo-exponentiation $\forall$ "Simple things"
how mutilated? ↑ what are they?
factorial modification

This ill-defined recipe
has a <u>unique</u> solution to weight = 12
whose <u>existence</u> defies belief.

- Products of primitives generated by H
- Take only the primitives of

$$1/H = \exp \sum_{n} \zeta(n)\, I_n(a,b,c,d)$$

- Keep only $n =$ odd
- $$1/H_{\text{odd}} = \prod_{n=3,5,7} \exp(\zeta(n) I(n))$$

$\exp(\zeta(3) I(3))\ \forall\ \zeta(n)$

$$= \left[1 + \frac{\zeta(3)I(3)}{1} + \frac{\zeta^2(3)I^2(3)}{2} + \frac{\zeta^3(3)I^3(3)}{6} \cdots\right]\cdot \zeta(n)$$

for $n \neq 3$

$$= \left[1 + \frac{\zeta(3)I(3)}{2} + \frac{\zeta^2(3)I^2(3)}{6} + \frac{\zeta^3(3)I^3(3)}{24} \cdots\right]\zeta(3)$$

for $n = 3$

But we still have a problem
if "simple" does not have $\zeta(5)\zeta(3)$
and we generate it twice by

$$\exp(\zeta(3)I(3))\cdot\zeta(5)$$

and $\exp(\zeta(5)I(5))\cdot\zeta(3)$

we cannot fit the data for $\zeta(5)\zeta(3)$
which is a complicated 5th order polynomial
in (a,b,c,d)

SOLUTION : KEEP ONLY DRESSING
OF $\zeta(m)$ by $\exp(\zeta(n)I(n))$
with $m \geq n$

& factorial modification for $m=n$

RATIONALE : THE IRREDUCIBLE $\zeta(5,3)$
COMPENSATES FOR THE OTHER DRESSING

STILL WE ARE NOT DONE !
IS THERE A 5-LINE SOLUTION
TO THESE 100 MEGABYTES OF DATA?

YES

WT=8 PRIMITIVE $= \zeta(5,3)$ ← NO PRODUCT
WT=10 PRIMITIVE $= \zeta(7,3) + 3\zeta^2(5)$
WT=11 PRIMITIVE $= \zeta(3,5,3) - \zeta(3)\zeta(5,3)$ IN 7-LOOP ϕ^4
WT=12 PRIMITIVE 1 $= \zeta(7,5)$ ← NO PRODUCT
PRIMITIVE 2 $= X$

Need to evaluate 139 pushdowns from
MZV's to EULER SUMS
FIT REMAINING 50 MEGABYTES WITH

GIFT FROM QFT → NUMBER THEORY

$X = \zeta(\bar{7},\bar{5}) - \zeta(\bar{5},\bar{7}) + \zeta(\bar{7})\zeta(\bar{5})$ to satisfy
with $O(12\times 4^9)$ conditions explaining
with 3 numbers $1:-1:1$ a million ten-digit numbers

4) Emumeration and "push-down" of Multiple Zeta Values / easy to find with PSLQ hard to prove with FORM 4

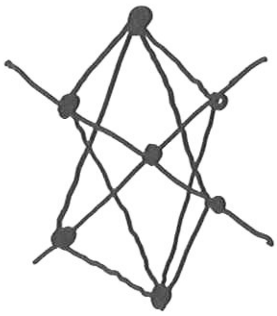
DRINFELD-DELIGNE CONJECTURE

# of IRREDUCIBLE MZV's generated by

$1/(1-x^2-x^3) =$ in QFT counter terms

1 | 1
$+x^2$ | $\cancel{\pi^2}$ G-scheme
$+x^3$ | $\zeta(3)$ 3-loop
$+x^4$ | $[\zeta(4)]$ postponed in QCD } 4-loop
$+2x^5$ | $\cancel{\pi^2\zeta(3)}$ $\zeta(5)$
$+2x^6$ | $[\zeta(6)]$ $\zeta^2(3)$ } 5-loop
$+3x^7$ | $\cancel{\pi^2\zeta(5)}$ $[\zeta(4)\zeta(3)]$ $\zeta(7)$
$+4x^8$ | $\cancel{\pi^2\zeta^2(3)}$ $[\zeta(8)]$ $\zeta(5)\zeta(3)$ $\zeta(5,3)$

$$\zeta(5,3) \equiv \sum_{m>n>0} \frac{1}{m^5}\frac{1}{n^3}$$ via counterterm from in 6-loop $\phi 4$ β-function



DJB DK

. . . .

$12 \times^{12}$

$\zeta(12)$

$3+10$ $\begin{cases} \zeta(2)\zeta(7,3) \\ \zeta(2)\zeta(7)\zeta(3) \\ \zeta(2)\zeta^2(5) \end{cases}$

$3+9$ $\begin{cases} \zeta(3)\zeta(9) \\ \zeta^4(3) \end{cases}$

$4+8$ $\begin{cases} \zeta(4)\zeta(5,3) \\ \zeta(4)\zeta(5)\zeta(3) \end{cases}$

$5+7$ $\{ \zeta(5)\zeta(7)$

$6+6$ $\{ \zeta(6)\zeta^2(3)$

+ 2 irreducible MZV's

Can take $\zeta(9,3)$ OR $\zeta(7,5)$

The 12th has depth 4, e.g.

$$\zeta(4,4,2,2)$$

unexpected discovery

DJB $\zeta(4,4,4,2)$ expressible as $\zeta(\bar{9},\bar{3})$ + MZVs depth < 4

$$\zeta(\bar{9},\bar{3}) \equiv \sum_{m>n>0} \frac{(-1)^m}{m^9}\frac{(-1)^n}{n^3}$$

is alternating Euler sum

Have found "pushdown"

MZV $\rightarrow$ EULER SUM of lesser depth

up to weight = 27 using PSLQ

But even the simplest is not derived.

[Jos could do it by handling my eqs for 3^{12} Euler sums]

5) Solution of Dyson-Schwinger equations

– subtractions

$$\Pi = \frac{1}{1-\Sigma}$$

Generates all rainbows
all chains in rainbows
rainbows in chains

$\left(\frac{g}{4\pi}\right)^2$

DJB + DK found the non-perturbative solution for anom. dimension $\gamma(a)$ and then $\Sigma(q^2, a)$ via

$$\frac{d \log(1-\Sigma)}{d \log q^2} = \gamma\left(\frac{a}{(1-\Sigma)^2}\right)$$

HOPF ALGEBRA

Borel resummation gives a parametric soln in terms of erfc function

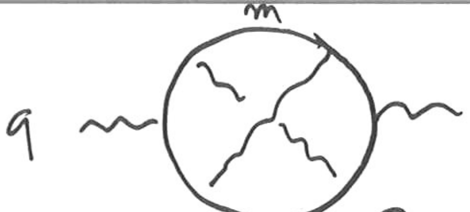
$$\sqrt{\frac{a}{2\pi}} = \exp\left[\lambda^2\right] \operatorname{erfc}[\lambda]$$

$$\lambda = \frac{\gamma + 2}{\sqrt{a}}$$

For ϕ^3 at $D=6$ need to solve

$$\begin{aligned} a = {} & -\gamma(\gamma+1)(\gamma+2)(\gamma+3) \leftarrow \text{rainbows} \\ & + 2a\gamma(2\gamma^2 + 6\gamma + 1)\gamma' \leftarrow \text{like Yukawa} \\ & + 4a^2\gamma\left(2\gamma(\gamma-3)\gamma'' + (\gamma-6)(\gamma')^2\right) \\ & + 8a^3\gamma\left(\gamma^2\gamma''' + 2\gamma\gamma'\gamma'' + (\gamma')^3\right). \end{aligned}$$

easy to generate 500 loops
but what is the solution?

1)


in D dimensions

DJB $q = 0 \Rightarrow$ Saalschutzian ${}_3F_2$

$\sim \sum_n$ pochhammers (n, D)

$\Rightarrow \varepsilon$ expansion in terms of

polylogs of ± 1

$\sum_{m>n} \frac{(-1)^m}{m^3} \frac{(-1)^n}{n} \rightarrow Li_4(\frac{1}{2}$

GPXT $m = 0$ gives only

$20\zeta(5) + \varepsilon\left[A\zeta(6) + B\zeta(3)^2\right] + \ldots$

Conjecture : new numbers at $O(\varepsilon^5)$ which are not MZVs

surely $\zeta(5,3) = \sum_{m>n} \frac{1}{m^5}\frac{1}{n^3}$ at $O(\varepsilon^3)$

2)
DJB


polylogs of -1


BOTH!


polylogs of $\frac{1+\sqrt{-3}}{2}$

LAPORTA
9 AUG 2002


Like ??

PSLQ

products of elliptic integrals with moduli in the quartic number field generated by

$K(m)K(\bar{m})$

$m = \frac{\Theta^2}{1+\Theta^2}$

$\bar{m} = \frac{\bar{\Theta}^2}{1+\bar{\Theta}^2}$

$\Theta = (2-\sqrt{3})^2(4-\sqrt{15})$

$\Theta + \frac{1}{\Theta} = (1+\sqrt{5})^4$

$\bar{\Theta} + \frac{1}{\bar{\Theta}} = (1-\sqrt{5})^4$

answer = Norm$[K]$ in golden subfield

3) Why does QCD hate $\zeta(4) = \frac{\pi^4}{90}$?

DJB Because $\zeta(3)$ is postponed in β

L loop counterterms up to $\zeta(2L-3)$
in QCD

But $\zeta(3)$ absent from 3-loop β in QCD

[and in quenched QED even $\beta_4 = 46$ is rational!]

Theorem: 4-loop R cannot involve $\zeta(4)$

Proof: Reorganize Chetyrkin + Tkachov integration by parts so that all MINCER out-put is

$$R_0(D) + R_3(D)\, Z_3(D) \leftarrow 6\zeta(3) + O(\varepsilon)$$
$$+ \sum_{i=1,2,3,4} \underbrace{R_{5i}(D)}_{\text{rational}}\, Z_{5i}(D) \quad 20\zeta(5) + O(\varepsilon)$$

$R_3(D)$ cannot be $O(1/\varepsilon)$

Scalar correlator gets $\zeta(4)$ from $\zeta(3)$ in 3-loop γ_m

Cannot be in logarithmic derivative

Does this contine at 5-loops in recent $O(\alpha_s^4 N_F^2)$ terms?

Scandals: Why is β_3^{QCD} rational?

Why is β_4^{QED} rational in quenched case?

(ward identities)