Progress in calculating massless Hexagon amplitudes at one-loop

T.B., J. Ph. Guillet, G. Heinrich, C. Schubert

- Motivation
- Where are we ...
- Summary

Thomas Binoth

RADCOR 02, Kloster Banz

Motivation:

HEP Experiments of the next 10 years are hadron colliders operating at the TeV scale

2002 ~ 2007+x Tevatron Run 2 2 TeV 2 fb^{-1}/year

~ 2007 ~ 2010+x LHC 14 TeV 30-100 fb^{-1}/year

$\boxed{\text{Decade of Multi particle / jet physics}}$

Jet rates at LHC: (LO*)

# jets	3	4	5	6	7	8
σ/nb	91.41	6.54	0.46	0.032	0.002	0.0002

Draggiotis / Kleiss / Papadopoulos hep-ph/020220

* $p_{T_j} > 60$ GeV, $\theta_{ij} > 30°$, $|\eta_j| < 3$

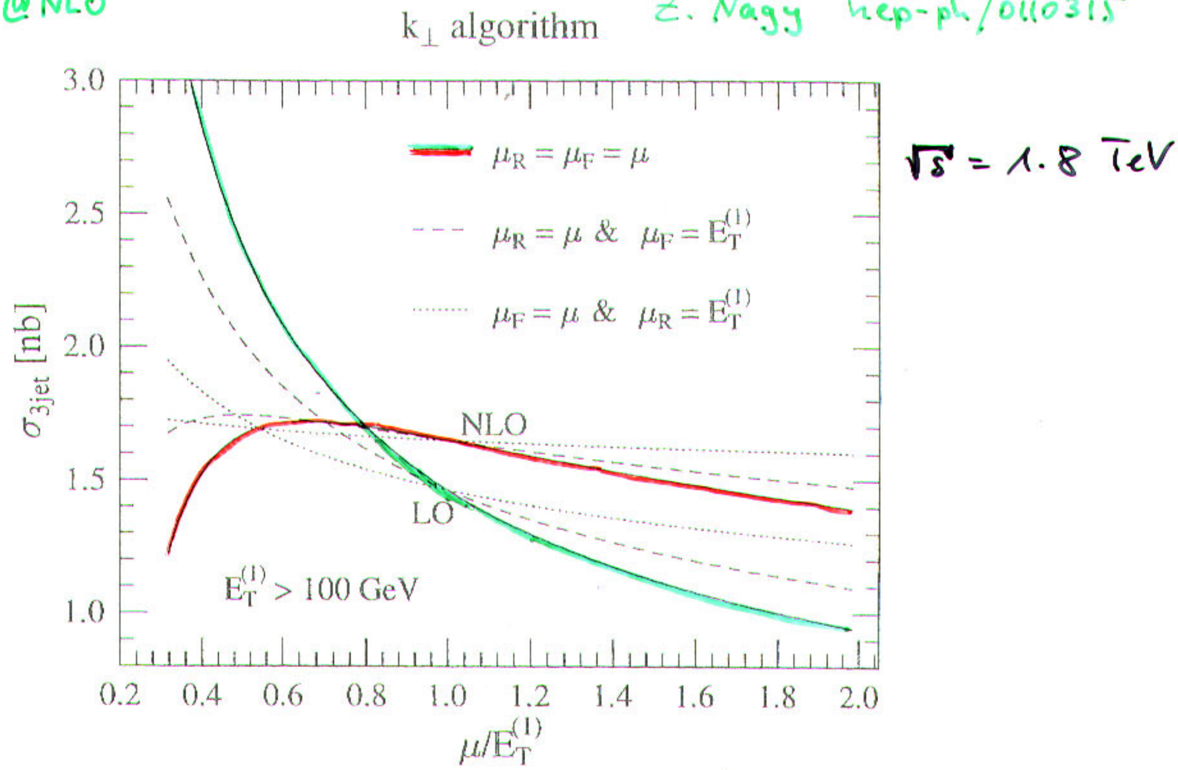
Problems with LO:

1) Large scale dependence

N-jet cross section $\sim \alpha_s(\mu)^N$

3 jet @NLO

Z. Nagy hep-ph/0110315

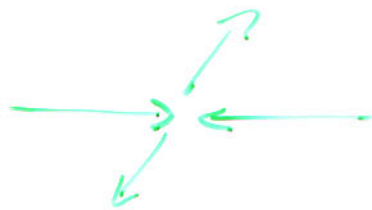


To make predictions of jet rates (at least) NLO mandatory!

e.g. $\Rightarrow$ reliable background estimates
for SM $H \to WW \to 4$ jets
susy $pp \to hA \to b\bar{b}b\bar{b}$

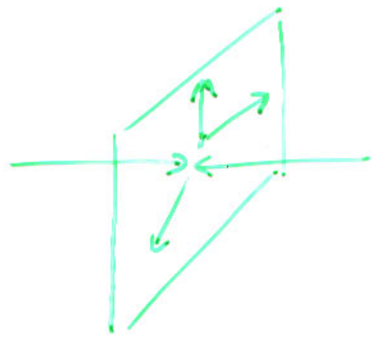
2) at LO degenerate kinematics:

2→2:



jets are forced to be back-to-back

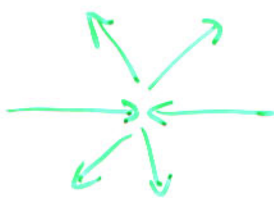
2→3:



jets lie in a plane

Constrained kinematics sensitive to extra parton emission

2→4:



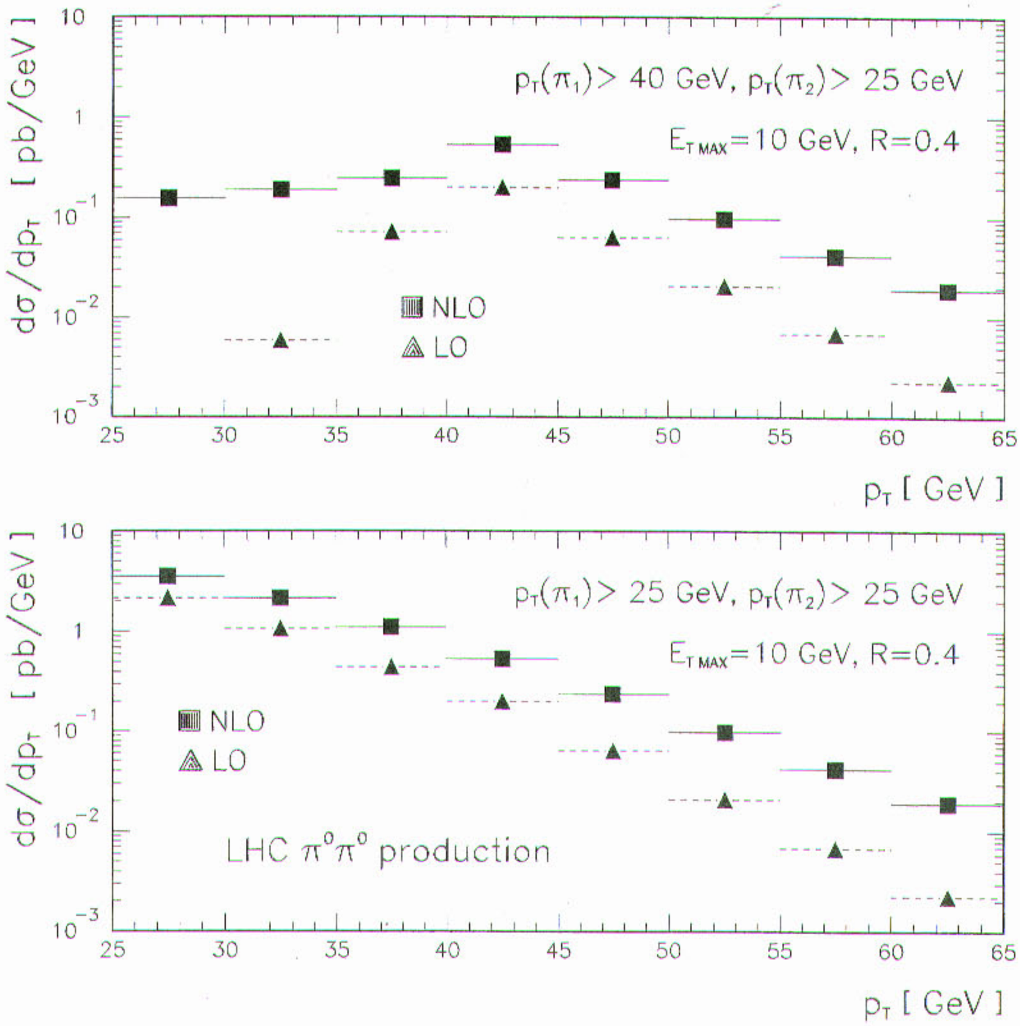
only restricted by 4-momentum conservation

- Backgrounds with severe cuts can be heavily under-estimated at LO!
- 2→4 NLO + 2→5 LO should give reliable QCD background predictions for all kinds of pair produced particles:

e.g. HZ, HA, ZZ, $\phi\phi$, ... plane-plane correlations

Example: LHC $\pi_0\pi_0$ production

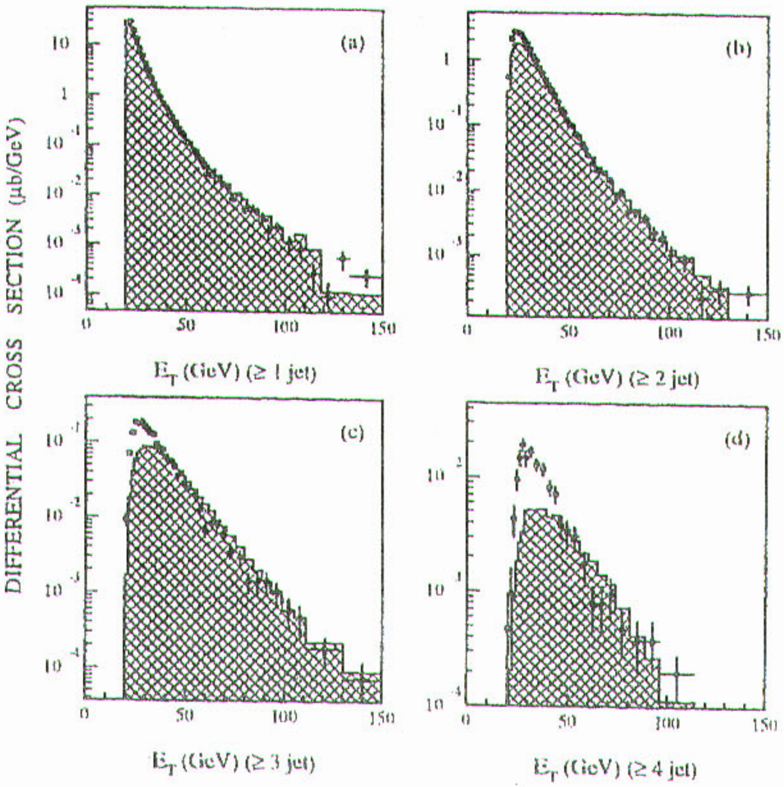
- important reducible background for $H \to \gamma\gamma$!



T.B., J.Ph. Guillet, E. Pilon, M. Werlen

3) Jet structure:

Jet structure theoretically better described the more partons are included in the matrix element.



Data vs. Pythia

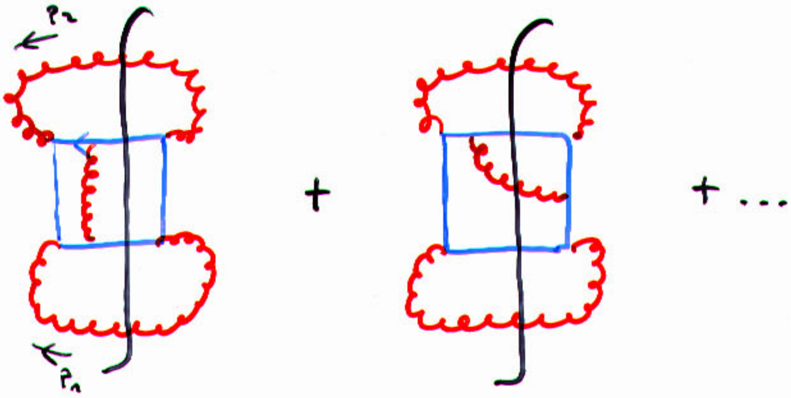
J. Krane (DØ, CDF)
hep-ex/0105069

N-point amplitudes : N = 4,5

What do we know up to now ?

- 2 → 2 parton scattering at NLO

  (R.K. Ellis , J.C. Sexton 1986)

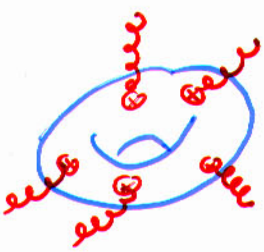


⇝ 4 point massless scalar integrals
rank 2 3-point tensorintegrals
1 → 2 phase space integrals

- 2 → 3 parton scattering at NLO

$gg \to ggg$
$q\bar{q} \to ggg$
$q\bar{q} \to q'\bar{q}'g$

Bern / Dixon / Kosower 93
Kunszt / Signer / Trócsányi 94

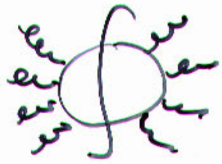


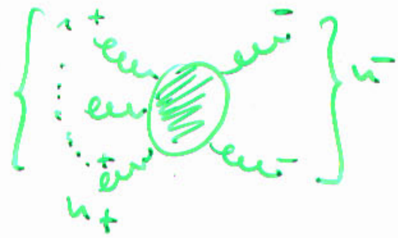
⇝
- „stringy" methods
- colour ordered subamplitudes
- susy relations
- helicity formalism at NLO

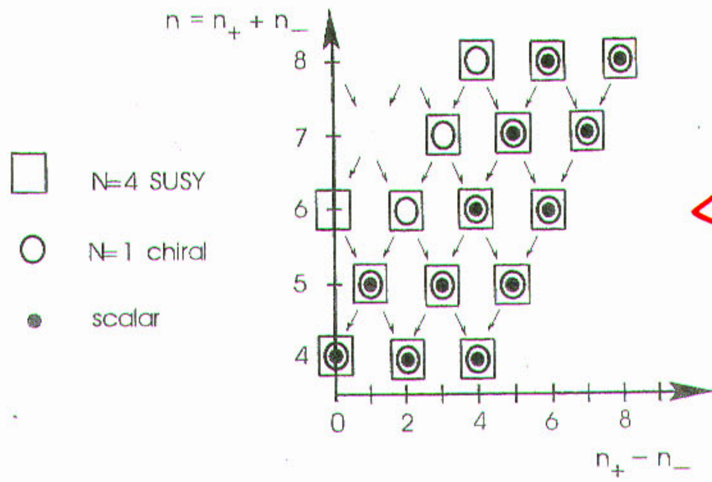
⇝ 5 point scalar integrals
Tensorreduction in Feynman parameter space

Bern, Dunbar, Dixon, Kosower, Chalmers, Morgan

What about Hexagon amplitudes ?

$\mathcal{A} \sim \int dPS$  „ Cut constructability "

one - loop n-gluon amplitudes : 



⟸ Hexagon

Lance Dixon, TASI 95

N = 4	susy	g ⟺ f ⟺ s
N = 1	susy chiral	f ⟺ s
non susy		f or s

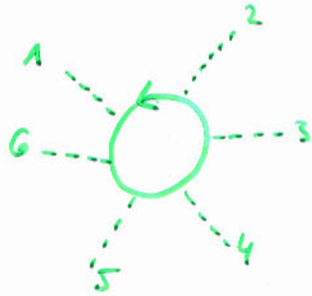
Needs to be solved !!!

Need : i) Scalar N-point integrals

ii) Tensorreduction

iii) Most efficient organization of calculation ?

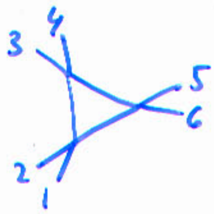
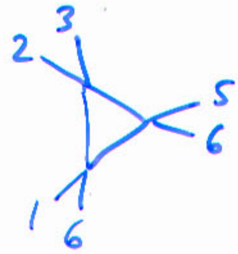
6 scalar scattering in the Yukawa model

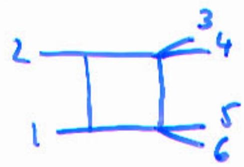
$A(1,\dots,6) =$  $= \int \frac{d^4k}{i\pi^{d/2}} \frac{\mathrm{tr}(q_1,\dots,q_6)}{q_1^2 \cdots q_6^2}$

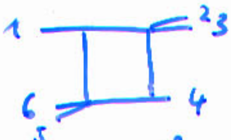
$$\mathcal{M}^{6s} \sim \sum_{\pi \in S_6} A(\pi_1,\dots,\pi_6)$$

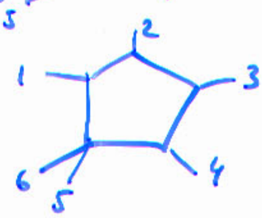
$q_j := k - p_1 - \dots - p_j$

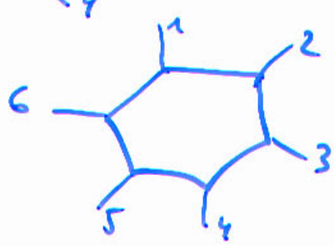
$\mathrm{tr}(12\dots) := \mathrm{tr}(\not{p}_1 \not{p}_2 \cdots)$

$A = 2$  $+ \; 2$ 

$+ \; \mathrm{tr}(12)$  $+ \; 5$ cyclic perms.

$+ \; \mathrm{tr}(14)$  $+ \; 2$ c.p.

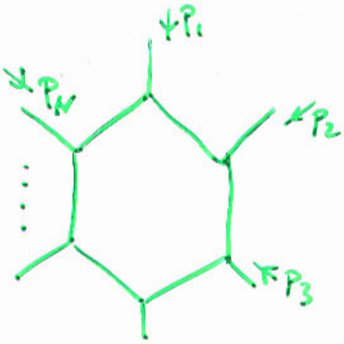
$+ \; \mathrm{tr}(1234)$  $+ \; 5$ c.p.

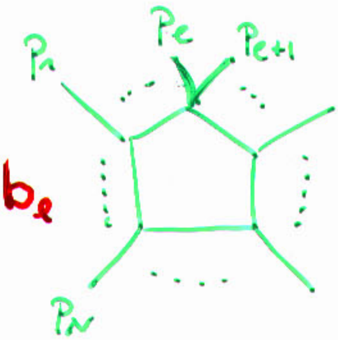
$+ \; \mathrm{tr}(123456)$ 

- No tensorreduction needed

⇝ reduce everything to  +  $|_{n+2}$
to get IR cancellations

Reduction formulas for N-point scalar integrals

$I_N^n =$ 

$= \sum_{l=1}^{N} b_l$ 

$+ R$

$$R = \begin{cases} -(1+2\varepsilon)\dfrac{\det G}{\det S}\, I_4\Big|_{n+2} & ; \quad N=4 \\ 2\varepsilon\left(\sum b_l\right) I_5^{n+2} & ; \quad N=5 \\ -(N-5+2\varepsilon)\underbrace{\left(\sum b_l\right)}_{\sim \det G \,\equiv\, 0\,!} I_N^{n+2} & ; \quad N \geq 6 \end{cases}$$

$$\boxed{I_N^n = (-1)^N\, \Gamma\!\left(N-\tfrac{n}{2}\right) \int_0^{\infty} d^N x\, \delta\!\left(1-\sum x_l\right) \frac{1}{(x \cdot S \cdot x)^{N-n/2}}}$$

$$\boxed{\begin{aligned} S_{ij} &= G_{ij} - G_{ii}/2 - G_{jj}/2 \\ G_{ij} &= 2\,(p_1+\ldots+p_i)\cdot(p_1+\ldots+p_j) \end{aligned}}$$

„kinematical information"

$\leadsto$ Each N-point integral is a linear combination of triangle integrals plus (n+2) dim. box integrals.

Bern, Chalmers
Bern, Dixon, Kosower
T.B., Guillet, Heinrich
O. Tarasov

Reduction coefficients for $N=6$ [9 scales]

$$S = \begin{bmatrix} 0 & 0 & s_{23} & s_{234} & s_{61} & 0 \\ 0 & 0 & 0 & s_{34} & s_{345} & s_{12} \\ s_{23} & 0 & 0 & 0 & s_{45} & s_{123} \\ s_{234} & s_{34} & 0 & 0 & 0 & s_{56} \\ s_{61} & s_{345} & s_{45} & 0 & 0 & 0 \\ 0 & s_{12} & s_{123} & s_{56} & 0 & 0 \end{bmatrix}$$

$$(S\cdot b)_j = 1 \quad \Leftrightarrow \quad b_j = \sum_{k=1}^{6} S^{-1}_{kj}$$

$$\sum_{e=1}^{6} b_e = -\frac{\det G}{\det S} = 0$$

$\mathrm{tr}(ij\ldots) := \mathrm{tr}(\not{k}_i \not{k}_j \cdots)$

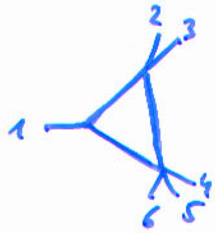
$$\boxed{b_1 = \frac{\mathrm{tr}(123456)\,\mathrm{tr}(4561) - 2\, s_{34} s_{45} s_{56}\, \mathrm{tr}(6123)}{4\, s_{12} s_{23} s_{34} s_{45} s_{56} s_{61} - \mathrm{tr}(123456)^2}}$$

$\sim$ long polynomial

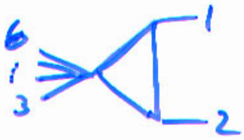
$$\mathrm{tr}(3456) + \mathrm{tr}(123456)\, b_1 + 2\, s_{34} s_{45} s_{56}\, b_4 = 0$$
$$\mathrm{tr}(14) + \mathrm{tr}(1234)\, b_2 + \mathrm{tr}(4561)\, b_5 = 0$$

$\leadsto$ nonlinear constraints between Mandelstam variables are represented linearly in terms of reduction coefficients !

Each graph IR, UV finite :



$\sim \frac{1}{\varepsilon}$

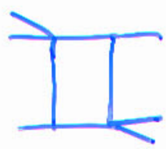


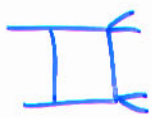
$\sim \frac{1}{\varepsilon^2}$

coefficients vanish due to linear relations of b's.

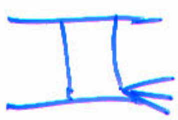
$\rightsquigarrow$ transparent.

Unexpected cancellations :

$F_{2B} \sim$  $\Big|_{n+2}$ $\rightsquigarrow$ vanish !

$F_{2A} \sim$  $\Big|_{n+2} \sim \underbrace{\mathrm{Li}_2\left(1-\frac{s_{ij}}{s_{ijk}}\right)}_{\text{vanish !}} + \log \ldots$

[related to IR cancellations]

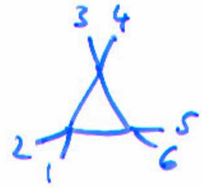
$F_1 \sim$  $\Big|_{n+2} \sim \underbrace{\mathrm{Li}_2}_{\text{vanish}} + \log \log + \underbrace{\pi^2}_{\text{vanish}}$

$\rightsquigarrow$ only log terms survive !

Result :

$$\mathcal{M}^{6s}(p_1,\dots,p_6) = -\frac{\lambda^6}{(4\pi)^2}\sum_{\pi\in S_6} G(\pi_1,\dots,\pi_6)$$

$$G(1,2,3,4,5,6) = \frac{2}{3}$$



$$+\left\{\frac{b_1}{s_{123}s_{345}-s_{12}s_{45}}\Big[\mathrm{tr}(6123) - 2s_{16}(s_{123}-s_{12})\Big]\right.$$

$$\left.+\frac{b_2}{s_{123}s_{234}-s_{23}s_{56}}\Big[\mathrm{tr}(1234)-2s_{34}(s_{123}-s_{23})\Big]\right\}\log\frac{s_{12}}{s_{123}}\log\frac{s_{23}}{s_{123}}$$

$$+\left\{-b_1+\frac{b_3}{2(s_{123}s_{234}-s_{23}s_{56})}\Big[\mathrm{tr}(1234)-2s_{34}(s_{123}-s_{23})\Big]\right.$$

$$\left.+\frac{b_6}{2(s_{234}s_{345}-s_{61}s_{34})}\Big[\mathrm{tr}(5612)-2s_{56}(s_{345}-s_{61})\Big]\right\}$$

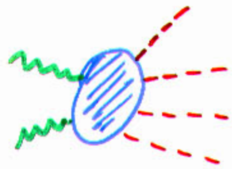
$$\times\left[\log\left(\frac{s_{12}}{s_{234}}\right)\log\left(\frac{s_{56}}{s_{234}}\right)+\log\left(\frac{s_{34}}{s_{234}}\right)\log\left(\frac{s_{12}}{s_{56}}\right)\right]$$

First (?) nontrivial non-susy Hexagon amplitude ...

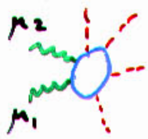
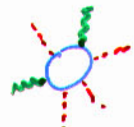
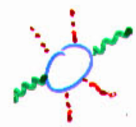
... probably most simple nontrivial example.

$\gamma\gamma \to$ 4 scalars in the gauged Yukawa model

$$\mathcal{A}_{\gamma\gamma\to 4s} = \varepsilon_1^{\mu_1}\varepsilon_2^{\mu_2}\,\mathcal{M}_{\mu_1\mu_2} \;;$$



(*) $$\mathcal{M}^{\mu_1\mu_2} = A\, g^{\mu_1\mu_2} + \sum_{j_1,j_2=1}^{5} B_{j_1 j_2}\, r_{j_1}^{\mu_1} r_{j_2}^{\mu_2} \;;\quad r_j = \sum_{e=1}^{j} p_e$$

(**) $$\mathcal{M}^{\mu_1\mu_2} = \sum_{\{3,4,5,6\}} \Big\{$$  +  +  + ... $\Big\}$

Define Projection operators on A, $B_{j_1 j_2}$:

$$\mathcal{P}^{\mu_1\mu_2} = \frac{1}{n-5}\left(g^{\mu_1\mu_2} - 2\, r_{j_1}^{\mu_1} H_{j_1 j_2} r_{j_2}^{\mu_2} \right)$$

$$R_e^{\mu_1} = 2\, H_{ej}\, r_j^{\mu_1} \;;\quad H = G^{-1};\; G_{ij} = 2 r_i \cdot r_j$$

$$\rightsquigarrow \quad \tilde{A}(\mathcal{M}) = \mathcal{P}^{\mu_1\mu_2}\mathcal{M}_{\mu_1\mu_2} = A$$

$$\tilde{B}_{j_1 j_2}(\mathcal{M}) = \left(R_{j_1}^{\mu_1} R_{j_2}^{\mu_2} - 2\, H_{j_1 j_2}\, \mathcal{P}^{\mu_1\mu_2} \right)\mathcal{M}_{\mu_1\mu_2}$$

$$= B_{j_1 j_2}$$

Properties: $$R_{j_1} \cdot r_{j_2} = \delta_{j_1 j_2}$$

$$\mathcal{P}^{\mu_1\mu_2} r_j^{\mu_2} = 0$$

$$\mathcal{P}^{\mu_1\mu_2} \mathcal{P}_{\mu_2\mu_1} = Tr(\mathcal{P}) = Tr(\mathcal{P}\mathcal{P}) = 1$$

$\leadsto$ R_e^μ n-dim. dual vector r_e^μ

$\mathcal{P}$ projector on $(n-5)$ dim. subspace $\perp$ to space spanned by $\{r_1, \dots, r_5\}$

$$\int d^n k \, \frac{(\mathcal{P}\cdot k)^\mu}{q_1^2 \dots q_N^2} = 0$$

$$\int d^n k \, \frac{R_e \cdot k}{q_1^2 \dots q_N^2} \sim \int d^N x \, \delta(1-\sum x) \frac{x_e}{(x\cdot S\cdot x)^{N-n/2}} \equiv I_N^n(e)$$

$$\int d^n k \, \frac{k\cdot \mathcal{P}\cdot k}{q_1^2 \dots q_N^2} \sim I_N^{n+2}$$

$\leadsto$ $\mathcal{P}$, R_e^μ map momentum space tensor integrals to Feynman parameter integrals

$\rightarrow$ Lorentz indices saturated!

{ string inspired methods produce Amplitude representations in terms of such integrals }

Reduction of $\underbrace{\text{Feynman}}^{N\text{-point}}$ parameter integrals:

$$\int_{-\infty}^{\infty} d^N x \, \frac{\partial}{\partial x_j} \left\{ \frac{\theta(x_1)\dots\theta(x_N)\,\delta(1-\Sigma x_e)\, x_{e_1}\dots x_{e_R}}{\left[\frac{1}{2}\, x\cdot S\cdot x\right]^{-N+n/2+1}} \right\} = 0$$

⇝ „integration by parts in parameter space"

recursion relation

$$I_N^n(l_0,\dots,l_{R-1}) = \sum_{k=0}^{P} S^{-1}_{l_0 l_k}\, I_N^{n+2}(l_1,\dots,\hat{l}_k,\dots,l_{R-1})$$
$$+ b_{l_0}(N-n-R)\, I_N^{n+2}(l_1,\dots,l_{R-1})$$
$$- \sum_{j=1}^{N} S^{-1}_{j l_0}\, I_{N-1,j}(l_1,\dots,l_{R-1})$$

[proof by induction over N]

N-point rank R ⇒ $\begin{cases} (N-1) \text{ point rank } R-1, \text{dim}=n \\ N \text{ point rank } R-1, R-2, \text{dim}=n+2 \end{cases}$

⇒ $\boxed{\text{Tensor integrals mapped to scalar integrals in higher dimensions.}}$

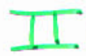
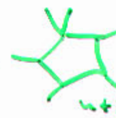
{no Gram determinants at that step...}

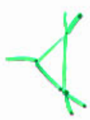
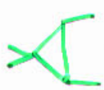
Reduction formula for scalar integrals induces mapping of $(n+2m)$-dim Integrals to a basis set:

$$\boxed{I_6^{n+2m},\ I_5^{n+2m},\ I_4^{n+2m},\ I_3^{n+2m},\ I_2^{n+2m}}$$

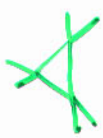
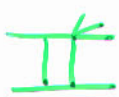
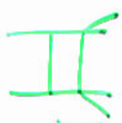
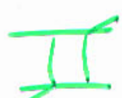
$$\Big\Downarrow \text{ reduction formulas}$$

$$\boxed{I_5^{n+2},\ I_4^{n+2},\ I_3^{n},\ I_2^{n}}$$

 $\rightsquigarrow$ $9\times$ , $20\times$ , $15\times$  $|_{n+2}$, $6\times$  $_{n+2}$

$\rightsquigarrow$ $1/\varepsilon$ poles in , ,  <u>cancel</u>

$\rightsquigarrow$  $|_{n+2}$ <u>cancel</u>

$\rightsquigarrow$ ,  $_{n+2}$,  $_{n+2}$,  $_{n+2}$ <u>remain</u>

$=: I_3,\quad F_1,\quad F_{2A},\quad F_{2B}$

⇝ Tensor coefficients A, B_{ij}

expressed in terms of I_3, F_1, F_{2A}, F_{2B}

⇝

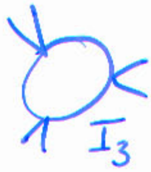
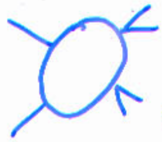
$$\boxed{\begin{aligned} \mathcal{M}^{++} &= A \; \varepsilon_1^+ \cdot \varepsilon_2^+ + \sum B_{jk} \; \varepsilon_1^+ \cdot r_j \; \varepsilon_2^+ \cdot r_k \\ \mathcal{M}^{+-} &= A \; \varepsilon_1^+ \cdot \varepsilon_2^- + \sum B_{jk} \; \varepsilon_1^+ \cdot r_j \; \varepsilon_2^- \cdot r_k \end{aligned}}$$

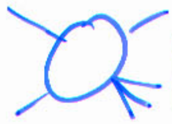
- gauge invariance + transversality:
  only 9 of 25 B_{ij} needed

- representation sufficient for practical purposes, still ...

Question: Is there a „optimal" representation in terms of reduction coefficients b_e ??

$\underline{\mathcal{M}^{++}}$:

 I_3 ,  F_{2A} $\sim \underline{\underline{0}}$!

$F_1 \sim$  $\Big|_{n+2} \sim \log \log + \pi^2$

$$\mathcal{M}^{++}_{F_1} = \sum_{\pi \in \{3456\}} C(12\pi_3\pi_4\pi_5\pi_6) \otimes$$

$$\Big\{ F_1(\pi_4\pi_5, \pi_5\pi_6, \pi_4\pi_5\pi_6) + F_1(1\pi_3, 2\pi_3, 12\pi_3) - 2\, F_1(1\pi_3, 1\pi_4, 1\pi_3\pi_4) \Big\}$$

$$C(123456) =$$

$$\frac{s_{23} + s_{45} - s_{123} - s_{12}s_{23}\mathbf{b_2} - s_{23}s_{45}\mathbf{b_3} - s_{45}s_{56}\mathbf{b_4}}{s_{16}}$$

$$+ \frac{s_{12} + s_{56} - s_{12}s_{23}\mathbf{b_1} - s_{45}s_{56}\mathbf{b_5} - s_{12}s_{56}\mathbf{b_6}}{s_{34}}$$

$$- 4 \left\{ \frac{\mathbf{b_5}}{s_{34}}\, \varepsilon(2456) - \frac{\mathbf{b_4}}{s_{16}}\, \varepsilon(1456) \right\}$$

$\left.\begin{matrix} \mathcal{M}^{+-} \\ \mathcal{M}^{++}_{F2B} \end{matrix}\right\}$ under construction ....

Summary:

- NLO description for $2 \to 4$ processes mandatory to make reliable predictions for jet rates and all kinds of backgrounds at LHC, Tevatron

- Amplitudes: $\phi\phi \to \phi\phi\phi\phi$, $\gamma\gamma \to \phi\phi\phi\phi$ computed to understand efficient organization of calculation

  - Projector method
  - Reduction of Feynman parameter integrals
  - Helicity methods
  - Linear amplitude representations in terms of reduction coefficients b_e
    $\rightsquigarrow$ „optimal representations" [conjecture?]

- Final goal: $qq \to qqqq$, $q\bar{q} \to qqqq$, ...

  ... still a long way!
  but we are on the track ...