The Asymptotic Expansion of Lattice Loop Integrals Around the Continuum Limit

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Continuum Lattice Perturbation Theory

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Overview

- Motivation
- Perturbation theory in lattice regularization. . .
- . . . with continuum methods
  - Asymptotic expansion
  - Integration-by-parts identities
- Applications
- Outlook

Phenomenology from the lattice

Typical QCD result:

"physical quantity"

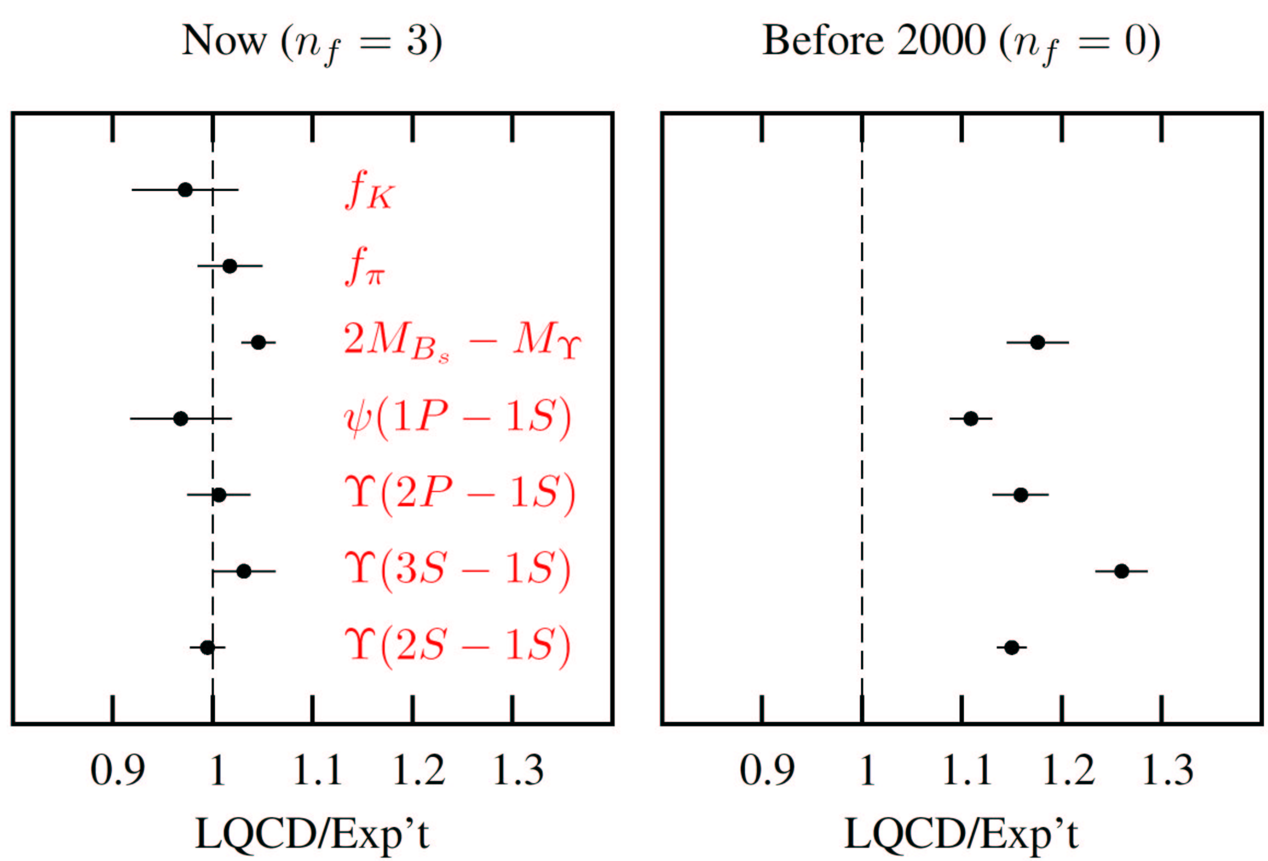
$=$ "short distance contribution" $\times$ "low energy part"

perturbative (asympt. freedom) | non-perturbative (lattice sim.)

- Splitting is regularization dependent.
- Matching requires perturbative calculation in lattice regularization.

Unquenching (©G. P. Lepage, HighPprecisionQCD collab.)

⇒ New results: (lattice QCD)/(experiment)



HPQCD+MILC: Very Preliminary

. G.P. Lepage, High Precision Nonperturbative QCD at the *SLAC Summer Institute* (August 2002). – p.65/77

HPQCD+MILC phenomenology program:

- Aim at few-% accuracy for π-, K-, B- and D-formfactors and mixing.
  - D-physics as a test case ($\leftarrow$ CLEO-C).
  - Apply to B-system, extract weak interaction parameters.
- Bottleneck: (1- and 2-loop) lattice PT.
  - Perturbatively improved actions,
  - precise matching calculations.

PT Calculations for HPQCD: G.P. Lepage

- Light quarks (Improved staggered)
  - Mass, wave function and coupling constant renormalization
  - Decay constants (currents)
  - $K - \bar{K}$ mixing (four quark operators)
- Heavy quarks (NRQCD)
  - Quarkonia leptonic decay widths
  - Radiative transition form factors
  - Mass renormalization
  - Decay constants f_D, f_B
  - $B - \bar{B}$ mixing (four quark operators)

(The trouble with) lattice PT

- Complicated propagators. . .

Simplest form of massive bosonic propagator:

$$G_B(k) = \frac{1}{(\widehat{k}^2 + m^2)}, \quad \text{with } \widehat{k}^2 = \sum_{\mu=1}^{4} 4\sin^2\frac{k_\mu}{2}\,.$$

e. g. self-energy integral:

$$\int_{-\pi}^{\pi} \frac{d^4k}{(2\pi)^4} \frac{1}{(\widehat{k}^2 + m^2)} \frac{1}{((\widehat{p+k})^2 + m^2)},$$

The trouble. . . (continued)

- . . . and complicated vertices, e.g. four gluon vertex:

$$
\begin{aligned}
&\Gamma^{ABCD}_{\mu\nu\lambda\rho}(p,q,r,s) = \\
&-g_0^2\Big[\sum_E f_{ABE}f_{CDE}\Big\{\delta_{\mu\lambda}\delta_{\nu\rho}[\cos\frac{1}{2}a(q-s)_\mu \cos\frac{1}{2}a(k-r)_\nu - \frac{a^4}{12}\tilde{k}_\nu\tilde{q}_\mu\tilde{r}_\nu\tilde{s}_\mu] \\
&\qquad - \delta_{\mu\rho}\delta_{\nu\lambda}[\cos\frac{1}{2}a(q-r)_\mu \cos\frac{1}{2}a(k-s)_\nu - \frac{a^4}{12}\tilde{k}_\nu\tilde{q}_\mu\tilde{r}_\mu\tilde{s}_\nu] \\
&\qquad + \frac{1}{6}\delta_{\nu\lambda}\delta_{\nu\rho}a^2(\widetilde{s-r})_\mu\tilde{k}_\nu\cos(\frac{1}{2}aq_\mu) \\
&\qquad - \frac{1}{6}\delta_{\mu\lambda}\delta_{\mu\rho}a^2(\widetilde{s-r})_\nu\tilde{q}_\mu\cos(\frac{1}{2}ak_\nu) \\
&\qquad + \frac{1}{6}\delta_{\mu\nu}\delta_{\mu\rho}a^2(\widetilde{q-k})_\lambda\tilde{r}_\rho\cos(\frac{1}{2}as_\lambda) \\
&\qquad - \frac{1}{6}\delta_{\mu\nu}\delta_{\mu\lambda}a^2(\widetilde{q-k})_\rho\tilde{s}_\lambda\cos(\frac{1}{2}ar_\rho) \\
&\qquad + \frac{1}{12}\delta_{\mu\nu}\delta_{\mu\lambda}\delta_{\mu\rho}a^2\sum_\sigma(\widetilde{q-k})_\sigma(\widetilde{s-r})_\sigma\Big\} \\
&\qquad + (B\leftrightarrow C, \nu\leftrightarrow\lambda, q\leftrightarrow r) + (B\leftrightarrow D, \nu\leftrightarrow\rho, q\leftrightarrow s)\Big] \\
&\qquad + \frac{g_0^2}{12}a^4\Big\{\frac{2}{3}(\delta_{AB}\delta_{CD}+\delta_{AC}\delta_{BD}+\delta_{AD}\delta_{BC}) \\
&\qquad + \sum_E(d_{ABE}d_{CDE}+d_{ACE}d_{BDE}+d_{ADE}d_{BCE})\Big\} \\
&\qquad \times\Big\{\delta_{\mu\nu}\delta_{\mu\lambda}\delta_{\mu\rho}\sum_\sigma\tilde{k}_\sigma\,\tilde{q}_\sigma\tilde{r}_\sigma\tilde{s}_\sigma - \delta_{\mu\nu}\delta_{\mu\lambda}\tilde{k}_\rho\tilde{q}_\rho\tilde{r}_\rho\tilde{s}_\mu \\
&\qquad - \delta_{\mu\nu}\delta_{\mu\rho}\tilde{k}_\lambda\tilde{q}_\lambda\tilde{s}_\lambda\tilde{r}_\mu - \delta_{\mu\lambda}\delta_{\mu\rho}\tilde{k}_\nu\tilde{r}_\nu\tilde{s}_\nu\tilde{q}_\mu - \delta_{\nu\lambda}\delta_{\nu\rho}\tilde{q}_\mu\tilde{r}_\mu\tilde{s}_\mu\tilde{k}_\nu \\
&\qquad + \delta_{\mu\nu}\delta_{\lambda\rho}\tilde{k}_\lambda\tilde{q}_\lambda\tilde{r}_\mu\tilde{s}_\mu + \delta_{\mu\lambda}\delta_{\nu\rho}\tilde{k}_\nu\tilde{r}_\nu\tilde{q}_\mu\tilde{s}_\mu + \delta_{\mu\rho}\delta_{\nu\lambda}\tilde{k}_\nu\tilde{s}_\nu\tilde{q}_\mu\tilde{r}_\mu\Big\}
\end{aligned}
$$

Methods for lattice PT

- Standard approach is numerics based.
  - Very flexible.
  - Needs significant computing resources.
  - Precision? Cancellations?
- Common perception: Tools of continuum PT not useful. *How about in the cont. limit??*

New approach

Use continuum techniques to minimize the amount of numerical integration.

- *Asymptotic expansion* around continuum limit.
- Splits integrals into
  - Soft part: Continuum integrals.
  - Hard part: Process-independent lattice tadpole integrals.
- *Integration-by-parts identities*
  - Reduce lattice tadpoles to a few master integrals. Solved using computer algebra.

Expansion around continuum limit

- In units of the lattice spacing a, momenta and masses go to zero in the limit $a \to 0$:

$$p^\mu = a\, p^\mu_{\text{phys}} \to 0 \qquad m = a\, m_{\text{phys}} \to 0$$

- Expansion does not commute with loop integration, but can be obtained on the level of the integrand.
- Use "asymptotic expansion", a technique developed for continuum integrals to perform the expansion.

Example: bosonic tadpole

To get started:

- Introduce intermediate *regulator* δ.
- *Change variables* $k_i = 2\tan\eta_i$:

$$\int_{-\pi}^{\pi} \frac{d^4k}{(2\pi)^4} \frac{1}{\left[\widehat{k}^2 + m^2\right]^{1+\delta}} = \frac{1}{\pi^4} \prod_{i=1}^{4} \int_{-\infty}^{\infty} \frac{d\eta_i}{1+\eta_i^2} \times \frac{1}{\left[\sum_i \frac{4\,\eta_i^2}{1+\eta_i^2} + m^2\right]^{1+\delta}}$$

$\rightarrow$ Maps integration region to $[-\infty, \infty]$.
Form becomes similar to continuum integrals.

Soft and hard regions

Expansion is obtained a sum of two contributions

- Soft: assume $\eta_i \sim m \ll 1$.
  - Expand the integrand in η_i and m.
  - $\rightsquigarrow$ Continuum loop integrals.
  - Independent of lattice action
- Hard: assume $\eta_i \sim 1 \gg m$.
  - Expand the integrand in m.
  - $\rightsquigarrow$ Massless lattice tadpoles.
  - Process independent

Note: No restriction on the integration region!

Soft and hard integrals

$$\prod_{i=1}^{4} \int_{-\infty}^{\infty} \frac{d\eta_i}{1+\eta_i^2} \times \frac{1}{\left[\sum_i \frac{4\,\eta_i^2}{1+\eta_i^2} + m^2\right]^{1+\delta}}$$

$\eta_i \sim m \ll 1$ $\quad$ $m \ll \eta_i \sim 1$

Soft integrals:

$$\int_{-\infty}^{\infty} d^4\eta \, \frac{\prod_i (\eta_i^2)^{\alpha_i}}{[4\,\eta^2 + m^2]^{n+\delta}}$$

continuum integrals

UV divergencies

Hard integrals: $H(\{a_i\}, n) =$

$$\prod_i \int_{-\infty}^{\infty} \frac{d\eta_i}{(1+\eta_i^2)^{a_i}} \times \frac{1}{\left[\sum_i \frac{\eta_i^2}{1+\eta_i^2}\right]^{n+\delta}}$$

massless lattice tadpoles

IR divergencies

Evaluation of $H(\{a_i\}, n)$

- Use algebraic relations to reduce the number of hard integrals. Carraciolo et. al.
  - Partial fractioning
  - Integration-by-parts

e. g. $0 = \int_{-\infty}^{\infty} d^4\eta \frac{\partial}{\partial \eta_1} \left\{ \eta_1 \prod_i \frac{1}{(1+\eta_i^2)^{a_i}} \times \frac{1}{\left[\sum_i \frac{4\,\eta_i^2}{1+\eta_i^2}\right]^{n+\delta}} \right\}$

Perform derivative, rewrite in $H(\{a_i\}, n)$'s:

$$0 = \left\{ 1 + 2\,a_1\,(\boldsymbol{a}_1^+ - 1) + 2\,(n+\delta)\,\boldsymbol{n}^+\,\boldsymbol{a}_1^+(\boldsymbol{a}_1^+ - 1) \right\} H(\{a_i\}, n)$$

Notation: operator $\boldsymbol{a}_1^+$ increases a_1 by one, etc.

Reduction to master integrals

- Use *computer algebra* to solve algebraic relations for $H(\{a_i\}, n)$ in the index range needed for a given calculation. Laporta

1. Start with a small number of equations.
2. Order integrals (and equations) by difficulty.
3. Solve, using Gauss's elimination method to express integrals in terms of simpler ones.
4. Add a few equations, go to 2.

Result: All hard integrals $H(\{a_i\}, n)$ are expressed in terms of four convergent ones.

Evaluation of the master integrals

- By solving the equations, we reduce all integrals to four convergent master integrals.

$$288\ (2-\delta)\ (1-\delta)^2\,\delta^2\,H(\{\vec{1}\},2) =$$
$$(-1728 + 8216\,\delta - 18752\,\delta^2 + 23630\,\delta^3 - 16346\,\delta^4 + 5810\,\delta^5 - 830\,\delta^6)\ H(\{\vec{1}\},0)$$
$$+ (6048 - 24124\,\delta + 46180\,\delta^2 - 46205\,\delta^3 + 24765\,\delta^4 - 6795\,\delta^5 + 755\,\delta^6)\ H(\{\vec{1}\},-1)$$
$$+ (-4680 + 17006\,\delta - 29440\,\delta^2 + 25510\,\delta^3 - 11520\,\delta^4 + 2618\,\delta^5 - 238\,\delta^6)\ H(\{\vec{1}\},-2)$$
$$+ (972 - 3321\,\delta + 5373\,\delta^2 - 4194\,\delta^3 + 1662\,\delta^4 - 325\,\delta^5 + 25\,\delta^6)\ H(\{\vec{1}\},-3)$$

- Evaluate master integrals numerically.

$$H(\{\vec{1}\},-n) \sim \int_{-\pi}^{\pi} d^4k\,(\widehat{k}^2)^{n+\delta}$$

  Expand the integrand in δ.

- Arbitrary precision is trivial.

Propagator at the origin

$$G(m) = \int_{-\pi}^{\pi} \frac{d^4k}{(2\pi)^4} \frac{1}{(\widehat{k}^2 + m^2)}$$

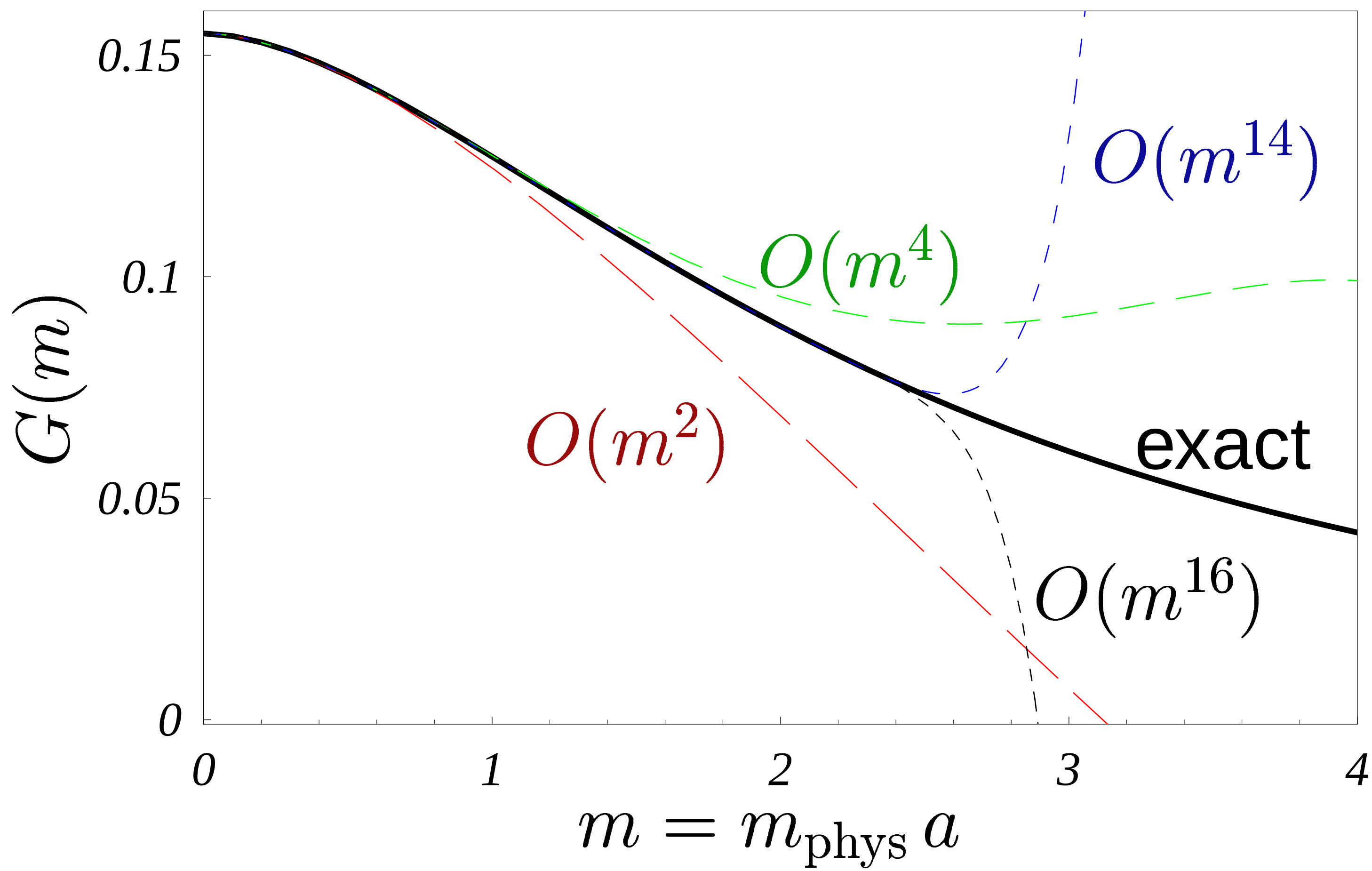
Result for the hard and the soft parts to $O(m^4)$:

$$G^{\mathrm{S}}(m) = \frac{m^2}{16\,\pi^2}\left(-1 - \frac{1}{\delta} + \ln\frac{m^2}{4}\right) + \frac{m^4}{128\pi^2}\left(\frac{3}{2} + \frac{1}{\delta} - \ln\frac{m^2}{4}\right)$$

$$G^{\mathrm{H}}(m) = b_1 + m^2\left(-b_2 + \frac{1}{16\pi^2}\left(\frac{1}{\delta} + \ln 4\right)\right) + m^4\left(b_3 - \frac{1}{128\pi^2}\left(\frac{1}{\delta} - \ln 4\right)\right)$$

- b_1, b_2 and b_3 from master integrals.
- UV singularities of the soft part G^{S} *cancel* against IR divergencies of the hard part G^{H}.

Propagator at the origin



Other propagators

Have applied the method to other propagators

Propagator	# of master ints.
boson (std.)	4
HQET	7
Wilson fermion Wilson f. & boson	10 16
Staggered f. & boson	16

and calculated various QCD self-energies.

Outlook

- Stategy is not limited to specific action, nor to one loop.
- Next step: Apply method to improved actions
  $\rightarrow$ 1-loop HPQCD calculations.
- Extension to two loops.
  - Computer algebra (solution of recurrence relations)?
  - Number of master integrals?

Summary

- Perturbative calculations in lattice reg. are becoming increasingly important.
- Calculations can be performed with continuum methods:
  - Asymptotic expansion
  - Integration-by-parts relations
  - $\rightsquigarrow$ Minimize the amount and complexity of numerical integrations.
  - $\rightsquigarrow$ More transparent results.
- Future: Improved actions. Two loops.