# Structural Relations between Harmonic Sums up to w=6

Johannes Blümlein, DESY

- Introduction
- Algebraic Relations
- Structural Relations
- Representation of some Observables
- Factorial Series
- The Basis
- Conclusions

Refs. J. Blümlein, DESY 07–042, and in preparation.

# 1. Introduction

- Single scale processes in massless Quantum Field Theories or being considered in the limit $m^2/Q^2 \to 0$ exhibit significant simplifications when calculated in Mellin space.
- This is, to some extent, due to structure of Feynman parameter intergrals which posess a Mellin symmetry.
- Harmonic sums form the appropriate language to derive compact expressions in the respective calculations.
- We will line out the relations of the harmonic sums, resp. their continuations to $N \epsilon \mathbf{Q}, \mathbf{R}, \mathbf{C}$.

x-space results :

Nielsen-type integrals, resp. harmonic polylogarithms (E. Remiddi and J. Vermaseren (1999))

$$S_{n,p,q}(x) = \frac{(-1)^{n+p+q-1}}{\Gamma(n)p!q!} \int_0^1 \frac{dz}{z} \ln^{(n-1)}(z) \ln^p(1-zx) \ln^q(1+zx)$$

# 2 Loop Wilson Coefficients

Order $\alpha_s^2$ contributions to the deep inelastic Wilson coefficient

W.L. van Neerven and E.B. Zijlstra
*Instituut-Lorentz, University of Leyden, P.O. Box 9506, NL-2300 RA Leyden, The Netherlands*

Volume 272, number 1,2 PHYSICS LETTERS B 28 November 1991

$$
\begin{aligned}
C_2^{(2),(+)}(x,1) = C_F^2 \Bigg[ & \frac{1+x^2}{1-x} \{4\ln^3(1-x) - (14\ln x + 9)\ln^2(1-x) \\
& - [4\,\mathrm{Li}_2(1-x) - 12\ln^2 x - 12\ln x + 16\zeta(2) + \tfrac{27}{2}]\ln(1-x) - \tfrac{4}{3}\ln^3 x - \tfrac{3}{2}\ln^2 x \\
& + [-24\,\mathrm{Li}_2(-x) + 24\zeta(2) + \tfrac{61}{2}]\ln x + 12\,\mathrm{Li}_3(1-x) - 12 S_{1,2}(1-x) \\
& + 48\,\mathrm{Li}_3(-x) - 6\,\mathrm{Li}_2(1-x) + 32\zeta(3) + 18\zeta(2) + \tfrac{51}{4}\} \\
& + (1+x)\{2\ln x\ln^2(1-x) + 4[\mathrm{Li}_2(1-x) - \ln^2 x]\ln(1-x) \\
& - 4[\mathrm{Li}_2(1-x) + \zeta(2)]\ln x + \tfrac{5}{3}\ln^3 x - 4\,\mathrm{Li}_3(1-x)\} \\
& + \left(40 + 8x - 48x^2 - \tfrac{72}{5}x^3 + \frac{8}{5x^2}\right)[\mathrm{Li}_2(-x) + \ln x\ln(1+x)] \\
& + (-8+40x)[\ln x\,\mathrm{Li}_2(-x) + S_{1,2}(1-x) - 2\,\mathrm{Li}_3(-x) - \zeta(2)\ln(1-x)] + (5+9x)\ln^2(1-x) \\
& + \tfrac{1}{2}(-91+141x)\ln(1-x) - (28+44x)\ln x\ln(1-x) - (14+30x)\,\mathrm{Li}_2(1-x) \\
& + (\tfrac{29}{2} + \tfrac{25}{2}x + 24x^2 + \tfrac{36}{5}x^3)\ln^2 x + \frac{1}{10}\left(13 - 407x + 144x^2 - \frac{16}{x}\right)\ln x + (-10 + 6x - 48x^2 - \tfrac{72}{5}x^3)\zeta(2) \\
& + \tfrac{407}{20} - \tfrac{1917}{20}x + \tfrac{72}{5}x^2 + \frac{8}{5x} + [6\zeta(2)^2 - 78\zeta(3) + 69\zeta(2) + \tfrac{331}{8}]\delta(1-x) \Bigg] \\
+ C_A C_F \Bigg[ & \frac{1+x^2}{1-x}\{-\tfrac{11}{3}\ln^2(1-x) + [4\,\mathrm{Li}_2(1-x) + 2\ln^2 x + \tfrac{44}{3}\ln x - 4\zeta(2) + \tfrac{367}{18}]\ln(1-x) \\
& - \ln^3 x - \tfrac{55}{6}\ln^2 x + [4\,\mathrm{Li}_2(1-x) + 12\,\mathrm{Li}_2(-x) - \tfrac{239}{6}]\ln x - 12\,\mathrm{Li}_3(1-x) + 12 S_{1,2}(1-x) - 24\,\mathrm{Li}_3(-x) \\
& + \tfrac{22}{3}\mathrm{Li}_2(1-x) + 2\zeta(3) + \tfrac{22}{3}\zeta(2) - \tfrac{3155}{108}\} \\
& + 4(1+x)[\mathrm{Li}_2(1-x) + \ln x\ln(1-x)] + \left(-20 - 4x + 24x^2 + \tfrac{36}{5}x^3 - \frac{4}{5x^2}\right)[\mathrm{Li}_2(-x) + \ln x\ln(1+x)] \\
& + (4-20x)[\ln x\,\mathrm{Li}_2(-x) + S_{1,2}(1-x) - 2\,\mathrm{Li}_3(-x) - \zeta(2)\ln(1-x)] + (\tfrac{133}{6} - \tfrac{1113}{18}x)\ln(1-x) \\
& + (-2 + 2x - 12x^2 - \tfrac{18}{5}x^3)\ln^2 x + \frac{1}{30}\left(13 + 1753x - 216x^2 + \frac{24}{x}\right)\ln x + (-2 - 10x + 24x^2 + \tfrac{36}{5}x^3)\zeta(2) \\
& - \tfrac{9687}{540} + \tfrac{59157}{540}x - \tfrac{36}{5}x^2 - \frac{4}{5x} + [\tfrac{71}{5}\zeta(2)^2 + \tfrac{140}{3}\zeta(3) - \tfrac{251}{3}\zeta(2) - \tfrac{5465}{72}]\delta(1-x) \Bigg] \\
+ n_f C_F \Bigg( & \frac{1+x^2}{1-x}[\tfrac{2}{3}\ln^2(1-x) - (\tfrac{8}{3}\ln x + \tfrac{29}{9})\ln(1-x) - \tfrac{4}{3}\mathrm{Li}_2(1-x) + \tfrac{5}{3}\ln^2 x + \tfrac{19}{3}\ln x - \tfrac{4}{3}\zeta(2) + \tfrac{247}{54}] \\
& + \tfrac{1}{3}(1+13x)\ln(1-x) - \tfrac{1}{3}(7+19x)\ln x - \tfrac{23}{18} - \tfrac{27}{2}x + [\tfrac{4}{3}\zeta(3) + \tfrac{38}{3}\zeta(2) + \tfrac{457}{36}]\delta(1-x)\Bigg), \qquad (9)
\end{aligned}
$$

where $C_A$, $C_F$ denote the colour factors and $n_f$ stands for the number of flavours. Here we have put $\mu^2 = Q^2$. The more general case ($\mu^2 \neq Q^2$) can be easily derived using renormalization group methods (see ref. [14]). In the above expression the terms of the type $\ln^i(1-x)/(1-x)$ have to be understood in the distributional sense [12]. The latter and the coefficient of the delta function can be derived from eq. (16) in ref. [13]. The second part in (8) is given by



$$
\begin{aligned}
C_2^{(2),G}(x,1) = n_f C_F \Bigg[ & 8(1+x)^2 \\
& \times[-4S_{1,2}(-x) - 4\ln(1+x)\,\mathrm{Li}_2(-x) - 2\zeta(2)\ln(1+x) - 2\ln x\ln^2(1+x) + \ln^2 x\ln(1+x)] \\
& + 4(1-x)^2\{\tfrac{5}{6}\ln^3(1-x) - (2\ln x + \tfrac{13}{4})\ln^2(1-x) + [2\,\mathrm{Li}_2(1-x) + 2\ln^2 x + 4\ln x + \tfrac{7}{2}]\ln(1-x) - \tfrac{5}{12}\ln^3 x \\
& + [\mathrm{Li}_2(1-x) - 4\,\mathrm{Li}_2(-x) + 3\zeta(2)]\ln x - 4\,\mathrm{Li}_3(1-x) - S_{1,2}(1-x) + 12\,\mathrm{Li}_3(-x) + 13\zeta(3) + \tfrac{13}{2}\zeta(2)\} \\
& + x^2\{\tfrac{10}{3}\ln^3(1-x) - 12\ln x\ln^2(1-x) + [16\ln^2 x - 16\zeta(2)]\ln(1-x) - 5\ln^3 x \\
& + [12\,\mathrm{Li}_2(1-x) + 20\zeta(2)]\ln x - 8\,\mathrm{Li}_3(1-x) + 12 S_{1,2}(1-x)\} \\
& + \left(48 + \tfrac{64}{3}x + \tfrac{96}{5}x^3 + \frac{8}{15x^2}\right)[\mathrm{Li}_2(-x) + \ln x\ln(1+x)] + (14x - 23x^2)\ln^2(1-x) \\
& + (-12x + 10x^2)\ln(1-x) + (-24x + 56x^2)\ln x\ln(1-x) + 64x\,\mathrm{Li}_3(-x) + (-10+24x)\mathrm{Li}_2(1-x) \\
& + (-\tfrac{3}{2} + \tfrac{22}{3}x - 36x^2 - \tfrac{48}{5}x^3)\ln^2 x + \frac{1}{15}\left(-236 + 339x - 648x^2 - \frac{8}{x}\right)\ln x + (64x + 36x^2)\zeta(3) \\
& + (-\tfrac{20}{3}x + 46x^2 + \tfrac{96}{5}x^3)\zeta(2) - \tfrac{647}{15} + \tfrac{239}{5}x - \tfrac{36}{5}x^2 + \frac{8}{15x}\Bigg] \\
+ n_f C_A \Bigg\{ & 4(1+x)^2[S_{1,2}(1-x) - 2\,\mathrm{Li}_3(-x) + 4S_{1,2}(-x) - 2\ln x\,\mathrm{Li}_2(1-x) + 4\ln(1+x)\,\mathrm{Li}_2(-x) \\
& + 2\ln x\,\mathrm{Li}_2(-x) + 2\zeta(2)\ln(1+x) + 2\ln x\ln^2(1+x) + \ln^2 x\ln(1+x)] \\
& + 8(1+2x+2x^2)\left[\mathrm{Li}_3\left(\frac{1-x}{1+x}\right) - \mathrm{Li}_3\left(-\frac{1-x}{1+x}\right) - \ln(1-x)\,\mathrm{Li}_2(-x) - \ln x\ln(1-x)\ln(1+x)\right] \\
& + \left(-24 + \tfrac{80}{3}x^2 - \frac{16}{3x}\right)[\mathrm{Li}_2(-x) + \ln x\ln(1+x)] + x^2[-4S_{1,2}(1-x) + 16\,\mathrm{Li}_3(-x) + 8\ln x\,\mathrm{Li}_2(1-x) \\
& + 8\ln^2 x\ln(1+x)] + \tfrac{2}{3}(1 - 2x + 2x^2)\ln^3(1-x) + (24x - 8x^2)\ln x\ln^2(1-x) \\
& + \left(-2 + 36x - \tfrac{122}{3}x^2 + \frac{8}{3x}\right)\ln^2(1-x) + (-4 - 32x + 8x^2)\ln^2 x\ln(1-x) \\
& + (8 - 144x + 148x^2)\ln x\ln(1-x) + (4 + 40x - 8x^2)\ln(1-x)\,\mathrm{Li}_2(1-x) \\
& + (-20 + 24x - 32x^2)\zeta(2)\ln(1-x) + \frac{1}{9}\left(-186 - 1362x + 1570x^2 + \frac{104}{x}\right)\ln(1-x) \\
& + (-4 - 72x + 8x^2)\,\mathrm{Li}_3(1-x) + \frac{1}{3}\left(12 - 192x + 176x^2 + \frac{16}{x}\right)\mathrm{Li}_2(1-x) + \tfrac{1}{3}(10+28x)\ln^3 x \\
& + (-1 + 88x - \tfrac{194}{3}x^2)\ln^2 x + (-48x + 16x^2)\zeta(2)\ln x + (58 + \tfrac{584}{3}x - \tfrac{2090}{9}x^2)\ln x - (10 + 12x + 12x^2)\zeta(3) \\
& + \frac{1}{3}\left(12 - 240x + 268x^2 - \frac{32}{x}\right)\zeta(2) + \tfrac{239}{9} + \tfrac{1072}{9}x - \tfrac{4493}{27}x^2 + \frac{344}{27x}\Bigg\}, \qquad (5)
\end{aligned}
$$

W.L. van Neerven et al.: (1992) 79 functions 80 objects would be maximal.

- The high complexity is partly caused applying the the IBP–Method.
- $x$–space usually is not the best space to work in.

## 3 Loop Anomalous Dimensions & Wilson Coefficients

- $\Longrightarrow$ Harmonic Sums in linear representation.
- Still high complexity of terms.
- Compactification possible applying algebraic and structural relations.
- Observation : In all single scale calculations the same Basic Functions occur in the resp. weight.

- $\Longrightarrow$ Derive these Universal Functions and their complex analysis.

# 2. Algebraic Relations

cf. J.Blümlein, Comput. Phys. Commun. **159** (2004) 19

Number of harmonic sums up to weight w : $3^{w-1}$.

Harmonic sums form a quasi-shuffle algebra through $\sqcup\!\sqcup$. (M.E. Hoffman, J. Algebraic Combin. **11** (2000) 49 )

$$\begin{aligned} S_{a_1,a_2} \sqcup\!\sqcup S_{a_3,a_4} &= S_{a_1,a_2,a_3,a_4} + S_{a_1,a_3,a_2,a_4} + S_{a_1,a_2,a_4,a_2} \\ &\quad + S_{a_3,a_4,a_1,a_2} + S_{a_3,a_1,a_4,a_2} + S_{a_3,a_1,a_2,a_4} \quad etc. \end{aligned}$$

Solve all the linear equations possible for the harmonic sums $\Longrightarrow$ algebraic basis.

Let $\{a, a, a, ..., b, b, ..., ..., z, z\}$ a set of $n_1$ $a$'s, $n_2$ $b$'s etc. The number of basis elements corresponding to all words formed by ALL the above letters is:

$$l_n(n_1, ..., n_q) = \frac{1}{n} \sum_{d|n_i} \mu(d) \frac{(n/d)!}{(n_1/d)! ... (n_d/d)!}, \quad \sum_i n_i = n$$

(E. Witt, 1937) $\Longrightarrow$ # Lyndon words

| w | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| $\#_c$ | 2 | 8 | 26 | 80 | 242 | 728 |
| $\#_r$ | 0 | 1 | 7 | 23 | 69 | 183 |

# Algebraic Relations



## Observation in Quantum Field Theory :

At least up to $O(\alpha_s^3)$ the contributing harmonic sums exhibit never any index $a_k = -1$ applying a compact representation.

The number of sums of this type is

$$N_{\neg\{-1\}}(w) = \frac{1}{2}\left[\left(1-\sqrt{2}\right)^w + \left(1+\sqrt{2}\right)^w\right]$$

$$N^{\text{basic}}_{\neg\{-1\}}(\mathsf{w}) \;=\; \frac{2}{\mathsf{w}} \sum_{d|\mathsf{w}} \mu\left(\frac{\mathsf{w}}{d}\right) N^{\text{basic}}_{\neg\{-1\}}(d)\ .$$

| w | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| $\#_c$ | 1 | 4 | 11 | 28 | 69 | 168 |
| $\#_r$ | 1 | 3 | 7 | 14 | 30 | 60 |

- Here $\#_c$ is smaller than $\#_r$ in the general case.

# Algebraic Relations

Remark:

Harmonic, Generalized Harmonic Polylogarithms and Multiple Polylogarithms also form shuffle algebras. As shuffle algebras are sub-sets of the quasi-shuffle algebra studied above, the respective algebraic relations can be derived directly.

- Form the index alphabet.
- Solve the shuffle-relations $\Longrightarrow$ Basis

As the relations in J.B., Comput. Phys. Commun. **159** (2004) 19 are of arbitrary weight (general alphabet) and depth $d \leq 6$ the corresponding relations can be read off there.

Algorithms to extend this scenario are available.

# 3. Structural Relations

w = 1:

$$\frac{1}{1-x} \qquad \& \qquad \frac{1}{1+x}$$

$$\frac{1}{1-x^2} = \frac{1}{2}\left[\frac{1}{1-x} + \frac{1}{1+x}\right]$$

$$\mathbf{M}\left[\left(\frac{1}{1-x}\right)_+\right]\left(\frac{N}{2}\right) = \mathbf{M}\left[\left(\frac{1}{1-x}\right)_+\right](N) + \mathbf{M}\left[\frac{1}{1+x}\right](N) + \ln(2)$$

$$-\psi\left(\frac{N}{2}\right) - \gamma_E \quad = \quad -\psi(N) - \gamma_E + \beta(N) + \ln(2); \qquad \beta(N) = \frac{1}{2}\left[\psi\left(\frac{N+1}{2}\right) - \psi\left(\frac{N}{2}\right)\right]$$

- $S_{-1}(N)$ depends on $S_1(N)$ for $N \epsilon \mathbf{Q}$

# Structural Relations

$N \epsilon \mathbf{R}$ :

$$S_2(N) = -\frac{d}{dN} S_1(N) + \zeta_2 \qquad \text{(etc.)}$$

For $N \epsilon \mathbf{R}$ : only one independent single sum occurs.

$$S_1(N) = \sum_{k=1}^{N} \frac{1}{k} = \psi(N+1) + \gamma_E$$

Single Harmonic sums $\cup$ $\zeta_{k_1,\ldots k_n}$ are closed under differentiation.

w = 2:

$$\mathbf{M}\left[\frac{\ln(1-x)}{1+x}\right](N) = -\mathbf{M}\left[\frac{\ln(1+x)}{1+x}\right](N) - [\psi(N) + \gamma_E + \ln(2)]\beta(N) + \beta'(N)$$

$$F_1(N) := \mathbf{M}\left[\frac{\ln(1+x)}{1+x}\right](N) \rightarrow S_{1,-1}(N)$$

# Structural Relations

The relations for w = 2 were explored by N. Nielsen (1906).

$$\xi(N) = \mathbf{M}\left[\left(\frac{\ln(1-x)}{1+x}\right)_+\right](N); \qquad \eta(N) = \mathbf{M}\left[\frac{\ln(1+x)-\ln(2)}{1-x}\right](N)$$

$$\xi_1(N) = \mathbf{M}\left[\frac{\ln(1+x)}{1+x}\right](N); \qquad -\xi_2(N) = \mathbf{M}\left[\frac{\ln(1-x)}{1+x}\right](N)$$

$$\begin{aligned}
[\psi(z)+\gamma_E][\psi(1-z)+\gamma_E] &= 2\zeta_2 - \xi(z) - \xi(1-z) \\
\beta(z)[\psi(z)+\gamma_E] &= \beta'(z) + \beta(z)\ln(2) - \xi_1(z) + \xi_2(z) \\
\beta(z)\beta(1-z) &= \eta(z) + \eta(1-z) \\
\beta^2(z) &= \psi'(z) - 2\eta(z)
\end{aligned}$$

# Structural Relations

If half-integer arguments in $N$ are allowed
$\mathbf{M}[\mathrm{Li}_k(-x)/(x \pm 1)](N)$ are not independent functions :

$$\frac{1}{2^{k-2}} \frac{\mathrm{Li}_k(x^2)}{1-x^2} = \frac{\mathrm{Li}_k(x)}{1-x} + \frac{\mathrm{Li}_k(x)}{1+x} + \frac{\mathrm{Li}_k(-x)}{1-x} + \frac{\mathrm{Li}_k(-x)}{1+x} \rightarrow \frac{\mathrm{Li}_k(-x)}{1-x}$$

- There always exists another IBP relation to express also $\mathrm{Li}_k(-x)/(1+x)$

$$\begin{aligned}
(-1)^N \mathbf{M}\left[\frac{\mathrm{Li}_2(-x)}{1+x}\right](N) &= -S_{2,-1}(N) - \ln(2)[S_2(N) - S_{-2}(N)] \\
&\quad -\frac{1}{2}\zeta_2 S_{-1}(N) + \frac{1}{4}\zeta_3 - \frac{1}{2}\zeta_2 \ln(2) \\
(-1)^N \mathbf{M}\left[\frac{-\mathrm{Li}_2(x) - \ln(x)\ln(1-x) + \zeta_2}{1+x}\right](N) &= -S_{-1,2}(N) + \zeta_2 S_{-1}(N) - \zeta_3 + \frac{3}{2}\zeta_2 \ln(2) \\
S_{-1,2}(N) + S_{2,-1}(N) &= S_{-1}(N) S_2(N) + S_{-3}(N)
\end{aligned}$$

# Structural Relations

$$
\begin{aligned}
(-1)^{(N+1)}\mathbf{M}\left[\frac{\mathrm{Li}_3(-x)}{1+x}\right](N) &= -S_{3,-1}(N) - \ln(2)[S_3(N) - S_{-3}(N)] \\
&\quad -\frac{1}{2}\zeta_2 S_{-2}(N) + \frac{3}{4}\zeta_3 S_{-1}(N) - \frac{1}{8}\zeta_2^2 + \frac{3}{4}\ln(2)\zeta_3 \\
(-1)^{N}\mathbf{M}\left[\frac{S_{1,2}(1-x)}{1+x}\right](N) &= -S_{-1,3}(N) + \zeta_3 S_{-1}(N) - \frac{19}{40}\zeta_2^2 + \frac{7}{4}\zeta_3\ln(2) \\
S_{1,2}(1-x) &= -\mathrm{Li}_3(x) + \log(x)\mathrm{Li}_2(x) + \frac{1}{2}\log(1-x)\log^2(x) + \zeta_3 \\
S_{-1,3}(N) + S_{3,-1}(N) &= S_{-1}(N)S_3(N) + S_{-4}(N)
\end{aligned}
$$

- At even w there exists an algebraic relation

$$
S_{\mathrm{w}/2,\mathrm{w}/2}(N) = \frac{1}{2}\left[S^2_{\mathrm{w}/2}(N) + S_{\mathrm{w}}(N)\right]
$$

which yields an additional relation for $\mathrm{Li}_k(x)/(1+x)$.

w = 3:

$$
\rightarrow \frac{\mathrm{Li}_2(x)}{x\pm 1}, \qquad \frac{\ln^2(1+x)}{x\pm 1}
$$

# Double Sums in General

- Applying differential operators one may show :

For $N \,\epsilon\, \mathbf{R}$ double harmonic sums can always be represented
by one basic function for even weight and two basic functions for odd weight.

$$\Longrightarrow \quad \frac{\mathrm{Li}_k(x)}{1+x}, \qquad \frac{\mathrm{Li}_k(x)}{1 \pm x}$$

Examples, which reduce :

$$S_{2,3}(N) = \mathbf{M}\left[\left(\frac{\ln(x)\,[S_{1,2}(1-x)-\zeta_3]+3\,[S_{1,3}(1-x)-\zeta_4]}{x-1}\right)_+\right](N)+3\zeta_4 S_1(N)$$

$$S_{-4,-2}(N) = -\mathbf{M}\left[\left(\frac{4\mathrm{Li}_5(-x)-\ln(x)\mathrm{Li}_4(-x)}{x-1}\right)_+\right](N)$$
$$+\frac{1}{2}\zeta_2\,[S_4(N)-S_{-4}(N)]-\frac{3}{2}\zeta_3 S_3(N)+\frac{21}{8}\zeta_4 S_2(N)-\frac{15}{4}\zeta_5 S_1(N)$$

$$S_{1,3}(1-x) = -\mathrm{Li}_4(x)+\log(x)\mathrm{Li}_3(x)-\frac{1}{2}\log^2(x)\mathrm{Li}_2(x)-\frac{1}{6}\log^3(x)\log(1-x)+\zeta_4$$

# Structural Relations

w $= 4; i \neq -1$ :

$$\frac{\mathrm{Li}_3(x)}{x+1}, \qquad \frac{S_{1,2}(x)}{x \pm 1}$$

The Mellin transform of

$$\left(\frac{\mathrm{Li}_3(x)}{x-1}\right)_+$$

reads

$$\mathbf{M}\left[\left(\frac{\mathrm{Li}_3(x)}{x-1}\right)_+\right](N) = \frac{1}{2}\left\{\frac{d}{dN}\mathbf{M}\left[\left(\frac{\mathrm{Li}_2(x)+\zeta_2}{x-1}\right)_+\right](N) - S_{2,2}(N-1) + \zeta_2 S_2(N-1) + 2\zeta_3 S_1(N-1)\right\}$$

and can be traced back to that of $(\mathrm{Li}_2(x)/(x-1))_+$

# Structural Relations



$\mathrm{w} = 5;\ i \neq -1\ :$

$$\frac{\mathrm{Li}_4(x)}{x \pm 1} \qquad \frac{S_{1,3}(x)}{x+1} \qquad \frac{S_{2,2}(x)}{x \pm 1} \qquad \frac{\mathrm{Li}_2^2(x)}{x+1} \qquad \frac{\ln(x) S_{1,2}(-x) - \mathrm{Li}_2^2(-x)/2}{x \pm 1}$$

$$\frac{\mathrm{Li}_2^2(x)}{x-1} \qquad \frac{S_{1,3}(x)}{x-1} \qquad [\text{Occur in the 3} - \text{loop Wils. Coeff. only}]$$

$\mathrm{w} = 6;\ i \neq -1\ :$

$$\frac{\mathrm{Li}_5(x)}{x+1} \qquad \frac{S_{3,2}(x)}{x \pm 1} \qquad \frac{S_{2,3}(x)}{x \pm 1} \qquad \frac{S_{1,4}(x)}{x \pm 1} \qquad \frac{\mathrm{Li}_2(x)\mathrm{Li}_3(x)}{x \pm 1}$$

$$\frac{S_{1,2}(x)\mathrm{Li}_2(x)}{x+1} \qquad \frac{A_1(x)}{x+1} \qquad \frac{A_2(x)}{x \pm 1} \qquad \frac{A_3(x)}{x+1} \qquad \frac{H_{0,-1,0,1,1}(x)}{x-1}$$

$$\frac{A_1(-x) + N_\alpha(x)}{x+1}\Big|_{\alpha=1..3}$$

New numerator functions :

$$A_1(x) = \int_0^x \frac{dy}{y} \mathrm{Li}_2^2(y), \qquad A_2(x) = \int_0^x \frac{dy}{y} \ln(1-y) S_{1,2}(y), \qquad A_3(x) = \int_0^x \frac{dy}{y} [\mathrm{Li}_4(1-y) - \zeta_4]$$

# Representation of Observables

- Unpolarized and Polarized Drell-Yan an Higgs-Boson Production Cross Section $O(\alpha_s^2)$ , w $= 4$ JB and V. Ravindran, Nucl. Phys. **B716** (2005) 128.
- Unpolarized and Polarized Time-like Anomalous Dimensions and Wilson Coefficients $O(\alpha_s^2)$, w $= 4$ JB and V. Ravindran, Nucl. Phys. **B749** (2006) 1.
- Anomalous Dimensions and Wilson Coefficients $O(\alpha_s^3)$ , w $= 5, 6,$

  from: S. Moch, J. Vermaseren, A. Vogt, Nucl. Phys. **B688** (2004) 101; **691** (2004) 129; **B724** (2005) 3 → J.B., DESY 07-042
- Polarized and Unpolarized Wilson Coefficients $O(\alpha_s^2)$ , w $= 4$ J.B. and S. Moch
- Polarized and Unpolarized asymptotic Heavy Flavor Wilson Coefficients $O(\alpha_s^{2(3)})$ , w $=$ 4,5,

  J.B., A. de Freitas, W. van Neerven, S. Klein, Nucl. Phys. **B755** (2006) 272; I. Bierenbaum, J.B., S. Klein, DESY 07-026,

  J.B. and S. Klein, DESY 07-027
- Virtual and soft corrections to Bhabha Scattering $O(\alpha^2)$ , w $= 4$,

  J.B. and S. Klein, arXiv:0706.2426 [hep-ph]

# 5. Factorial Series



Consider

$$\Omega(z) = \int_0^1 dt\ t^{z-1}\ \varphi(t); \qquad \varphi(1-t) = \sum_{k=0}^{\infty} a_k t^k$$

$$Re(z) > 0, \qquad \Omega(z) = \sum_{k=0}^{\infty} \frac{a_{k+1} k!}{z(z+1)\ldots(z+k)}$$

- $\Omega(z)$ is meromorphic in $z \,\epsilon\, \mathbf{C}$, obeys a recursion $z \to z+1$ and has an analytic asymptotic representation.
- The poles are situated at the non-potitive integers.

Examples:

$$\begin{aligned}
F_5(z) &= \mathbf{M}\left[\frac{\mathrm{Li}_2(z)}{1+z}\right](z) \\
F_5(z+1) &= -F_5(z) + \frac{1}{z}\left[\zeta_2 - \frac{\psi(z+1)+\gamma_E}{z}\right] \\
\text{Asymp. ser. : } \mathrm{Li}_2(z) &\to -\mathrm{Li}_2(1-z) - \ln(z)\ln(1-z) + \zeta_2 \\
\mathbf{M}\left[\frac{\mathrm{Li}_2(1-z)}{1+z}\right](N) &\sim \frac{1}{2N^2} + \frac{1}{4N^3} - \frac{7}{24}\frac{1}{N^4} - \frac{1}{3}\frac{1}{N^5} + \frac{73}{120}\frac{1}{N^6} \ldots
\end{aligned}$$

# Factorial Series

$$F_{13}(z) = \mathbf{M}\left[\left(\frac{\mathrm{Li}_2^2(z)}{z-1}\right)_+\right](z)$$

$$F_{13}(z+1) = -F_{13}(z) + \frac{\zeta_2^2}{z} + \frac{4\zeta_3}{z^2} + \frac{2\zeta_2}{z^2}S_1(z) + \frac{2S_{2,1}(z)}{z^2} + \frac{2}{z^3}\left[S_1^2(z) + S_2(z)\right]$$

$$\text{Asymp. ser. : } \mathrm{Li}_2^2(z) \rightarrow \mathrm{Li}_2^2(1-z) + \ln^2(z)\ln^2(1-z) + \zeta_2^2 + 2\mathrm{Li}_2(1-z)\ln(z)\ln(1-z) + \dots$$

$$\mathbf{M}\left[\left(\frac{\mathrm{Li}_2^2(1-z)}{z-1}\right)_+\right](N) \sim \frac{1}{z^2} - \frac{7}{24}\frac{1}{z^4} + \frac{1}{12}\frac{1}{z^5} + \frac{223}{1080}\frac{1}{z^6} - \frac{7}{45}\frac{1}{z^7} - \frac{3767}{15120}\frac{1}{z^8} + \frac{38}{105}\frac{1}{z^9}$$
$$+\frac{14327}{31500}\frac{1}{z^{10}} - \frac{198}{175}\frac{1}{z^{11}} - \frac{138673}{118800}\frac{1}{z^{12}} + \frac{3263}{693}\frac{1}{z^{13}} + \frac{5265804043}{1324323000}\frac{1}{z^{14}}$$
$$-\frac{13399637}{525525}\frac{1}{z^{15}} - \frac{143341487}{8408400}\frac{1}{z^{16}} + \frac{25092}{143}\frac{1}{z^{17}} + \frac{34809672614}{402026625}\frac{1}{z^{18}}$$
$$-\frac{5749693892}{3828825}\frac{1}{z^{19}} + O\left(\frac{1}{z^{20}}\right)$$

# 6. The Basis

| | | | |
|---|---|---|---|
| $w=1$ | $1/(x-1)_+$ | | |
| $w=2$ | $\ln(1+x)/(x+1)$ | | |
| $w=3$ | $\mathrm{Li}_2(x)/(x\pm1)$ | | |
| $w=4$ | $\mathrm{Li}_3(x)/(x+1)$ | $S_{1,2}(x)/(x\pm1)$ | |
| $w=5$ | $\mathrm{Li}_4(x)/(x\pm1)$ | $S_{1,3}(x)/(x\pm1)$ | $S_{2,2}(x)/(x\pm1)$ |
| | $\mathrm{Li}_2^2(x)/(x\pm1)$ | $[\ln(x)S_{1,2}(-x)-\mathrm{Li}_2^2(-x)/2]/(x\pm1)$ | |
| $w=6$ | $\mathrm{Li}_5(x)/(x+1)$ | $S_{1,4}(x)/(x\pm1)$ | $S_{2,3}(x)/(x\pm1)$ |
| | $S_{3,2}(x)/(x\pm1)$ | $\mathrm{Li}_2(x)\mathrm{Li}_3(x)/(x\pm1)$ | $S_{1,2}(x)\mathrm{Li}_2(x)/(x+1)$ |
| | $A_1(x)/(x+1)$ | $A_2(x)/(x\pm1)$ | $A_3(x)/(x+1)$ |
| | $H_{0,-1,0,1,1}(x)/(x-1)$ | $[A_1(-x)+N_\alpha(x)]/(x+1)\vert_{\alpha=1..3}$ | |

- $O(\alpha)$ Wilson Coefficients/anom. dim. #1
- $O(\alpha^2)$ Anomalous Dimensions #2
- $O(\alpha^2)$ Wilson Coefficients $\# \leq 5$
- $O(\alpha^3)$ Anomalous Dimensions #15
- $O(\alpha^3)$ Wilson Coefficients #35

# 7. Conclusions

- The single-scale quantities in Quantum Field Theories to 3 Loop Order $\Leftrightarrow$ w $=$ 6 can be represented in a polynomial ring spanned by a few Mellin transforms of the above basic functions, which are the same for all known processes. This points to their general nature.

- The basic Mellin transforms are meromorphic functions with single poles at the non-positive integers.

- The total amount of harmonic sums reduces due to algebraic relations [index structure], and structural relations $N \epsilon \mathbf{Q}$, $N \epsilon \mathbf{R}$.

- They can be represented in terms of factorial series up to simple "soft components". This allows an exact analytic continuation.

- Up to w $=$ 6 physical (pseudo-) observables are free of harmonic sums with index $= \{$-1$\}$. Up to w $=$ 5 all numerator functions are Nielsen integrals.