New 3-loop Wilson coefficients for deep-Inelastic heavy flavor production

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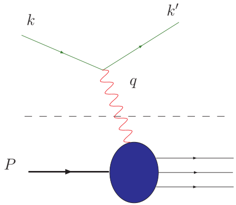
NTU Singapore, 14.02.2014

Contents

Introduction

Unpolarized Deep–Inelastic Scattering (DIS):



$\longrightarrow \quad L_{\mu\nu}$

$\longrightarrow \quad W_{\mu\nu}$

$$Q^2 := -q^2, \quad x := \frac{Q^2}{2P.q} \quad \text{Bjorken–x}$$

$$\frac{d\sigma}{dQ^2\,dx} \sim W_{\mu\nu} L^{\mu\nu}$$

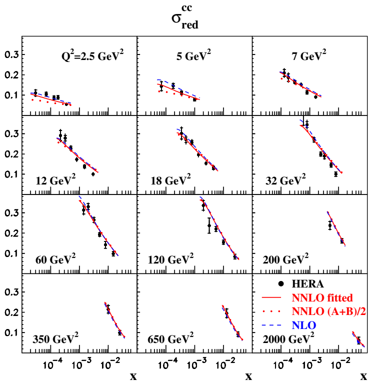
$$W_{\mu\nu}(q,P,s) = \frac{1}{4\pi}\int d^4\xi \exp(iq\xi)\langle P,s \mid [J_\mu^{em}(\xi), J_\nu^{em}(0)] \mid P,s\rangle =$$

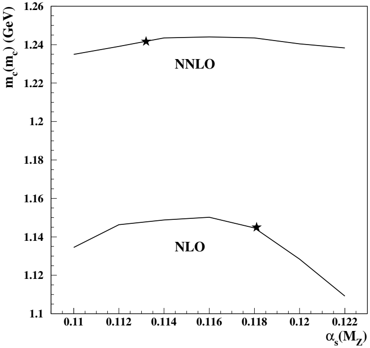
$$\frac{1}{2x}\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) F_L(x,Q^2) + \frac{2x}{Q^2}\left(P_\mu P_\nu + \frac{q_\mu P_\nu + q_\nu P_\mu}{2x} - \frac{Q^2}{4x^2} g_{\mu\nu}\right) F_2(x,Q^2) \; .$$

Structure Functions: $F_{2,L}$

contain light and heavy quark contributions.

Deep–Inelastic Scattering (DIS):





NNLO:

S. Alekhin, J. Blümlein, K. Daum, K. Lipka, Phys.Lett. B720 (2013) 172 [1212.2355]

$$m_c(m_c) = 1.24 \pm 0.03(exp)\ {}^{+0.03}_{-0.02}\ (scale)\ {}^{+0.00}_{-0.07}\ (thy),$$

$$\alpha_s(M_Z^2) = 0.1132 \pm 0.011$$

Yet approximate NNLO treatment [Kawamura et al. [1205.5227]].

$\alpha_s(M_Z^2)$ from NNLO DIS(+) analyses [from ABM13 and update]

	$\alpha_s(M_Z^2)$	
BBG	$0.1134\ ^{+0.0019}_{-0.0021}$	valence analysis, NNLO
GRS	0.112	valence analysis, NNLO
ABKM	0.1135 ± 0.0014	HQ: FFNS $N_f = 3$
JR	0.1128 ± 0.0010	dynamical approach
JR	0.1140 ± 0.0006	including NLO-jets
MSTW	0.1171 ± 0.0014	(2009)
Thorne	0.1136	[DIS+DY, HT*] (2014)
ABM11$_J$	$0.1134 - 0.1149 \pm 0.0012$	Tevatron jets (NLO) incl.
ABM13	0.1133 ± 0.0011	
ABM13	0.1132 ± 0.0011	(without jets)
CTEQ	0.1159..0.1162	
CTEQ	0.1140	(without jets)
NN21	$0.1174 \pm 0.0006 \pm 0.0001$	
Gehrmann et al.	$0.1131\ ^{+\ 0.0028}_{-\ 0.0022}$	e^+e^- thrust
Abbate et al.	0.1140 ± 0.0015	e^+e^- thrust
ATLAS/CMS	0.1151 ± 0.0033	$t\bar{t}$
BBG	$0.1141\ ^{+0.0020}_{-0.0022}$	valence analysis, $\mathrm{N^3LO}$

$$\Delta_{\rm TH}\alpha_s = \alpha_s({\rm N^3LO}) - \alpha_s({\rm NNLO}) + \Delta_{\rm HQ} = +0.0009 \pm 0.0006_{\rm HQ}$$

NNLO accuracy is needed to analyze the world data. $\Longrightarrow$ NNLO HQ corrections needed.

Goals

- Complete the NNLO heavy flavor Wilson coefficients for twist-2 in the dynamical safe region $Q^2 > 20 GeV^2$ (no higher twist) for $F_2(x, Q^2)$
- Measure m_c and α_s as precisely as possible
- Provide precise CC heavy flavor corrections
- Consequences for LHC:
  - NNLO VFNS will be provided
  - better constraint on sea quarks and the gluon
  - precise m_c and α_s on input

Factorization of the Structure Functions

At leading twist the structure functions factorize in terms of a Mellin convolution

$$F_{(2,L)}(x, Q^2) = \sum_j \underbrace{\mathbb{C}_{j,(2,L)}\left(x, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2}\right)}_{perturbative} \otimes \underbrace{f_j(x, \mu^2)}_{nonpert.}$$

into (pert.) Wilson coefficients and (nonpert.) parton distribution functions (PDFs).

$\otimes$ denotes the Mellin convolution

$$f(x) \otimes g(x) \equiv \int_0^1 dy \int_0^1 dz \, \delta(x - yz) f(y) g(z) \; .$$

The subsequent calculations are performed in Mellin space, where $\otimes$ reduces to a multiplication, due to the Mellin transformation

$$\hat{f}(N) = \int_0^1 dx \, x^{N-1} f(x) \; .$$

Wilson coefficients:

$$\mathbb{C}_{j,(2,L)}\left(N,\frac{Q^2}{\mu^2},\frac{m^2}{\mu^2}\right)=C_{j,(2,L)}\left(N,\frac{Q^2}{\mu^2}\right)+H_{j,(2,L)}\left(N,\frac{Q^2}{\mu^2},\frac{m^2}{\mu^2}\right)\ .$$

At $Q^2 \gg m^2$ the heavy flavor part

$$H_{j,(2,L)}\left(N,\frac{Q^2}{\mu^2},\frac{m^2}{\mu^2}\right)=\sum_i C_{i,(2,L)}\left(N,\frac{Q^2}{\mu^2}\right)A_{ij}\left(\frac{m^2}{\mu^2},N\right)$$

[Buza, Matiounine, Smith, van Neerven 1996 Nucl.Phys.B]

factorizes into the light flavor Wilson coefficients C and the massive operator matrix elements (OMEs) of local operators O_i between partonic states j

$$A_{ij}\left(\frac{m^2}{\mu^2},N\right)=\langle j \mid O_i \mid j\rangle\ .$$

$\rightarrow$ additional Feynman rules with local operator insertions for partonic matrix elements.

The unpolarized light flavor Wilson coefficients are known up to NNLO [Moch, Vermaseren, Vogt, 2005 Nucl.Phys.]].

For $F_2(x,Q^2)$: at $Q^2 \gtrsim 10m^2$ the asymptotic representation holds at the 1% level.

The Wilson Coefficients at large Q^2

$$
\begin{aligned}
L_{q,(2,L)}^{\mathsf{NS}}(N_F+1) &= a_s^2\Big[A_{qq,Q}^{(2),\mathsf{NS}}(N_F+1)\ \delta_2 + \hat{C}_{q,(2,L)}^{(2),\mathsf{NS}}(N_F)\Big] \\
&+ a_s^3\Big[A_{qq,Q}^{(3),\mathsf{NS}}(N_F+1)\ \delta_2 + A_{qq,Q}^{(2),\mathsf{NS}}(N_F+1) C_{q,(2,L)}^{(1),\mathsf{NS}}(N_F+1) + \hat{C}_{q,(2,L)}^{(3),\mathsf{NS}}(N_F)\Big] \\
L_{q,(2,L)}^{\mathsf{PS}}(N_F+1) &= a_s^3\Big[\ A_{qq,Q}^{(3),\mathsf{PS}}(N_F+1)\ \delta_2 + A_{gq,Q}^{(2)}(N_F)\ \ N_F\tilde{C}_{g,(2,L)}^{(1)}(N_F+1) + N_F\hat{\tilde{C}}_{q,(2,L)}^{(3),\mathsf{PS}}(N_F)\Big] \\
L_{g,(2,L)}^{\mathsf{S}}(N_F+1) &= a_s^2 A_{gg,Q}^{(1)}(N_F+1) N_F\tilde{C}_{g,(2,L)}^{(1)}(N_F+1) + a_s^3\Big[\ A_{qg,Q}^{(3)}(N_F+1)\ \delta_2 \\
&+A_{gg,Q}^{(1)}(N_F+1)\ \ N_F\tilde{C}_{g,(2,L)}^{(2)}(N_F+1) + A_{gg,Q}^{(2)}(N_F+1)\ \ N_F\tilde{C}_{g,(2,L)}^{(1)}(N_F+1) \\
&+\ A_{Qg}^{(1)}(N_F+1)\ \ N_F\tilde{C}_{q,(2,L)}^{(2),\mathsf{PS}}(N_F+1) + N_F\hat{\tilde{C}}_{g,(2,L)}^{(3)}(N_F)\Big]\ , \\
H_{q,(2,L)}^{\mathsf{PS}}(N_F+1) &= a_s^2\Big[\ A_{Qq}^{(2),\mathsf{PS}}(N_F+1)\ \delta_2 +\ \tilde{C}_{q,(2,L)}^{(2),\mathsf{PS}}(N_F+1)\Big] + a_s^3\Big[\ A_{Qq}^{(3),\mathsf{PS}}(N_F+1)\ \delta_2 \\
&+\ \tilde{C}_{q,(2,L)}^{(3),\mathsf{PS}}(N_F+1) + A_{gq,Q}^{(2)}(N_F+1)\ \tilde{C}_{g,(2,L)}^{(1)}(N_F+1) \\
&+A_{Qq}^{(2),\mathsf{PS}}(N_F+1)\ C_{q,(2,L)}^{(1),\mathsf{NS}}(N_F+1)\Big]\ , \\
H_{g,(2,L)}^{\mathsf{S}}(N_F+1) &= a_s\Big[\ A_{Qg}^{(1)}(N_F+1)\ \delta_2 +\ \tilde{C}_{g,(2,L)}^{(1)}(N_F+1)\Big] + a_s^2\Big[\ A_{Qg}^{(2)}(N_F+1)\ \delta_2 \\
&+\ A_{Qg}^{(1)}(N_F+1)\ C_{q,(2,L)}^{(1),\mathsf{NS}}(N_F+1) +\ A_{gg,Q}^{(1)}(N_F+1)\ \tilde{C}_{g,(2,L)}^{(1)}(N_F+1) \\
&+\ \tilde{C}_{g,(2,L)}^{(2)}(N_F+1)\Big] +\ a_s^3\Big[\ A_{Qg}^{(3)}(N_F+1)\ \delta_2 +\ A_{Qg}^{(2)}(N_F+1)\ C_{q,(2,L)}^{(1),\mathsf{NS}}(N_F+1) \\
&+\ A_{gg,Q}^{(2)}(N_F+1)\ \tilde{C}_{g,(2,L)}^{(1)}(N_F+1) +\ A_{Qg}^{(1)}(N_F+1)\Big\{C_{q,(2,L)}^{(2),\mathsf{NS}}(N_F+1) \\
&+\ \tilde{C}_{q,(2,L)}^{(2),\mathsf{PS}}(N_F+1)\Big\} +\ A_{gg,Q}^{(1)}(N_F+1)\ \tilde{C}_{g,(2,L)}^{(2)}(N_F+1) +\ \tilde{C}_{g,(2,L)}^{(3)}(N_F+1)\Big]
\end{aligned}
$$

J. Ablinger, J. Blümlein, S. Klein, C. Schneider and F. Wißbrock, Nucl. Phys. B **844** (2011) 26;

The Wilson Coefficients at large Q^2

$$
\begin{aligned}
L_{q,(2,L)}^{\mathsf{NS}}(N_F+1) &= a_s^2\Big[A_{qq,Q}^{(2),\mathsf{NS}}(N_F+1)\ \delta_2+\hat{C}_{q,(2,L)}^{(2),\mathsf{NS}}(N_F)\Big] \\
&+ a_s^3\Big[A_{qq,Q}^{(3),\mathsf{NS}}(N_F+1)\ \delta_2+A_{qq,Q}^{(2),\mathsf{NS}}(N_F+1)C_{q,(2,L)}^{(1),\mathsf{NS}}(N_F+1)+\hat{C}_{q,(2,L)}^{(3),\mathsf{NS}}(N_F)\Big] \\
L_{q,(2,L)}^{\mathsf{PS}}(N_F+1) &= a_s^3\Big[\ A_{qq,Q}^{(3),\mathsf{PS}}(N_F+1)\ \delta_2+A_{gq,Q}^{(2)}(N_F)\ \ N_F\tilde{C}_{g,(2,L)}^{(1)}(N_F+1)+N_F\hat{\tilde{C}}_{q,(2,L)}^{(3),\mathsf{PS}}(N_F)\Big] \\
L_{g,(2,L)}^{\mathsf{S}}(N_F+1) &= a_s^2A_{gg,Q}^{(1)}(N_F+1)N_F\tilde{C}_{g,(2,L)}^{(1)}(N_F+1)+a_s^3\Big[\ A_{qg,Q}^{(3)}(N_F+1)\ \delta_2 \\
&+A_{gg,Q}^{(1)}(N_F+1)\ \ N_F\tilde{C}_{g,(2,L)}^{(2)}(N_F+1)+A_{gg,Q}^{(2)}(N_F+1)\ \ N_F\tilde{C}_{g,(2,L)}^{(1)}(N_F+1) \\
&+\ A_{Qg}^{(1)}(N_F+1)\ \ N_F\tilde{C}_{q,(2,L)}^{(2),\mathsf{PS}}(N_F+1)+N_F\hat{\tilde{C}}_{g,(2,L)}^{(3)}(N_F)\Big]\ , \\
H_{q,(2,L)}^{\mathsf{PS}}(N_F+1) &= a_s^2\Big[\ A_{Qq}^{(2),\mathsf{PS}}(N_F+1)\ \delta_2+\ \tilde{C}_{q,(2,L)}^{(2),\mathsf{PS}}(N_F+1)\Big]+a_s^3\Big[\ A_{Qq}^{(3),\mathsf{PS}}(N_F+1)\ \delta_2 \\
&+\tilde{C}_{q,(2,L)}^{(3),\mathsf{PS}}(N_F+1)+A_{gq,Q}^{(2)}(N_F+1)\ \tilde{C}_{g,(2,L)}^{(1)}(N_F+1) \\
&+A_{Qq}^{(2),\mathsf{PS}}(N_F+1)\ C_{q,(2,L)}^{(1),\mathsf{NS}}(N_F+1)\Big]\ , \\
H_{g,(2,L)}^{\mathsf{S}}(N_F+1) &= a_s\Big[\ A_{Qg}^{(1)}(N_F+1)\ \delta_2+\ \tilde{C}_{g,(2,L)}^{(1)}(N_F+1)\Big]+a_s^2\Big[\ A_{Qg}^{(2)}(N_F+1)\ \delta_2 \\
&+\ A_{Qg}^{(1)}(N_F+1)\ C_{q,(2,L)}^{(1),\mathsf{NS}}(N_F+1)+\ A_{gg,Q}^{(1)}(N_F+1)\ \tilde{C}_{g,(2,L)}^{(1)}(N_F+1) \\
&+\tilde{C}_{g,(2,L)}^{(2)}(N_F+1)\Big]+\ a_s^3\Big[\ A_{Qg}^{(3)}(N_F+1)\ \delta_2+\ A_{Qg}^{(2)}(N_F+1)\ C_{q,(2,L)}^{(1),\mathsf{NS}}(N_F+1) \\
&+\ A_{gg,Q}^{(2)}(N_F+1)\ \tilde{C}_{g,(2,L)}^{(1)}(N_F+1)+\ A_{Qg}^{(1)}(N_F+1)\Big\{C_{q,(2,L)}^{(2),\mathsf{NS}}(N_F+1) \\
&+\tilde{C}_{q,(2,L)}^{(2),\mathsf{PS}}(N_F+1)\Big\}+\ A_{gg,Q}^{(1)}(N_F+1)\ \tilde{C}_{g,(2,L)}^{(2)}(N_F+1)+\ \tilde{C}_{g,(2,L)}^{(3)}(N_F+1)\Big]
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&+ a_s^3\Big[A_{qq,Q}^{(3),\rm NS}(N_F+1)\,\delta_2+A_{qq,Q}^{(2),\rm NS}(N_F+1)C_{q,(2,L)}^{(1),\rm NS}(N_F+1)+\hat{C}_{q,(2,L)}^{(3),\rm NS}(N_F)\Big]\\
L_{q,(2,L)}^{\rm PS}(N_F+1) &= a_s^3\Big[\;A_{qq,Q}^{(3),\rm PS}(N_F+1)\,\delta_2+A_{gq,Q}^{(2)}(N_F)\;\;N_F\tilde{C}_{g,(2,L)}^{(1)}(N_F+1)+N_F\hat{\tilde{C}}_{q,(2,L)}^{(3),\rm PS}(N_F)\Big]\\
L_{g,(2,L)}^{\rm S}(N_F+1) &= a_s^2A_{gg,Q}^{(1)}(N_F+1)N_F\tilde{C}_{g,(2,L)}^{(1)}(N_F+1)+a_s^3\Big[\;A_{qg,Q}^{(3)}(N_F+1)\,\delta_2\\
&+A_{gg,Q}^{(1)}(N_F+1)\;\;N_F\tilde{C}_{g,(2,L)}^{(2)}(N_F+1)+A_{gg,Q}^{(2)}(N_F+1)\;\;N_F\tilde{C}_{g,(2,L)}^{(1)}(N_F+1)\\
&+A_{Qg}^{(1)}(N_F+1)\;\;N_F\tilde{C}_{q,(2,L)}^{(2),\rm PS}(N_F+1)+N_F\hat{\tilde{C}}_{g,(2,L)}^{(3)}(N_F)\Big]\,,\\
H_{q,(2,L)}^{\rm PS}(N_F+1) &= a_s^2\Big[\;A_{Qq}^{(2),\rm PS}(N_F+1)\,\delta_2+\;\tilde{C}_{q,(2,L)}^{(2),\rm PS}(N_F+1)\Big]+a_s^3\Big[\;A_{Qq}^{(3),\rm PS}(N_F+1)\,\delta_2\\
&+\tilde{C}_{q,(2,L)}^{(3),\rm PS}(N_F+1)+A_{gq,Q}^{(2)}(N_F+1)\;\tilde{C}_{g,(2,L)}^{(1)}(N_F+1)\\
&+A_{Qq}^{(2),\rm PS}(N_F+1)\;C_{q,(2,L)}^{(1),\rm NS}(N_F+1)\Big]\,,\\
H_{g,(2,L)}^{\rm S}(N_F+1) &= a_s\Big[\;A_{Qg}^{(1)}(N_F+1)\,\delta_2+\;\tilde{C}_{g,(2,L)}^{(1)}(N_F+1)\Big]+a_s^2\Big[\;A_{Qg}^{(2)}(N_F+1)\,\delta_2\\
&+A_{Qg}^{(1)}(N_F+1)\;C_{q,(2,L)}^{(1),\rm NS}(N_F+1)+\;A_{gg,Q}^{(1)}(N_F+1)\;\tilde{C}_{g,(2,L)}^{(1)}(N_F+1)\\
&+\tilde{C}_{g,(2,L)}^{(2)}(N_F+1)\Big]+\;a_s^3\Big[\;A_{Qg}^{(3)}(N_F+1)\,\delta_2+\;A_{Qg}^{(2)}(N_F+1)\;C_{q,(2,L)}^{(1),\rm NS}(N_F+1)\\
&+A_{gg,Q}^{(2)}(N_F+1)\;\tilde{C}_{g,(2,L)}^{(1)}(N_F+1)+\;A_{Qg}^{(1)}(N_F+1)\Big\{C_{q,(2,L)}^{(2),\rm NS}(N_F+1)\\
&+\tilde{C}_{q,(2,L)}^{(2),\rm PS}(N_F+1)\Big\}+\;A_{gg,Q}^{(1)}(N_F+1)\;\tilde{C}_{g,(2,L)}^{(2)}(N_F+1)+\;\tilde{C}_{g,(2,L)}^{(3)}(N_F+1)\Big]
\end{aligned}
$$

J. Ablinger, J. Blümlein, S. Klein, C. Schneider and F. Wißbrock, Nucl. Phys. B **844** (2011) 26; J. Ablinger et al., 2013

The Wilson Coefficients at large Q^2

$$
\begin{aligned}
L^{\mathsf{NS}}_{q,(2,L)}(N_F+1) &= a_s^2\Big[A^{(2),\mathsf{NS}}_{qq,Q}(N_F+1)\;\delta_2+\hat{C}^{(2),\mathsf{NS}}_{q,(2,L)}(N_F)\Big] \\
&+ a_s^3\Big[A^{(3),\mathsf{NS}}_{qq,Q}(N_F+1)\;\delta_2+A^{(2),\mathsf{NS}}_{qq,Q}(N_F+1)C^{(1),\mathsf{NS}}_{q,(2,L)}(N_F+1)+\hat{C}^{(3),\mathsf{NS}}_{q,(2,L)}(N_F)\Big] \\
L^{\mathsf{PS}}_{q,(2,L)}(N_F+1) &= a_s^3\Big[\;A^{(3),\mathsf{PS}}_{qq,Q}(N_F+1)\;\delta_2+A^{(2)}_{gq,Q}(N_F)\;\;N_F\tilde{C}^{(1)}_{g,(2,L)}(N_F+1)+N_F\hat{\tilde{C}}^{(3),\mathsf{PS}}_{q,(2,L)}(N_F)\Big] \\
L^{\mathsf{S}}_{g,(2,L)}(N_F+1) &= a_s^2A^{(1)}_{gg,Q}(N_F+1)N_F\tilde{C}^{(1)}_{g,(2,L)}(N_F+1)+a_s^3\Big[\;A^{(3)}_{qg,Q}(N_F+1)\;\delta_2 \\
&+A^{(1)}_{gg,Q}(N_F+1)\;\;N_F\tilde{C}^{(2)}_{g,(2,L)}(N_F+1)+A^{(2)}_{gg,Q}(N_F+1)\;\;N_F\tilde{C}^{(1)}_{g,(2,L)}(N_F+1) \\
&+\;A^{(1)}_{Qg}(N_F+1)\;\;N_F\tilde{C}^{(2),\mathsf{PS}}_{q,(2,L)}(N_F+1)+N_F\hat{\tilde{C}}^{(3)}_{g,(2,L)}(N_F)\Big]\;, \\
H^{\mathsf{PS}}_{q,(2,L)}(N_F+1) &= a_s^2\Big[\;A^{(2),\mathsf{PS}}_{Qq}(N_F+1)\;\delta_2+\;\tilde{C}^{(2),\mathsf{PS}}_{q,(2,L)}(N_F+1)\Big]+a_s^3\Big[\;A^{(3),\mathsf{PS}}_{Qq}(N_F+1)\;\delta_2 \\
&+\;\tilde{C}^{(3),\mathsf{PS}}_{q,(2,L)}(N_F+1)+A^{(2)}_{gq,Q}(N_F+1)\;\tilde{C}^{(1)}_{g,(2,L)}(N_F+1) \\
&+A^{(2),\mathsf{PS}}_{Qq}(N_F+1)\;C^{(1),\mathsf{NS}}_{q,(2,L)}(N_F+1)\Big]\;, \\
H^{\mathsf{S}}_{g,(2,L)}(N_F+1) &= a_s\Big[\;A^{(1)}_{Qg}(N_F+1)\;\delta_2+\;\tilde{C}^{(1)}_{g,(2,L)}(N_F+1)\Big]+a_s^2\Big[\;A^{(2)}_{Qg}(N_F+1)\;\delta_2 \\
&+\;A^{(1)}_{Qg}(N_F+1)\;C^{(1),\mathsf{NS}}_{q,(2,L)}(N_F+1)+\;A^{(1)}_{gg,Q}(N_F+1)\;\tilde{C}^{(1)}_{g,(2,L)}(N_F+1) \\
&+\;\tilde{C}^{(2)}_{g,(2,L)}(N_F+1)\Big]+\;a_s^3\Big[\;A^{(3)}_{Qg}(N_F+1)\;\delta_2+\;A^{(2)}_{Qg}(N_F+1)\;C^{(1),\mathsf{NS}}_{q,(2,L)}(N_F+1) \\
&+\;A^{(2)}_{gg,Q}(N_F+1)\;\tilde{C}^{(1)}_{g,(2,L)}(N_F+1)+\;A^{(1)}_{Qg}(N_F+1)\Big\{C^{(2),\mathsf{NS}}_{q,(2,L)}(N_F+1) \\
&+\;\tilde{C}^{(2),\mathsf{PS}}_{q,(2,L)}(N_F+1)\Big\}+\;A^{(1)}_{gg,Q}(N_F+1)\;\tilde{C}^{(2)}_{g,(2,L)}(N_F+1)+\;\tilde{C}^{(3)}_{g,(2,L)}(N_F+1)\Big]
\end{aligned}
$$

J. Ablinger, J. Blümlein, S. Klein, C. Schneider and F. Wißbrock, Nucl. Phys. B **844** (2011) 26; J. Ablinger et al., 2013

Variable Flavor Number Scheme

$$
\begin{aligned}
f_k(n_f+1,\mu^2) + f_{\overline{k}}(n_f+1,\mu^2) &= A_{qq,Q}^{\rm NS}\left(n_f,\frac{\mu^2}{m^2}\right) \otimes \left[f_k(n_f,\mu^2) + f_{\overline{k}}(n_f,\mu^2)\right] \\
&\quad + \tilde{A}_{qq,Q}^{\rm PS}\left(n_f,\frac{\mu^2}{m^2}\right) \otimes \Sigma(n_f,\mu^2) + \tilde{A}_{qg,Q}^{\rm S}\left(n_f,\frac{\mu^2}{m^2}\right) \otimes G(n_f,\mu^2) \\
f_{Q+\bar{Q}}(n_f+1,\mu^2) &= \tilde{A}_{Qq}^{\rm PS}\left(n_f,\frac{\mu^2}{m^2}\right) \otimes \Sigma(n_f,\mu^2) + \tilde{A}_{Qg}^{\rm S}\left(n_f,\frac{\mu^2}{m^2}\right) \otimes G(n_f,\mu^2)\ . \\
G(n_f+1,\mu^2) &= A_{gq,Q}^{\rm S}\left(n_f,\frac{\mu^2}{m^2}\right) \otimes \Sigma(n_f,\mu^2) + A_{gg,Q}^{S}\left(n_f,\frac{\mu^2}{m^2}\right) \otimes G(n_f,\mu^2)\ . \\
\Sigma(n_f+1,\mu^2) &= \sum_{k=1}^{n_f+1}\left[f_k(n_f+1,\mu^2) + f_{\overline{k}}(n_f+1,\mu^2)\right] \\
&= \left[A_{qq,Q}^{\rm NS}\left(n_f,\frac{\mu^2}{m^2}\right) + n_f \tilde{A}_{qq,Q}^{\rm PS}\left(n_f,\frac{\mu^2}{m^2}\right) + \tilde{A}_{Qq}^{\rm PS}\left(n_f,\frac{\mu^2}{m^2}\right)\right] \\
&\quad \otimes \Sigma(n_f,\mu^2) \\
&\quad + \left[n_f \tilde{A}_{qg,Q}^{\rm S}\left(n_f,\frac{\mu^2}{m^2}\right) + \tilde{A}_{Qg}^{\rm S}\left(n_f,\frac{\mu^2}{m^2}\right)\right] \otimes G(n_f,\mu^2)
\end{aligned}
$$

The choice of matching scales is not free and varies with the process in case of precision observables. Blümlein, van Neerven [hep-ph/9811351]

$\Longrightarrow$ More complicated for 2 masses J. Blümlein, Wißbrock, 2013

Status of OME calculations

Leading Order: [Witten 1976, Babcock, Sivers 1978, Shifman, Vainshtein, Zakharov 1978, Leveille, Weiler 1979, Glück, Reya 1979, Glück, Hoffmann, Reya 1982]

Next-to-Leading Order:

[Laenen, van Neerven, Riemersma, Smith 1993]

$Q^2 \gg m^2$: via IBP [Buza, Matiounine, Smith, Migneron, van Neerven 1996]

Compact results via ${}_pF_q$'s [Bierenbaum, Blümlein, Klein, 2007]

$O(\alpha_s^2 \varepsilon)$ (for general N) [Bierenbaum, Blümlein, Klein 2008, 2009]

Next-to-Next-to-Leading Order: $Q^2 \gg m^2$

Moments for F_2: $N = 2...10(14)$ [Bierenbaum, Blümlein, Klein 2009]

Contributions to transversity: $N = 1...13$ [Blümlein, Klein, Tödtli 2009]

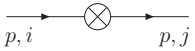
Terms $\propto n_f$ to F_2 (general N): [Ablinger, Blümlein, Klein, Schneider, Wißbrock 2011]

At 3-loop order known

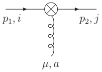
- $A_{qq,Q}^{\rm PS}, A_{qg,Q}$: complete [[1008.3347] Nucl.Phys. B844 (2011) 26]
- $A_{qq,Q}^{\rm NS}$, $A_{qq,Q}^{\rm NS,\,TR}$, $A_{gq,Q}$ and $A_{Qq}^{\rm PS}$: also complete
- A_{Qg}, $A_{gg,Q}$: all terms of $O(n_f T_F^2 C_{A/F})$,
- CC $O(\alpha_s^2)$ corrections [[1401.4352] Nucl. Phys. B in print]
- First contributions to $O(T_F^2 C_{A/F})$
- Two masses $m_1 \neq m_2$, $N = 2, 4, 6$

2. Calculation of the 3-loop operator matrix elements

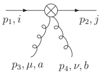
The OMEs are calculated using the QCD Feynman rules together with the following operator insertion Feynman rules:



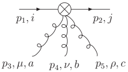
$$\delta^{ij} \Delta\!\!\!/ \gamma_\pm (\Delta \cdot p)^{N-1} , \quad N \geq 1$$



$$g t^a_{ji} \Delta^\mu \Delta\!\!\!/ \gamma_\pm \sum_{j=0}^{N-2} (\Delta \cdot p_1)^j (\Delta \cdot p_2)^{N-j-2} , \quad N \geq 2$$



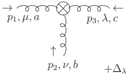
$$g^2 \Delta^\mu \Delta^\nu \Delta\!\!\!/ \gamma_\pm \sum_{j=0}^{N-3} \sum_{l=j+1}^{N-2} (\Delta p_2)^j (\Delta p_1)^{N-l-2}$$
$$\left[(t^a t^b)_{ji} (\Delta p_1 + \Delta p_4)^{l-j-1} + (t^b t^a)_{ji} (\Delta p_1 + \Delta p_3)^{l-j-1} \right] ,$$
$$N \geq 3$$



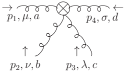
$$g^3 \Delta_\mu \Delta_\nu \Delta_\rho \Delta\!\!\!/ \gamma_\pm \sum_{j=0}^{N-4} \sum_{l=j+1}^{N-3} \sum_{m=l+1}^{N-2} (\Delta . p_2)^j (\Delta . p_1)^{N-m-2}$$
$$\Big[\; (t^a t^b t^c)_{ji} (\Delta . p_4 + \Delta . p_5 + \Delta . p_1)^{l-j-1} (\Delta . p_5 + \Delta . p_1)^{m-l-1}$$
$$+ (t^a t^c t^b)_{ji} (\Delta . p_4 + \Delta . p_5 + \Delta . p_1)^{l-j-1} (\Delta . p_4 + \Delta . p_1)^{m-l-1}$$
$$+ (t^b t^a t^c)_{ji} (\Delta . p_3 + \Delta . p_5 + \Delta . p_1)^{l-j-1} (\Delta . p_5 + \Delta . p_1)^{m-l-1}$$
$$+ (t^b t^c t^a)_{ji} (\Delta . p_3 + \Delta . p_5 + \Delta . p_1)^{l-j-1} (\Delta . p_3 + \Delta . p_1)^{m-l-1}$$
$$+ (t^c t^a t^b)_{ji} (\Delta . p_3 + \Delta . p_4 + \Delta . p_1)^{l-j-1} (\Delta . p_4 + \Delta . p_1)^{m-l-1}$$
$$+ (t^c t^b t^a)_{ji} (\Delta . p_3 + \Delta . p_4 + \Delta . p_1)^{l-j-1} (\Delta . p_3 + \Delta . p_1)^{m-l-1} \Big] ,$$
$$N \geq 4$$

$$\gamma_+ = 1 , \quad \gamma_- = \gamma_5 .$$

$$\tfrac{1+(-1)^N}{2} \delta^{ab} (\Delta \cdot p)^{N-2}$$
$$\left[g_{\mu\nu} (\Delta \cdot p)^2 - (\Delta_\mu p_\nu + \Delta_\nu p_\mu) \Delta \cdot p + p^2 \Delta_\mu \Delta_\nu \right] , \quad N \geq 2$$



$$-ig \tfrac{1+(-1)^N}{2} f^{abc} \Bigg($$
$$\left[(\Delta_\nu g_{\lambda\mu} - \Delta_\lambda g_{\mu\nu}) \Delta \cdot p_1 + \Delta_\mu (p_{1,\nu} \Delta_\lambda - p_{1,\lambda} \Delta_\nu) \right] (\Delta \cdot p_1)^{N-2}$$
$$+ \Delta_\lambda \left[\Delta \cdot p_1 p_{2,\mu} \Delta_\nu + \Delta \cdot p_2 p_{1,\nu} \Delta_\mu - \Delta \cdot p_1 \Delta \cdot p_2 g_{\mu\nu} - p_1 \cdot p_2 \Delta_\mu \Delta_\nu \right]$$
$$\times \sum_{j=0}^{N-3} (-\Delta \cdot p_1)^j (\Delta \cdot p_2)^{N-3-j}$$
$$+ \left\{ \begin{smallmatrix} p_1 \to p_2 \to p_3 \to p_1 \\ \mu \to \nu \to \lambda \to \mu \end{smallmatrix} \right\} + \left\{ \begin{smallmatrix} p_1 \to p_3 \to p_2 \to p_1 \\ \mu \to \lambda \to \nu \to \mu \end{smallmatrix} \right\} \Bigg) , \quad N \geq 2$$



$$g^2 \tfrac{1+(-1)^N}{2} \Bigg(f^{abe} f^{cde} O_{\mu\nu\lambda\sigma}(p_1, p_2, p_3, p_4)$$
$$+ f^{ace} f^{bde} O_{\mu\lambda\nu\sigma}(p_1, p_3, p_2, p_4) + f^{ade} f^{bce} O_{\mu\sigma\nu\lambda}(p_1, p_4, p_2, p_3) \Bigg) ,$$
$$O_{\mu\nu\lambda\sigma}(p_1, p_2, p_3, p_4) = \Delta_\nu \Delta_\lambda \Bigg\{ -g_{\mu\sigma} (\Delta \cdot p_3 + \Delta \cdot p_4)^{N-2}$$
$$+ [p_{4,\mu} \Delta_\sigma - \Delta \cdot p_4 g_{\mu\sigma}] \sum_{i=0}^{N-3} (\Delta \cdot p_3 + \Delta \cdot p_4)^i (\Delta \cdot p_4)^{N-3-i}$$
$$- [p_{1,\sigma} \Delta_\mu - \Delta \cdot p_1 g_{\mu\sigma}] \sum_{i=0}^{N-3} (-\Delta \cdot p_1)^i (\Delta \cdot p_3 + \Delta \cdot p_4)^{N-3-i}$$
$$+ [\Delta \cdot p_1 \Delta \cdot p_4 g_{\mu\sigma} + p_1 \cdot p_4 \Delta_\mu \Delta_\sigma - \Delta \cdot p_4 p_{1,\sigma} \Delta_\mu - \Delta \cdot p_1 p_{4,\mu} \Delta_\sigma]$$
$$\times \sum_{i=0}^{N-4} \sum_{j=0}^{i} (-\Delta \cdot p_1)^{N-4-i} (\Delta \cdot p_3 + \Delta \cdot p_4)^{i-j} (\Delta \cdot p_4)^j \Bigg\}$$
$$- \left\{ \begin{smallmatrix} p_1 \leftrightarrow p_2 \\ \mu \leftrightarrow \nu \end{smallmatrix} \right\} - \left\{ \begin{smallmatrix} p_3 \leftrightarrow p_4 \\ \lambda \leftrightarrow \sigma \end{smallmatrix} \right\} + \left\{ \begin{smallmatrix} p_1 \leftrightarrow p_2, \; p_3 \leftrightarrow p_4 \\ \mu \leftrightarrow \nu, \; \lambda \leftrightarrow \sigma \end{smallmatrix} \right\} , \quad N \geq 2$$

The diagrams are generated using QGRAF [Nogueira 1993 J. Comput. Phys].

	$A_{qq,Q}^{(3),\mathrm{NS}}$	$A_{gq,Q}^{(3)}$	$A_{Qq}^{(3),\mathrm{PS}}$	$A_{gg,Q}^{(3)}$	$A_{Qg}^{(3)}$
No. diagrams	110	86	123	642	1233

A Form program was written in order to perform the γ-matrix algebra in the numerator of all diagrams, which are then expressed as a linear combination of scalar integrals.

$$\begin{aligned} A_{qq,Q}^{(3),\mathrm{NS}} &\rightarrow 7426 \text{ scalar integrals.} \\ A_{gq,Q}^{(3)} &\rightarrow 12529 \text{ scalar integrals.} \\ A_{Qq}^{(3),\mathrm{PS}} &\rightarrow 5470 \text{ scalar integrals.} \end{aligned}$$

$\Longrightarrow$ Need to use integration by parts identities.

Integration by parts

We use Reduze [A. von Manteuffel, C. Studerus, 2012] to express all scalar integrals required in the calculation in terms of a small(er) set of master integrals.

Reduze is a C++ program based on Laporta's algorithm. It is somewhat difficult to adapt this algorithm to the case where we have operator insertions, due to the dependance on the arbitrary parameter N. For this reason we apply the following trick:

$$(\Delta \cdot k)^N \to \sum_{N=0}^{\infty} x^N (\Delta \cdot k)^N = \frac{1}{1 - x\Delta \cdot k}$$

This can be then treated as an additional propagator, and Laporta's algorithm can be applied without further modification.

If we denote the master integrals by M_i, then the reduction algorithm will allow us to express any given integral I as

$$I = \sum_i c_i(x) M_i(x)$$

In fact, any given diagram D will be written this way: $D = \sum_i c_i(x) M_i(x)$
In general the coefficients $c_i(x)$ will be functions of the form

$$c_i(x) = \sum_{j=1}^{n_1} \frac{a_i}{(1 - x\Delta \cdot p)^j} + \sum_{j=-n_2}^{n_3} b_j (x\Delta \cdot p)^j$$

where a_j and b_j depend only on ϵ, and n_1, n_2 and n_3 are non-negative integers.
We want to obtain each diagram $D(N)$ as a function of N. We proceed as follows:

1. Calculate the master integrals $M_i(N)$ as functions of N.
2. Evaluate $M_i(x) = \sum_{N=0}^{\infty} x^N M_i(N)$.
3. Insert the results in $D(x) = \sum_i c_i(x) M_i(x)$.
4. Obtain $D(N)$ by extracting the Nth term in the Taylor expansion of $D(x)$.

Step 1 is done using a variety of techniques to be described shortly.
Steps 2 to 4 are done using the Mathematica packages "HarmonicSums.m", "SumProduction.m" and "EvaluateMultiSums.m" by J. Ablinger and C. Schneider.

Number of master integrals:

$$\begin{aligned} A_{qq,Q}^{(3),\mathrm{NS}} &\rightarrow 35 \text{ master integrals.} \\ A_{gq,Q}^{(3)} &\rightarrow 41 \text{ master integrals.} \\ A_{Qq}^{(3),\mathrm{PS}} &\rightarrow 66 \text{ master integrals.} \end{aligned}$$

If we also include $A_{gg,Q}^{(3)}$ and $A_{Qg}^{(3)}$, there is a total of more than 600 master integrals for the entire project.

24 integral families are required and implemented in Reduze.

Calculation of the master integrals

For the calculation of the master integrals we use a wide variety of tools:

- Hypergeometric functions.
- Summation methods based on Zeilberger's algorithm, implemented in the Mathematica program Sigma [C. Schneider, 2005–].
  - Reduction of the sums to a small number of key sums.
  - Expansion the summands in ε.
  - Harmonic sums are algebraically reduced using the package HarmonicSums (Ablinger) [Ablinger, Blümlein, Schneider 2011].
- Mellin-Barnes representations.
- Differential (difference) equations.
- In the case of convergent massive 3-loop Feynman integrals, they can be performed in terms of Hyperlogarithms [Generalization of a method by F. Brown, 2008, to non-vanishing masses and local operators].

Hypergeometric functions and Sigma

Consider the following master integral:

$$\int \frac{d^D k_1}{(2\pi)^D} \frac{d^D k_2}{(2\pi)^D} \frac{d^D k_3}{(2\pi)^D} \frac{(\Delta \cdot k_1)^N}{D_1^2 D_2 D_3 D_4 D_5}$$

where

$$D_1 = (k_1 - p)^2, \quad D_2 = (k_2 - p)^2, \quad D_3 = k_3^2 - m^2,$$
$$D_4 = (k_1 - k_3)^2 - m^2 \quad \text{and} \quad D_5 = (k_2 - k_3)^2 - m^2$$

After Feynman parametrization we obtain

$$-\int_0^1 dx \int_0^1 dy \int_0^1 dz_1 \int_0^1 dz_2\, \Gamma\left(-\frac{3}{2}\epsilon\right) \sum_{j=0}^{N} (-1)^j \binom{N}{j} x^{j-1+\epsilon/2} (1-x)^{\epsilon/2}$$
$$\times y^{\epsilon/2} (1-y)^{\epsilon/2} z_1^{-\epsilon/2} z_2^{-1-\epsilon/2} (1 - z_1 - z_2)^j$$
$$\times \theta(1 - z_1 - z_2) \left(1 + z_1 \frac{x}{1-x} + z_2 \frac{y}{1-y}\right)^{\frac{3}{2}\epsilon}$$

Now we use the following integral representation of the Appell hypergeometric function:

$$\int_0^1 dw_1 \int_0^1 dw_2 \, \frac{\theta(1-w_1-w_2) w_1^{b-1} w_2^{b'-1} (1-w_1-w_2)^{c-b-b'-1}}{(1-w_1 x - w_2 y)^a}$$
$$= \Gamma\left[\begin{matrix} b, b', c-b-b' \\ c \end{matrix}\right] F_1\left[a; b, b'; c; x, y\right] .$$

so, our integral becomes

$$-\int_0^1 dx \int_0^1 dy \, \Gamma\left(-\frac{3}{2}\epsilon\right) \sum_{j=0}^{N} (-1)^j \binom{N}{j} x^{j-1+\epsilon/2} (1-x)^{\epsilon/2} y^{\epsilon/2} (1-y)^{\epsilon/2}$$
$$\times \Gamma\left[\begin{matrix} 1-\epsilon/2, -\epsilon/2, j+1 \\ j+2-\epsilon \end{matrix}\right] F_1\left(-\frac{3}{2}\epsilon; 1-\epsilon/2, -\epsilon/2; j+2-\epsilon; \frac{x}{x-1}, \frac{y}{y-1}\right)$$

we want to expand the F_1 function, so we use the following analytic continuation:

$$F_1\left[a;,b,b';c;\frac{x}{1-x},\frac{y}{1-y}\right]=(1-x)^b(1-y)^{b'}F_1\left[c-a;b,b';c;x,y\right]$$

so, our integral is now

$$-\int_0^1 dx\int_0^1 dy\,\Gamma\left(-\frac{3}{2}\epsilon\right)\sum_{j=0}^{N}(-1)^j\binom{N}{j}x^{j-1+\epsilon/2}(1-x)y^{\epsilon/2}$$
$$\times\Gamma\left[\begin{matrix}1-\epsilon/2,-\epsilon/2,j+1\\ j+2-\epsilon\end{matrix}\right]F_1\left(j+2+\epsilon/2;1-\epsilon/2,-\epsilon/2;j+2-\epsilon;x,y\right)$$

We can now use the series representation of the Appell hypergeometric function. We obtain

$$-\Gamma\left(-\frac{3}{2}\epsilon\right)\sum_{j=0}^{N}(-1)^j\frac{N!}{(N-j)!}\sum_{m,n=0}^{\infty}\frac{\Gamma(m+j+\epsilon/2)}{\Gamma(m+j+2+\epsilon/2)}\frac{1}{(n+1+\epsilon/2)}$$
$$\times\frac{\Gamma(m+n+j+2+\epsilon/2)\Gamma(m+1-\epsilon/2)\Gamma(n-\epsilon/2)}{m!n!\Gamma(j+2+\epsilon/2)\Gamma(m+n+j+2-\epsilon)}$$

We can now expand in ϵ, and the resulting sums can be performed using Sigma. The final result is

$$-\frac{8}{3\epsilon^3}-\frac{4}{3\epsilon^2}\left(\frac{N}{N+1}-S_1(N)\right)+\frac{1}{3\epsilon}\left(\frac{2(N-1)S_1(N)}{N+1}-S_1(N)^2+S_2(N)\right.$$
$$\left.+\frac{2N}{(N+1)^2}-3\zeta(2)-8\right)+\left(\frac{(-N-7)S_2(N)}{6(N+1)}+\frac{4N^2+9N+7}{3(N+1)^2}\right)S_1(N)$$
$$+\frac{\zeta(2)}{2}\left(S_1(N)-\frac{N}{N+1}\right)+\frac{S_1(N)^3}{18}+\frac{(1-N)S_1(N)^2}{6(N+1)}+\frac{(-5N-7)S_2(N)}{6(N+1)}$$
$$+\frac{(N-8)S_3(N)}{9(N+1)}+\frac{(N+4)S_{2,1}(N)}{3(N+1)}-\frac{N\left(4N^2+8N+5\right)}{3(N+1)^3}+\frac{2N\zeta(3)}{N+1}-\frac{5\zeta(3)}{3}$$

Mellin-Barnes integral representations

Let's consider now the following master integral:

$$\int \frac{d^D k_1}{(2\pi)^D} \frac{d^D k_2}{(2\pi)^D} \frac{d^D k_3}{(2\pi)^D} \frac{(\Delta \cdot k_1)^N}{D_1^2 D_2 D_3 D_4 D_5}$$

where

$$D_1 = (k_1 - p)^2 - m^2, \quad D_2 = (k_2 - p)^2 - m^2, \quad D_3 = k_3^2,$$

$$D_4 = (k_1 - k_3)^2 \quad \text{and} \quad D_5 = (k_2 - k_3)^2$$

After Feynman parametrization we obtain

$$B = -\int_0^1 dx \int_0^1 dy \int_0^1 dz_1 \int_0^1 dz_2 \, \Gamma\left(-\frac{3}{2}\epsilon\right) (x-y)^N x^{-1+\epsilon/2} (1-x)^{\epsilon/2}$$

$$\times y^{\epsilon/2} (1-y)^{\epsilon/2} z_1^{-\epsilon/2} (1-z_1)^{-1-\epsilon/2} z_2^{-1-\epsilon/2} (1-z_2)^N \left(\frac{z_1}{x(1-x)} + \frac{1-z_1}{y(1-y)}\right)^{\frac{3}{2}\epsilon}$$

Now we make use of

$$\frac{1}{(A+B)^\nu} = \frac{1}{2i\pi}\int_{-i\infty}^{i\infty} d\tau \; \frac{\Gamma(-\tau)\Gamma(\tau+\nu)}{\Gamma(\nu)} \frac{A^\tau}{B^{\tau+\nu}}$$

to express our integral as

$$\int_{-i\infty}^{i\infty} d\tau \; \sum_{j=0}^{N}(-1)^j \binom{N}{j} \Gamma(-\tau)\Gamma(\tau+\nu)\frac{\Gamma(-\tau+j+\epsilon/2)\Gamma(-\tau+1+\epsilon/2)}{\Gamma(-2\tau+j+1+\epsilon)}$$
$$\times \frac{\Gamma(\tau+N-j+1-\epsilon)\Gamma(\tau+1-\epsilon)}{\Gamma(2\tau+N-j+2-2\epsilon)} \frac{\Gamma(\tau+1-\epsilon/2)\Gamma(-\tau+\epsilon)}{\Gamma(1+\epsilon/2)}$$
$$\times \frac{\Gamma(1+\epsilon/2)\Gamma(N+1)}{\Gamma(N+2+\epsilon/2)}$$

We use the Mathematica MB.m package by M. Czakon [Comput.Phys.Commun. 175 (2006)] together with the MBresolve.m addition of V. Smirnov et. al. [Eur.Phys.J. C62 (2009)], to resolve the singularities in $\epsilon = D - 4$ for this expression, after which we can expand in ϵ.

After we expand in ϵ, we solve the resulting integrals by closing the contour to the right and taking residues. This leads to a linear combination of several multiple sums. For example,

$$\sum_{k=n+1}^{2n} \sum_{n=0}^{[N/2]} (-1)^{N+k} \binom{N}{N-2n} \frac{\Gamma(2k-2n-1)\Gamma(-k+2n+1)\Gamma(k-2n+N)}{(N+1)\Gamma(2k-2n+N+1)} \times \Big(2\psi(2k-2n-1) - \psi(-k+2n+1) + \psi(k-2n+N) - 2\psi(2k-2n+N+1)\Big)$$

Here $[N/2]$ is the integer part of $N/2$. Other sums arising in this example are even more complicated. Individually, they lead to cyclotomic sums:

$$S_{\{a_1,b_1,c_1\},\ldots\{a_l,b_l,c_l\}}(s_1,\ldots,s_l,N) = \sum_{k_1=1}^{N} \frac{s_1^k}{(a_1 k_1 + b_1)^{c_1}} S_{\{a_2,b_2,c_2\},\ldots\{a_l,b_l,c_l\}}(s_2,\ldots,s_l,N)$$

However, when we combine all the sums to obtain the final expression for our master integral, the result turns out to be expressed in terms of standard harmonic sums:

$$\frac{1+(-1)^N}{(N+1)(N+2)}\Bigg[-\frac{8}{3\epsilon^3}-\frac{8}{3\epsilon^2(N+2)}+\frac{1}{\epsilon}\Bigg(\frac{2}{3}S_{-2}(N)-2S_2(N)-\zeta(2)$$
$$-\frac{8\left(2N^2+6N+5\right)}{3(N+1)^2(N+2)^2}\Bigg)-\frac{1}{3}S_{-3}(N)+S_{-2}(N)\left(S_{-1}(N)-\frac{N+4}{3(N+1)(N+2)}\right)$$
$$-S_{-1}(N)S_2(N)-\frac{NS_2(N)}{(N+1)(N+2)}-\frac{1}{3}S_3(N)-S_{-2,-1}(N)+S_{2,-1}(N)$$
$$-\frac{8\left(2N^2+6N+5\right)}{3(N+1)^2(N+2)^3}-\frac{\zeta(2)}{N+2}-\frac{8\zeta(3)}{3}\Bigg]$$

These methods were enough to calculate $A_{qq,Q}^{(3)\mathrm{NS}}$, $A_{gq}^{(3)}$ and transversity.

Differential (difference) equations

In general our N-dependent Feynman master integrals will have the following structure

$$M(N) = F(N)(m^2)^{-a+\frac{3}{2}D}(\Delta \cdot p)^N$$

Differentiating the master integrals w.r.t. m^2 or $\Delta \cdot p$ doesn't gives any new information, since we already know the functional dependence on these invariants.

However, in the x representation of the integrals

$$M(x) = \sum_{N=0}^{\infty} x^N F(N)(m^2)^{-a+\frac{3}{2}D}(\Delta \cdot p)^N\,,$$

we can differentiate w.r.t. x, since as we saw before, this turns the operator insertions into artificial propagators.

Differentiation will rise the powers of the artificial propagators and the resulting integrals can be re-expressed in terms of master integrals.

$$\frac{d}{dx}M(x) = \sum_i \frac{p_i(x)}{q_i(x)} M_i(x)$$

Integrals in a given sector will produce a system of coupled differential equations.

Let's consider the following example,

$$\begin{aligned} M_1(N) &= \int \frac{d^D k_1}{(2\pi)^D} \frac{d^D k_2}{(2\pi)^D} \frac{d^D k_3}{(2\pi)^D} \frac{\sum_{j=0}^{N} (\Delta \cdot k_3)^j (\Delta \cdot k_3 - \Delta \cdot k_1)^{N-j}}{D_1 D_2 D_3 D_4 D_5}, \\ M_2(N) &= \int \frac{d^D k_1}{(2\pi)^D} \frac{d^D k_2}{(2\pi)^D} \frac{d^D k_3}{(2\pi)^D} \frac{\sum_{j=0}^{N} (\Delta \cdot k_3)^j (\Delta \cdot k_3 - \Delta \cdot k_1)^{N-j}}{D_1^2 D_2 D_3 D_4 D_5}, \end{aligned}$$

where $D_1 = (k_1 - p)^2$, $D_2 = (k_2 - p)^2$, $D_3 = k_3^2 - m^2$, $D_4 = (k_3 - k_1)^2 - m^2$ and $D_4 = (k_3 - k_2)^2 - m^2$.

In the x representation, they become

$$\begin{aligned} M_1(x) &= \int \frac{d^D k_1}{(2\pi)^D} \frac{d^D k_2}{(2\pi)^D} \frac{d^D k_3}{(2\pi)^D} \frac{1}{D_1 D_2 D_3 D_4 D_5 D_6 D_7}\,, \\ M_2(x) &= \int \frac{d^D k_1}{(2\pi)^D} \frac{d^D k_2}{(2\pi)^D} \frac{d^D k_3}{(2\pi)^D} \frac{1}{D_1^2 D_2 D_3 D_4 D_5 D_6 D_7}\,, \end{aligned}$$

with $D_6 = 1 - x\Delta \cdot k_3$ and $D_7 = 1 - x(\Delta \cdot k_3 - \Delta \cdot k_1)$. So,

$$\begin{aligned} \frac{d}{dx} M_1(x) &= \frac{1}{1-x}\left(2+\epsilon-\frac{1}{x}\right) M_1(x) + \frac{2x}{1-x} M_2(x) + K_1(x) \\ \frac{d}{dx} M_2(x) &= -\frac{1}{1-x}\left(\frac{1-2\epsilon}{x}+\frac{3}{2}\epsilon-2\right) M_2(x) \\ &\quad + \frac{\epsilon}{4}(2+3\epsilon)\frac{1}{1-x}\left(\frac{1}{x^2}-\frac{1}{x}\right) M_1(x) + K_2(x) \end{aligned}$$

where $K_1(x)$ and $K_2(x)$ are linear combinations of (already solved) subsector master integrals.

$K_1(x)$ and $K_2(x)$ can be turned into the N representation using the Linz Mathematica packages. Then using the fact that

$$
\begin{aligned}
M_1(x) &= \sum_{N=0}^{\infty} x^N F_1(N)(m^2)^{-a+\frac{3}{2}D}(\Delta \cdot p)^N\,, \\
M_2(x) &= \sum_{N=0}^{\infty} x^N F_2(N)(m^2)^{-a+\frac{3}{2}D}(\Delta \cdot p)^N\,,
\end{aligned}
$$

we get the following system of coupled difference equations:

$$
\begin{aligned}
&(N+2)F_1(N+1) - (N+2+\epsilon)F_1(N) - 2F_2(N-1) = K_1(N)\,, \\
&(N+2-2\epsilon)F_2(N+1) - \left(N+2-\frac{3}{2}\epsilon\right)F_1(N) \\
&-\frac{\epsilon}{4}(2+3\epsilon)\left(F_1(N+2) - F_1(N+1)\right) = K_2(N)\,,
\end{aligned}
$$

The system can be solved using the Linz Mathematica packages. The results are again given in terms of standard harmonic sums.

Main Results:

- 3 Loop Anomalous Dimensions
- Wilson Coefficients and OMEs

3-Loop Anomalous Dimensions $\propto T_F$

Transversity:

$$\gamma_{\rm NS,TR}^{(0),qq}(N) = 2C_F\left[4S_1 - 3\right]$$

$$\gamma_{\rm NS,TR}^{(k),\pm}(N) = \gamma_{\rm NS,TR}^{(k),qq}(N) \pm \gamma_{\rm NS,TR}^{(k),q\bar{q}}(N), \quad k = 1,2.$$

$$\begin{aligned}\gamma_{\rm NS,TR}^{(1),qq}(N) = &\frac{1}{2}C_F\left(C_F - \frac{C_A}{2}\right)\Bigg[128S_{-2,1} + \frac{4(17N^2+17N-12)}{3N(N+1)} - 128S_{-2}S_1 \\ &-\frac{2144}{9}S_1 + \frac{352}{3}S_2 - 64S_3 - 64S_{-3}\Bigg] \\ &+\frac{1}{2}C_F^2\left[S_1\left(\frac{2144}{9} - 64S_2\right) - \frac{208}{3}S_2 - \frac{86}{3}\right] + C_FT_FN_F\left[-\frac{160}{9}S_1 + \frac{32}{3}S_2 + \frac{4}{3}\right]\end{aligned}$$

$$\gamma_{\rm NS,TR}^{(1),q\bar{q}}(N) = C_F\left(C_F - \frac{C_A}{2}\right)\frac{8}{N(N+1)}$$

$$\begin{aligned}\hat{\gamma}_{\rm NS,TR}^{(2),qq}(N) = &C_F^2T_F\Bigg\{-\frac{256}{3}S_{3,1} + \left[-\frac{8(1331N^2+1331N-36)}{27N(N+1)} - 128\zeta_3 + \frac{1280}{9}S_2 - \frac{128}{3}S_3\right]S_1 \\ &-\frac{4(153N^2+153N-176)}{9N(N+1)} - \frac{128}{3}S_2^2 + \frac{9968}{27}S_2 - \frac{832}{9}S_3 + \frac{128}{3}S_4 + 96\zeta_3\Bigg\} \\ &+C_FT_F\left(C_F - \frac{C_A}{2}\right)\Bigg\{\left[-\frac{512}{3}S_{-2,1} + \frac{32(209N^2+209N-9)}{27N(N+1)} - 128S_3 + 256\zeta_3\right]S_1 \\ &+\frac{512}{3}S_{3,1} - \frac{2560}{9}S_{-2,1} - \frac{256}{3}S_{-2,2} + \frac{1024}{3}S_{-2,1,1} + \frac{32(15N^3+30N^2+12N-5)}{3N(N+1)^2} \\ &+\left(\frac{1280}{9} - \frac{256}{3}S_1\right)S_{-3} + \left(\frac{2560}{9}S_1 - \frac{256}{3}S_2\right)S_{-2} - \frac{10688}{27}S_2 + \frac{896}{3}S_3 - \frac{640}{3}S_4 \\ &-\frac{256}{3}S_{-4} - 192\zeta_3\Bigg\} \\ &+C_FT_F^2(2N_F+1)\left[+\frac{8(17N^2+17N-8)}{9N(N+1)} - \frac{128}{27}S_1 - \frac{640}{27}S_2 + \frac{128}{9}S_3\right]\end{aligned}$$

$$\hat{\gamma}_{\rm NS,TR}^{(2),q\bar{q}}(N) = C_F\left(C_F - \frac{C_A}{C_F}\right)\left[\frac{64}{3N(N+1)}S_1 - \frac{32(13N+7)}{9N(N+1)^2}\right]$$

- Independent confirmation of full two-loop results.
- 1st ab initio calculation of the contribution $\propto T_F$ at 3 loops.
- Note a typo in the 15th moment in 1203.1022.
- Independent calculation of the anmalous dimensions ($\propto T_F$) $\gamma_{qq}^{\rm NS\pm}$ and γ_{gq} at 3 loops.

Wilson Coefficient $L_{q,2}^{(3),\mathrm{PS}}$

$$
\begin{aligned}
L_{q,2}^{(3),PS}(N) \;=\; & C_F N_F T_F^2 \times \\
& \Bigg\{ -\frac{32\left(N^2+N+2\right)^2}{9(N-1)N^2(N+1)^2(N+2)} \ln^3\left(\frac{m^2}{Q^2}\right) - \frac{32P_3}{9(N-1)N^3(N+1)^3(N+2)^2} \ln^2\left(\frac{m^2}{Q^2}\right) \\
& + \Bigg[-\frac{32P_5}{27(N-1)N^4(N+1)^4(N+2)^3} + \frac{64P_1}{3(N-1)N^3(N+1)^3(N+2)^2} S_1 \\
& + \frac{32\left(N^2+N+2\right)^2}{3(N-1)N^2(N+1)^2(N+2)} \left[S_1^2 - S_2\right] \Bigg] \ln\left(\frac{m^2}{Q^2}\right) \\
& - \frac{32P_7}{243(N-1)N^5(N+1)^5(N+2)^4} - \frac{16P_2}{27(N-1)N^3(N+1)^3(N+2)^2} S_1^2 \\
& - \frac{16P_4}{27(N-1)N^3(N+1)^3(N+2)^2} S_2 + \Bigg[\frac{32P_6}{81(N-1)N^4(N+1)^4(N+2)^3} \\
& + \frac{32\left(N^2+N+2\right)^2 S_2}{9(N-1)N^2(N+1)^2(N+2)} \Bigg] S_1 - \frac{64\left(N^2+N+2\right)^2}{27(N-1)N^2(N+1)^2(N+2)} S_1^3 \\
& + \frac{160\left(N^2+N+2\right)^2 S_3}{27(N-1)N^2(N+1)^2(N+2)} + \frac{256\left(N^2+N+2\right)^2}{9(N-1)N^2(N+1)^2(N+2)} \zeta_3 \Bigg\} + N_f \hat{\tilde{C}}_{q,2}^{\mathrm{PS}(3)}(N, N_f)
\end{aligned}
$$

Wilson Coefficient $L_{g,2}^{(3),\mathrm{S}}$

$$L_{g,2}^{(3),S} = N_F T_F^3 \Big[-\frac{64\left(N^3-4N^2-N-2\right)}{9N^2(N+1)(N+2)} - \frac{64\left(N^2+N+2\right)}{9N(N+1)(N+2)} S_1 \Big] \ln^2\left(\frac{m^2}{Q^2}\right)$$

$$+N_F T_F^2 C_A \Big\{ + \Big[\frac{32\left(N^2+N+2\right)}{9N(N+1)(N+2)} S_1 - \frac{64\left(N^2+N+1\right)\left(N^2+N+2\right)}{9(N-1)N^2(N+1)^2(N+2)^2} \Big] \ln^3\left(\frac{m^2}{Q^2}\right)$$

$$+ \Big[\frac{16\left(N^2+N+2\right)}{3N(N+1)(N+2)} S_1^2 + \frac{32P_3}{9(N-1)N(N+1)^2(N+2)} S_1 + \frac{8P_{17}}{9(N-1)N^3(N+1)^3(N+2)^3}$$

$$- \frac{16\left(N^2+N+2\right)}{3N(N+1)(N+2)} S_2 - \frac{32\left(N^2+N+2\right)}{3N(N+1)(N+2)} S_{-2}$$

$$\Big] \ln^2\left(\frac{m^2}{Q^2}\right) + \Big[\frac{32P_6}{9(N-1)N(N+1)^2(N+2)^2} S_1^2 + \Big[\frac{16P_{16}}{27(N-1)N^3(N+1)^2(N+2)^3}$$

$$+ \frac{64\left(N^2+N+2\right)}{3N(N+1)(N+2)} S_2 \Big] S_1 + \frac{16P_{18}}{27(N-1)N^4(N+1)^4(N+2)^3} + \Big[\frac{64(N-1)}{3N(N+1)} S_1$$

$$- \frac{64P_7}{9(N-1)N(N+1)^2(N+2)^2} \Big] S_{-2} - \frac{32P_8}{9(N-1)N(N+1)^2(N+2)^2} S_2$$

$$\frac{64}{3(N+2)} S_3 - \frac{128}{3N(N+1)(N+2)} S_{-3} + \frac{256}{3N(N+1)(N+2)} S_{-2,1} - \frac{32(N-1)}{N(N+1)} \zeta_3 \Big] \ln\left(\frac{m^2}{Q^2}\right)$$

$$- \frac{4\left(N^2+N+2\right)}{27N(N+1)(N+2)} S_1^4 + \frac{32P_1}{81N(N+1)^2(N+2)^2} S_1^3 + \Big[\frac{16P_{11}}{81N(N+1)^3(N+2)^3}$$

$$+ \frac{8\left(N^2+N+2\right)}{9N(N+1)(N+2)} S_2 \Big] S_1^2 + \Big[\frac{8P_{22}}{243(N-1)N^4(N+1)^4(N+2)^4} - \frac{32P_1}{27N(N+1)^2(N+2)^2} S_2$$

$$+ \frac{160\left(N^2+N+2\right)}{27N(N+1)(N+2)} S_3 - \frac{64\left(N^2+N+2\right)}{9N(N+1)(N+2)} S_{2,1} \Big] S_1 - \frac{4\left(N^2+N+2\right)}{9N(N+1)(N+2)} S_2^2$$

$$+ \frac{8P_{23}}{243(N-1)N^5(N+1)^5(N+2)^5} - \frac{16P_{12}}{81N(N+1)^3(N+2)^3} S_2 + \frac{64P_2}{81N(N+1)^2(N+2)^2} S_3$$

$$- \frac{56\left(N^2+N+2\right)}{9N(N+1)(N+2)} S_4 - \frac{32\left(121N^3+293N^2+414N+224\right)}{81N(N+1)^2(N+2)} S_{-2}$$

$$- \frac{128\left(N^2+N+2\right)}{9N(N+1)(N+2)} S_{-4} + \frac{128\left(5N^2+8N+10\right)}{27N(N+1)(N+2)} S_{-3} + \frac{128P_1}{27N(N+1)^2(N+2)^2} S_{2,1}$$

$$- \frac{128\left(N^2+N+2\right)}{9N(N+1)(N+2)} S_{3,1} + \frac{64\left(N^2+N+2\right)}{9N(N+1)(N+2)} S_{2,1,1} + \Big[\frac{512\left(N^2+N+1\right)\left(N^2+N+2\right)}{9(N-1)N^2(N+1)^2(N+2)^2}$$

$$- \frac{256\left(N^2+N+2\right)}{9N(N+1)(N+2)} S_1 \Big] \zeta_3 \Big\}$$

$$+N_F T_F^2 C_F \Big\{ + \Big[\frac{8\left(N^2+N+2\right)P_9}{9(N-1)N^3(N+1)^3(N+2)^2} - \frac{32\left(N^2+N+2\right)}{9N(N+1)(N+2)} S_1 \Big] \ln^3\left(\frac{m^2}{Q^2}\right)$$

$$+ \Big[\frac{16\left(N^2+N+2\right)}{3N(N+1)(N+2)} [S_1^2+S_2] - \frac{16P_{15}}{9(N-1)N^3(N+1)^3(N+2)^2} S_1 + \frac{4P_{19}}{9(N-1)N^4(N+1)^4(N+2)^3} \Big]$$

$$\times \ln^2\left(\frac{m^2}{Q^2}\right) + \Big[-\frac{32\left(N^2+N+2\right)}{3N(N+1)(N+2)} S_1^3 - \frac{8P_{14}}{9(N-1)N^3(N+1)^3(N+2)^2} S_1^2$$

$$+ (\frac{32\left(N^2+N+2\right)}{3N(N+1)(N+2)} S_2 - \frac{16P_{20}}{27(N-1)N^4(N+1)^4(N+2)^3} \Big] S_1$$

$$+ \frac{4P_{24}}{27(N-2)(N-1)N^5(N+1)^5(N+2)^4(N+3)} + \Big[\frac{64P_{13}}{3(N-2)(N-1)N^2(N+1)^2(N+2)^2(N+3)}$$

$$+ \frac{512}{3N(N+1)(N+2)} S_1 \Big] S_{-2} + \frac{8\left(N^2+N+2\right)P_{10}}{9(N-1)N^3(N+1)^3(N+2)^2} S_2 - \frac{64(N-1)}{3N(N+1)} S_3$$

$$+ \frac{256}{3N(N+1)(N+2)} S_{-3} + \frac{64\left(N^2+N+2\right)}{3N(N+1)(N+2)} S_{2,1} - \frac{512}{3N(N+1)(N+2)} S_{-2,1}$$

$$+ \frac{64(N-1)}{N(N+1)} \zeta_3 \Big] \ln\left(\frac{m^2}{Q^2}\right) + \frac{4\left(N^2+N+2\right)}{27N(N+1)(N+2)} S_1^4 - \frac{16\left(10N^3+13N^2+29N+6\right)}{81N^2(N+1)(N+2)} S_1^3$$

$$+ \Big[\frac{8P_5}{81N^2(N+1)^2(N+2)} + \frac{8\left(N^2+N+2\right)}{9N(N+1)(N+2)} S_2 \Big] S_1^2 + \Big[-\frac{4P_{21}}{243(N-1)N^5(N+1)^5(N+2)^2}$$

$$- \frac{16\left(10N^3+13N^2+29N+6\right)}{27N^2(N+1)(N+2)} S_2 + \frac{32\left(N^2+N+2\right)}{27N(N+1)(N+2)} S_3 \Big] S_1 + \frac{4\left(N^2+N+2\right)}{9N(N+1)(N+2)} S_2^2$$

$$+ \frac{P_{25}}{243(N-1)N^6(N+1)^6(N+2)^5} + \frac{8P_4}{27N^2(N+1)^2(N+2)} S_2 + \frac{32\left(5N^3-16N^2+N-6\right)}{81N^2(N+1)(N+2)} S_3$$

$$- \frac{56\left(N^2+N+2\right)}{9N(N+1)(N+2)} S_4 + \Big[\frac{256\left(N^2+N+2\right)}{9N(N+1)(N+2)} S_1 - \frac{64\left(N^2+N+2\right)P_9}{9(N-1)N^3(N+1)^3(N+2)^2} \Big] \zeta_3$$

$$\Big\} + N_F \hat{\tilde{C}}_{2,g}^{(3)}(N, N_F)$$

3-Loop OME: Non-Singlet

$$a_{NS}^{(3)}(N) = C_F^2 T_F \Big\{ \Big[\frac{128}{27}S_2 - \frac{16(2N+1)}{27N^2(N+1)^2}\Big] S_1^3 + \Big[(-1)^N \frac{64(2N^2+2N+1)}{9N^3(N+1)^3}$$

$$-\frac{8(2N^5+7N^4+3N^3-9N^2-7N+2)}{9N^3(N+1)^3} - \frac{64}{9N(N+1)}S_2 + \frac{64}{3}S_3 - \frac{128}{9}S_{2,1} - \frac{256}{9}S_{-2,1}\Big] S_1^2$$

$$+\Big[-\frac{64}{9}S_2^2 + \frac{16(448N^4+896N^3+484N^2+54N+45)}{81N^2(N+1)^2}S_2 - (-1)^N\frac{32P_1}{27N^4(N+1)^4}$$

$$+\frac{8P_2}{81N^4(N+1)^4} - \frac{64(40N^2+40N+9)}{27N(N+1)}S_3 + \frac{704}{9}S_4 + \frac{128}{9N(N+1)}S_{2,1} - \frac{320}{9}S_{3,1}$$

$$-\frac{256(10N^2+10N-3)}{27N(N+1)}S_{-2,1} - \frac{256}{9}S_{-2,2} + \frac{64}{3}S_{2,1,1} + \frac{1024}{9}S_{-2,1,1}\Big] S_1$$

$$-\frac{16(31N^2+31N-6)}{27N(N+1)}S_2^2 + (-1)^N\frac{16P_3}{81N^5(N+1)^5} + \frac{P_4}{162N^5(N+1)^5} + \Big[\frac{16(3N^2+3N+2)}{3N(N+1)}$$

$$-\frac{64}{3}S_1\Big]B_4 + \Big[\frac{256}{9}S_1 - \frac{128(10N^2+10N+3)}{27N(N+1)}\Big]S_{-4} + \Big[96S_1$$

$$-\frac{24(3N^2+3N+2)}{N(N+1)}\Big]\zeta_4 + \Big[\frac{128}{9}[S_1^2+S_2] - \frac{128(10N^2+10N+3)}{27N(N+1)}S_1$$

$$+\frac{64(112N^3+224N^2+169N+39)}{81N(N+1)^2}\Big]S_{-3} - \frac{176(17N^2+17N+6)}{27N(N+1)}S_4$$

$$+\frac{8(1301N^4+2602N^3+2177N^2+492N-84)}{81N^2(N+1)^2}S_3 + \frac{512}{9}S_5 + \frac{256}{9}S_{-5} + \Big[\frac{256}{27}S_1^3$$

$$-\frac{128}{9N(N+1)}S_1^2 + \frac{128(112N^4+224N^3+121N^2+9N+9)}{81N^2(N+1)^2}S_1$$

$$-\frac{64(181N^4+266N^3+82N^2-3N+18)}{81N^3(N+1)^3} - \frac{512}{9}S_{2,1} - \frac{1280}{27}S_2 + \frac{512}{27}S_3\Big]S_{-2}$$

$$+\frac{16(7N^4+14N^3+3N^2-4N-4)}{9N^2(N+1)^2}S_{2,1} + \frac{256}{9}S_{2,3} - \frac{512}{9}S_{2,-3} + \frac{16(89N^2+89N+30)}{27N(N+1)}S_{3,1}$$

$$-\frac{512}{9}S_{4,1} - \frac{128(112N^3+112N^2-39N+18)}{81N^2(N+1)}S_{-2,1} + \Big[\frac{64(-1)^N(2N^2+2N+1)}{9N^3(N+1)^3}$$

$$-\frac{8P_5}{81N^3(N+1)^3} + \frac{256}{27}S_3 + \frac{256}{3}S_{-2,1}\Big]S_2 - \frac{128(10N^2+10N-3)}{27N(N+1)}S_{-2,2} + \frac{512}{9}S_{-2,3}$$

$$-\frac{16(3N^2+3N+2)}{3N(N+1)}S_{2,1,1} + \frac{512}{9}S_{2,1,-2} + \frac{256}{9}S_{3,1,1} + \frac{512(10N^2+10N-3)}{27N(N+1)}S_{-2,1,1}$$

$$+\frac{512}{9}S_{-2,2,1} - \frac{2048}{9}S_{-2,1,1,1} + \Big[(-1)^N\frac{16(2N^2+2N+1)}{3N^3(N+1)^3} + \frac{P_6}{3N^3(N+1)^3} + \Big[\frac{64}{3}S_1$$

$$-\frac{32}{3N(N+1)}\Big]S_{-2} - \frac{8(3N^2+3N+2)}{3N(N+1)}S_2 + \Big[\frac{8(15N^4+30N^3+15N^2-4N-2)}{3N^2(N+1)^2} + \frac{32}{3}S_2\Big]S_1$$

$$+\frac{32}{3}S_3 + \frac{32}{3}S_{-3} - \frac{64}{3}S_{-2,1}\Big]\zeta_2 + \Big[\frac{2(561N^4+1122N^3+767N^2+302N+48)}{9N^2(N+1)^2} - \frac{1208}{9}S_1$$

$$-\frac{64}{3}S_2\Big]\zeta_3\Big\}$$

$$+C_F C_A T_F\Big\{-\frac{64}{27}S_2S_1^3 + \Big[-(-1)^N\frac{32(2N^2+2N+1)}{9N^3(N+1)^3} + \frac{4P_9}{9N^3(N+1)^3} + \frac{32}{9N(N+1)}S_2 - \frac{80}{9}S_3$$

$$+\frac{128}{9}S_{2,1} + \frac{128}{9}S_{-2,1}\Big]S_1^2 + \Big[\frac{80(2N+1)^2}{9N(N+1)}S_3 + \frac{112}{9}S_2^2 + (-1)^N\frac{16P_1}{27N^4(N+1)^4} + \frac{4P_{10}}{729N^4(N+1)^4}$$

$$-\frac{16(N-1)(2N^3-N^2-N-2)}{9N^2(N+1)^2}S_2 - \frac{208}{9}S_4 - \frac{8(9N^2+9N+16)}{9N(N+1)}S_{2,1} + \frac{64}{3}S_{3,1}$$

$$+\frac{128(10N^2+10N-3)}{27N(N+1)}S_{-2,1} + \frac{128}{9}S_{-2,2} - 32S_{2,1,1} - \frac{512}{9}S_{-2,1,1}\Big]S_1$$

$$-\frac{4(15N^2+15N+14)}{9N(N+1)}S_2^2 + \frac{24(N-1)(N^2+2)}{5N(N+1)^2}\zeta_2^2 - (-1)^N\frac{8P_3}{81N^5(N+1)^5} + \frac{P_{11}}{1458N^5(N+1)^5}$$

$$+\Big[\frac{12(5N^3+13N^2+8N+6)}{N(N+1)^2} - 96S_1\Big]\zeta_4 + \Big[\frac{64(10N^2+10N+3)}{27N(N+1)} - \frac{128}{9}S_1\Big]S_{-4}$$

$$+\Big[\frac{32}{3}S_1 - \frac{8(3N^2+3N+2)}{3N(N+1)}\Big]B_4 + \Big[-\frac{64}{9}[S_1^2+S_2] + \frac{64(10N^2+10N+3)}{27N(N+1)}S_1$$

$$-\frac{32(112N^3+224N^2+169N+39)}{81N(N+1)^2}\Big]S_{-3} - \frac{8P_{12}}{81N^2(N+1)^2}S_3$$

$$+\frac{4(311N^2+311N+78)}{27N(N+1)}S_4 - \frac{224}{9}S_5 - \frac{128}{9}S_{-5} - \frac{4(2N^3-35N^2-37N-24)}{9N^3(N+1)^2}[S_1^2+S_2]$$

$$-\frac{8P_{13}}{9N^2(N+1)^2}S_{2,1} + \Big[-\frac{64(112N^4+224N^3+121N^2+9N+9)}{81N^2(N+1)^2}S_1 - \frac{128}{27}S_1^3 + \frac{64}{9N(N+1)}S_1^2$$

$$+\frac{640}{27}S_2 - \frac{256}{27}S_3 + \frac{256}{9}S_{2,1} + \frac{32(181N^4+266N^3+82N^2-3N+18)}{81N^3(N+1)^3}\Big]S_{-2}$$

$$-\frac{128}{3}S_{2,3} + \frac{256}{9}S_{2,-3} - \frac{8(13N+4)(13N+9)}{27N(N+1)}S_{3,1} + \frac{256}{9}S_{4,1} + \Big[-(-1)^N\frac{32(2N^2+2N+1)}{9N^3(N+1)^3}$$

$$-\frac{4P_{14}}{81N^3(N+1)^3} + \frac{496}{27}S_3 - \frac{64}{3}S_{2,1} - \frac{128}{3}S_{-2,1}\Big]S_2 + \frac{64(10N^2+10N-3)}{27N(N+1)}S_{-2,2}$$

$$+\frac{64(112N^3+112N^2-39N+18)}{81N^2(N+1)}S_{-2,1} - \frac{256}{9}S_{-2,3} + \frac{8(3N^2+3N+2)}{N(N+1)}S_{2,1,1} - \frac{256}{9}S_{2,1,-2}$$

$$+\frac{64}{3}S_{2,2,1} - \frac{256}{9}S_{3,1,1} - \frac{256(10N^2+10N-3)}{27N(N+1)}S_{-2,1,1} - \frac{256}{9}S_{-2,2,1} + \frac{224}{9}S_{2,1,1,1}$$

$$+\frac{1024}{9}S_{-2,1,1,1} + \Big[-(-1)^N\frac{8(2N^2+2N+1)}{3N^3(N+1)^3} + \frac{P_{15}}{27N^3(N+1)^3} + \Big(\frac{16}{3N(N+1)} - \frac{32}{3}S_1\Big)S_{-2}$$

$$-\frac{16}{27}S_1 - \frac{88}{9}S_2 - \frac{16}{3}S_3 - \frac{16}{3}S_{-3} + \frac{32}{3}S_{-2,1}\Big]\zeta_2 + \Big[-16S_1^2 + \frac{4(637N^2+637N+108)}{27N(N+1)}S_1$$

$$+\frac{P_{16}}{27N^2(N+1)^2} + 16S_2\Big]\zeta_3\Big\}$$

$$+C_F T_F^2\Big\{-\frac{4P_7}{729N^4(N+1)^4} - \frac{19424}{729}S_1 + \frac{1856}{81}S_2 - \frac{640}{81}S_3 + \frac{128}{27}S_4$$

$$+N_F\Big[\frac{2P_8}{729N^4(N+1)^4} - \frac{55552}{729}S_1 + \frac{640}{27}S_2 - \frac{320}{81}S_3 + \frac{64}{27}S_4\Big]$$

$$+\Big[\frac{4(3N^4+6N^3+47N^2+20N-12)}{27N^2(N+1)^2} - \frac{160}{27}S_1 + \frac{32}{9}S_2\Big](2+N_F)\zeta_2$$

$$+\Big[\frac{256(3N^2+3N+2)}{27N(N+1)} - \frac{1024}{27}S_1 + N_F\Big(\frac{448}{27}S_1 - \frac{112(3N^2+3N+2)}{27N(N+1)}\Big)\Big]\zeta_3\Big\}$$

3-Loop OME: Transversity

$$\begin{aligned}
a_{qq}^{\mathrm{NS,TR}(3)} = {} & C_F^2 T_F \Big\{ \frac{128}{27} S_2 S_1^3 + \Big[\frac{64}{3} S_3 - \frac{128}{9} S_{2,1} - \frac{256}{9} S_{-2,1} - \frac{16}{9N} - \frac{32(-1)^N}{9N(N+1)} \Big] S_1^2 \\
& + \Big[-\frac{64}{9} S_2^2 + \frac{7168 S_2}{81} + \frac{32(-1)^N(13N+7)}{27N(N+1)^2} - \frac{2560 S_3}{27} + \frac{704 S_4}{9} - \frac{320}{9} S_{3,1} \\
& - \frac{2560}{27} S_{-2,1} - \frac{256}{9} S_{-2,2} + \frac{64}{3} S_{2,1,1} + \frac{1024}{9} S_{-2,1,1} \\
& + \frac{8\big(769N^4 + 1547N^3 + 787N^2 - 15N - 12\big)}{27N^2(N+1)^2} \Big] S_1 - \frac{496}{27} S_2^2 \\
& - \frac{16(-1)^N\big(133N^4 + 188N^3 + 46N^2 - 45N - 18\big)}{81N^3(N+1)^3} \\
& - \frac{2\big(6327N^6 + 18981N^5 + 18457N^4 + 5687N^3 - 260N^2 + 144N + 144\big)}{81N^3(N+1)^3} \\
& + \Big[16 - \frac{64}{3} S_1 \Big] B_4 + \Big[\frac{256}{9} S_1 - \frac{1280}{27} \Big] S_{-4} + \Big[96 S_1 - 72 \Big] \zeta_4 + \Big[\frac{128}{9} S_1^2 - \frac{1280}{27} S_1 \\
& + \frac{128}{9} S_2 + \frac{7168}{81} \Big] S_{-3} + \frac{10408}{81} S_3 - \frac{2992}{27} S_4 + \frac{512}{9} S_5 + \frac{256}{9} S_{-5} + \Big[\frac{256}{27} S_1^3 \\
& + \frac{14336}{81} S_1 - \frac{1280}{27} S_2 + \frac{512}{27} S_3 - \frac{512}{9} S_{2,1} - \frac{64}{9N(N+1)} \Big] S_{-2} + \frac{112}{9} S_{2,1} + \frac{256}{9} S_{2,3} \\
& - \frac{512}{9} S_{2,-3} + \frac{1424}{27} S_{3,1} - \frac{512}{9} S_{4,1} - \frac{14336}{81} S_{-2,1} + \Big[-\frac{16\big(169N^2 + 169N + 6\big)}{27N(N+1)} \\
& + \frac{256 S_3}{27} + \frac{256}{3} S_{-2,1} - \frac{32(-1)^N}{9N(N+1)} \Big] S_2 - \frac{1280}{27} S_{-2,2} + \frac{512}{9} S_{-2,3} - 16 S_{2,1,1} + \frac{512}{9} S_{2,1,-2} \\
& + \frac{256}{9} S_{3,1,1} + \frac{5120}{27} S_{-2,1,1} + \frac{512}{9} S_{-2,2,1} - \frac{2048}{9} S_{-2,1,1,1} \\
& + \Big[-\frac{2\big(45N^2 + 45N - 4\big)}{3N(N+1)} + \frac{64}{3} S_{-2} S_1 - 8 S_2 + \Big[\frac{32}{3} S_2 + 40 \Big] S_1 + \frac{32}{3} S_3 + \frac{32}{3} S_{-3} \\
& - \frac{64}{3} S_{-2,1} - \frac{8(-1)^N}{3N(N+1)} \Big] \zeta_2 + \Big[-\frac{1208}{9} S_1 - \frac{64}{3} S_2 + \frac{350}{3} \Big] \zeta_3 \Big\} \\
& + C_F T_F^2 \Big\{ \frac{8\big(157N^4 + 314N^3 + 277N^2 - 24N - 72\big)}{243N^2(N+1)^2} - \frac{19424}{729} S_1 + \frac{1856}{81} S_2 - \frac{640}{81} S_3 \\
& + \frac{128}{27} S_4 + N_F \Big[\frac{32\big(308N^4 + 616N^3 + 323N^2 - 3N - 9\big)}{243N^2(N+1)^2} - \frac{55552}{729} S_1 + \frac{640}{27} S_2 \\
& - \frac{320}{81} S_3 + \frac{64}{27} S_4 \Big] + \Big[-\frac{320}{27} S_1 + \frac{64}{9} S_2 + N_F \Big[-\frac{160}{27} S_1 + \frac{32}{9} S_2 + \frac{16}{9} \Big] + \frac{32}{9} \Big] \zeta_2 \\
& + \Big[-\frac{1024}{27} S_1 + N_F \Big[\frac{448}{27} S_1 - \frac{112}{9} \Big] + \frac{256}{9} \Big] \zeta_3 \Big\}
\end{aligned}$$

$$\begin{aligned}
& + C_A C_F T_F \Big\{ -\frac{64}{27} S_2 S_1^3 + \Big[\frac{4(3N+2)}{9N(N+1)} - \frac{80}{9} S_3 + \frac{128}{9} S_{2,1} + \frac{128}{9} S_{-2,1} + \frac{16(-1)^N}{9N(N+1)} \Big] S_1^2 \\
& + \Big[\frac{112}{9} S_2^2 - \frac{16(N-2)(2N+3)}{9(N+1)(N+2)} S_2 - \frac{16(-1)^N(13N+7)}{27N(N+1)^2} \\
& + \frac{4\big(6197N^3 + 18591N^2 + 15850N + 4320\big)}{729N(N+1)(N+2)} + \frac{320}{9} S_3 - \frac{208}{9} S_4 - 8 S_{2,1} + \frac{64}{3} S_{3,1} \\
& + \frac{1280}{27} S_{-2,1} + \frac{128}{9} S_{-2,2} - 32 S_{2,1,1} - \frac{512}{9} S_{-2,1,1} \Big] S_1 - \frac{20}{3} S_2^2 \\
& + \frac{8(-1)^N\big(133N^4 + 188N^3 + 46N^2 - 45N - 18\big)}{81N^3(N+1)^3} \\
& + \frac{-1013N^6 - 3039N^5 - 5751N^4 - 2981N^3 + 1752N^2 + 1872N + 432}{243N^3(N+1)^3} \\
& + \Big[72 - 96 S_1 \Big] \zeta_4 + \Big[\frac{640}{27} - \frac{128}{9} S_1 \Big] S_{-4} + \Big[\frac{32}{3} S_1 - 8 \Big] B_4 + \Big[-\frac{64}{9} S_1^2 + \frac{640}{27} S_1 \\
& - \frac{64}{9} S_2 - \frac{3584}{81} \Big] S_{-3} - \frac{8\big(27N^3 + 560N^2 + 1365N + 778\big)}{81(N+1)(N+2)} S_3 + \frac{1244}{27} S_4 - \frac{224}{9} S_5 \\
& - \frac{128}{9} S_{-5} - \frac{32\big(3N^3 + 7N^2 + 7N + 6\big)}{9(N+1)(N+2)} S_{2,1} + \Big[-\frac{128}{27} S_1^3 - \frac{7168}{81} S_1 \\
& + \frac{640}{27} S_2 - \frac{256}{27} S_3 + \frac{256}{9} S_{2,1} + \frac{32}{9N(N+1)} \Big] S_{-2} - \frac{128}{3} S_{2,3} + \frac{256}{9} S_{2,-3} - \frac{1352}{27} S_{3,1} \\
& + \frac{256}{9} S_{4,1} + \Big[-\frac{4\big(364N^3 + 1227N^2 + 872N + 36\big)}{81N(N+1)(N+2)} + \frac{496}{27} S_3 - \frac{64}{3} S_{2,1} \\
& - \frac{128}{3} S_{-2,1} + \frac{16(-1)^N}{9N(N+1)} \Big] S_2 + \frac{7168}{81} S_{-2,1} + \frac{640}{27} S_{-2,2} - \frac{256}{9} S_{-2,3} + 24 S_{2,1,1} \\
& - \frac{256}{9} S_{2,1,-2} + \frac{64}{3} S_{2,2,1} - \frac{256}{9} S_{3,1,1} - \frac{2560}{27} S_{-2,1,1} - \frac{256}{9} S_{-2,2,1} + \frac{224}{9} S_{2,1,1,1} \\
& + \frac{1024}{9} S_{-2,1,1,1} + \Big[\frac{2\big(35N^2 + 35N - 6\big)}{9N(N+1)} - \frac{32}{3} S_{-2} S_1 - \frac{16}{27} S_1 - \frac{88}{9} S_2 - \frac{16}{3} S_3 \\
& - \frac{16}{3} S_{-3} + \frac{32}{3} S_{-2,1} + \frac{4(-1)^N}{3N(N+1)} \Big] \zeta_2 + \Big[-16 S_1^2 + \frac{2548 S_1}{27} \\
& + \frac{2\big(108N^3 - 239N^2 - 1137N - 646\big)}{9(N+1)(N+2)} + 16 S_2 \Big] \zeta_3 \Big\}
\end{aligned}$$

3-Loop OME: A_{gq} [[1402.0359], Nucl. Phys. B in print.]

$$a_{gq}^{(3)}(N) = \frac{1}{2}[1+(-1)^N]\Big\{ C_F^2 T_F \Big\{ \bar{p}_{gq}\left(\frac{64}{3}B_4 - 96\zeta_4\right) - 2\Big[-\frac{29}{27}\bar{p}_{gq} S_1^4$$

$$+ \frac{2(275N^4+472N^3+951N^2+598N+96)}{81(N-1)N^2(N+1)^2} S_1^3 + \Big[-\frac{2P_1}{81(N-1)N^3(N+1)^3}$$

$$+ \frac{14}{9}\bar{p}_{gq}S_2\Big] S_1^2 + \Big[-\frac{2(209N^3+376N^2+669N+418)}{27(N-1)N(N+1)^2}S_2 - \frac{4P_0}{243(N-1)N^4(N+1)^4}$$

$$+ \frac{104}{27}\bar{p}_{gq}S_3 - \frac{16}{9}\bar{p}_{gq}S_{2,1}\Big]S_1 + \frac{1}{3}\bar{p}_{gq}S_2^2 + \frac{2P_2}{243(N-2)(N-1)^2N^5(N+1)^5(N+2)^4}$$

$$+ \frac{2P_3}{81(N-2)(N-1)^2N^4(N+1)^4(N+2)^2}S_2 - \frac{64\bar{p}_{gq}}{(N-1)N(N+1)(N+2)}S_{-1}S_2$$

$$- \frac{4P_4}{81(N-1)^2N^3(N+1)^3(N+2)}S_3 + \frac{110}{9}\bar{p}_{gq}S_4 + \Big[\frac{64\bar{p}_{gq}}{(N-1)N(N+1)(N+2)}S_{-1}$$

$$+ \frac{16P_5}{3(N-2)(N-1)^2N^3(N+1)^3(N+2)^2}\Big]S_{-2} - \frac{64\bar{p}_{gq}}{3(N-1)N(N+1)(N+2)}[S_{-3}$$

$$-3S_{2,-1}+3S_{-2,-1}] + \frac{8(35N^3+64N^2+111N+70)}{27(N-1)N(N+1)^2}S_{2,1} - \frac{16}{9}\bar{p}_{gq}[3S_{3,1}-S_{2,1,1}]\Big]$$

$$-2\Big[\frac{2(17N^4+28N^3+69N^2+46N+24)}{9(N-1)N^2(N+1)^2}S_1 + \frac{P_6}{9(N-1)^2N^3(N+1)^3(N+2)^2}$$

$$-\frac{1}{3}\bar{p}_{gq}(10S_1^2-14S_2)\Big]\zeta_2 + 2\Big[\frac{2P_7}{9(N-1)^2N^3(N+1)^3(N+2)} + \frac{152}{9}\bar{p}_{gq}S_1\Big]\zeta_3\Big\}$$

$$+C_F T_F^2\Big\{-2N_F\Big[\frac{8}{27}\bar{p}_{gq}S_1^3 - \frac{8(8N^3+13N^2+27N+16)}{27(N-1)N(N+1)^2}[S_1^2+S_2]$$

$$+\Big[\frac{16(35N^4+97N^3+178N^2+180N+70)}{27(N-1)N(N+1)^3} + \frac{8}{9}\bar{p}_{gq}S_2\Big]S_1$$

$$-\frac{16(1138N^5+4237N^4+8861N^3+11668N^2+8236N+2276)}{243(N-1)N(N+1)^4} + \frac{16}{27}\bar{p}_{gq}S_3\Big]$$

$$-2\Big[3\Big[\frac{16(39N^4+101N^3+201N^2+205N+78)}{81(N-1)N(N+1)^3} + \frac{16}{27}\bar{p}_{gq}S_2\Big]S_1$$

$$-\frac{16(8N^3+13N^2+27N+16)}{27(N-1)N(N+1)^2}[S_1^2+S_2] + \frac{16}{27}\bar{p}_{gq}[S_1^3+2S_3]$$

$$-\frac{8(1129N^5+3814N^4+8618N^3+11884N^2+8425N+2258)}{243(N-1)N(N+1)^4}\Big]$$

$$-2(2+N_F)\Big[\frac{8}{3}\bar{p}_{gq}S_1 - \frac{8(8N^3+13N^2+27N+16)}{9(N-1)N(N+1)^2}\Big]\zeta_2 + \bar{p}_{gq}\Big[\frac{512}{9}-\frac{224}{9}N_F\Big]\zeta_3\Big\}$$

$$+C_A C_F T_F\Big\{\bar{p}_{gq}\left(96\zeta_4 - \frac{32}{3}B_4\right) - 2\Big[\frac{29}{27}\bar{p}_{gq}S_1^4 - \frac{2P_8}{81(N-1)^2N^2(N+1)^2(N+2)}S_1^3$$

$$+\Big[\frac{2P_9}{81(N-1)^2N^3(N+1)^3(N+2)^2} + \frac{58}{9}\bar{p}_{gq}S_2\Big]S_1^2 + \Big[\frac{32}{9}\bar{p}_{gq}\Big[\frac{53}{12}S_3$$

$$+S_{2,1}\Big] - \frac{4P_{10}}{243(N-1)^2N^4(N+1)^4(N+2)^3} - \frac{2P_{11}}{27(N-1)^2N^2(N+1)^2(N+2)}S_2$$

$$-16\bar{p}_{gq}S_{-2,1}\Big]S_1 + \frac{2P_{12}}{243(N-2)(N-1)^2N^5(N+1)^5(N+2)^4} + \frac{61}{9}\bar{p}_{gq}S_2^2 + \frac{16}{9}\bar{p}_{gq}S_{-2}^2$$

$$+\Big[\frac{152}{9}\bar{p}_{gq}S_1 - \frac{8P_{13}}{27(N-1)^2N^2(N+1)^2(N+2)}\Big]S_{-3} + \frac{32\bar{p}_{gq}}{(N-1)N(N+1)(N+2)}S_{-1}S_2$$

$$+\frac{2P_{14}}{27(N-2)(N-1)^2N^3(N+1)^3(N+2)^2}S_2 - \frac{8P_{15}}{81(N-1)^2N^2(N+1)^2(N+2)}S_3$$

$$+\frac{178}{9}\bar{p}_{gq}S_4 + \Big[\frac{88}{9}\bar{p}_{gq}[S_1^2+S_2] - \frac{16(52N^4+95N^3+210N^2+137N+36)}{27(N-1)N^2(N+1)^2}S_1$$

$$-\frac{32\bar{p}_{gq}}{(N-1)N(N+1)(N+2)}S_{-1} + \frac{8P_{16}}{27(N-2)(N-1)^2N^3(N+1)^3(N+2)^2}\Big]S_{-2}$$

$$-\frac{8(14N^5+15N^4+4N^3+81N^2-10N+88)}{9(N-1)^2N^2(N+1)^2(N+2)}S_{2,1} - \frac{32\bar{p}_{gq}}{(N-1)N(N+1)(N+2)}S_{2,-1}$$

$$-\frac{16}{3}S_{3,1} + \frac{160}{9}\bar{p}_{gq}S_{-4} + \frac{16(26N^4+49N^3+126N^2+85N+36)}{27(N-1)N^2(N+1)^2}S_{-2,1} - \frac{112}{9}\bar{p}_{gq}S_{-2,2}$$

$$+\frac{32\bar{p}_{gq}}{(N-1)N(N+1)(N+2)}S_{-2,-1} - \frac{136}{9}S_{-3,1} - 8\bar{p}_{gq}S_{2,1,1} + \frac{176}{9}\bar{p}_{gq}S_{-2,1,1}\Big]$$

$$-\frac{16}{9}\frac{(-2-2N-N^2+N^3)}{(N-1)N(N+1)}[17S_{-3,1}+6S_{3,1}]$$

$$-2\Big[\bar{p}_{gq}\left(\frac{10}{3}S_1^2+2S_2+4S_{-2}\right) - \frac{2(59N^5+94N^4+59N^3-84N^2-224N+168)}{9(N-1)^2N^2(N+1)(N+2)}S_1$$

$$+\frac{2P_{17}}{9(N-1)N^3(N+1)^3(N+2)^2}\Big]\zeta_2 - 2\Big[\frac{2P_{18}}{9(N-1)^2N^2(N+1)^2(N+2)} + \frac{56}{9}\bar{p}_{gq}S_1\Big]\zeta_3\Big\}\Big\},$$

3-Loop OME: A_{Qq}^{PS}

$$
\begin{aligned}
a_{Qq}^{(3),\mathrm{PS}}(N) \quad = \quad & C_F^2 T_F \Bigg\{ \frac{64(N^2+N+2)^2}{(N-1)N^2(N+1)^2(N+2)} S_{2,2}\left(2, \frac{1}{2}\right) - \frac{64(N^2+N+2)^2}{(N-1)N^2(N+1)^2(N+2)} S_{3,1}\left(2, \frac{1}{2}\right) \\
& + 2^N \left[- \frac{32 P_3 S_{2,1}\left(1, \frac{1}{2}, N\right)}{(N-1)^2 N^3 (N+1)^2 (N+2)} - \frac{32 P_3 S_{1,1,1}\left(\frac{1}{2}, 1, 1, N\right)}{(N-1)^2 N^3 (N+1)^2 (N+2)} + \frac{32 P_4 S_{1,1}\left(1, \frac{1}{2}, N\right)}{(N-1)^3 N^4 (N+1)^2 (N+2)} + \cdots \right] \\
& + 2^{-N} \left[- \frac{64(N^2+N+2)^2 S_{1,2,1}\left(2, \frac{1}{2}, 1, N\right)}{(N-1)N^2(N+1)^2(N+2)} + \frac{64(N^2+N+2)^2 S_{1,2,1}\left(2, 1, \frac{1}{2}, N\right)}{(N-1)N^2(N+1)^2(N+2)} + \cdots \right] + \cdots \Bigg\} \\
& + C_F T_F^2 N_F \Bigg\{ - \frac{16(N^2+N+2)^2 S_1(N)^3}{27(N-1)N^2(N+1)^2(N+2)} + \frac{16 P_9 S_1(N)^2}{27(N-1)N^3(N+1)^3(N+2)^2} \\
& + \left[- \frac{208(N^2+N+2)^2}{9(N-1)N^2(N+1)^2(N+2)} S_2 - \frac{32 P_{23}}{81(N-1)N^4(N+1)^4(N+2)^3} \right] S_1 \\
& + \frac{32 P_{31}}{243(N-1)N^5(N+1)^5(N+2)^4} + \frac{224(N^2+N+2)^2}{9(N-1)N^2(N+1)^2(N+2)} \zeta_3 + \cdots \Bigg\} \\
& + C_F C_A T_F \Bigg\{ \frac{2(N^2+N+2)^2 S_1(N)^4}{9(N-1)N^2(N+1)^2(N+2)} + \frac{4(N^2+N+2) P_6 S_1(N)^3}{27(N-1)^2 N^3 (N+1)^3 (N+2)^2} \\
& + 2^{-N} \left[\frac{16 P_2 S_3(2, N)}{(N-1)N^3(N+1)^2} - \frac{16 P_2 S_{1,2}(2, 1, N)}{(N-1)N^3(N+1)^2} + \frac{16 P_2 S_{2,1}(2, 1, N)}{(N-1)N^3(N+1)^2} - \frac{16 P_2 S_{1,1,1}(2, 1, 1, N)}{(N-1)N^3(N+1)^2} \right] \\
& - \frac{32(N^2+N+2)^2 S_{1,1,2}\left(2, \frac{1}{2}, 1, N\right)}{(N-1)N^2(N+1)^2(N+2)} + \frac{32(N^2+N+2)^2 S_{1,1,2}\left(2, 1, \frac{1}{2}, N\right)}{(N-1)N^2(N+1)^2(N+2)} \\
& + \frac{32(N^2+N+2)^2 S_{1,2,1}\left(2, \frac{1}{2}, 1, N\right)}{(N-1)N^2(N+1)^2(N+2)} - \frac{32(N^2+N+2)^2 S_{1,2,1}\left(2, 1, \frac{1}{2}, N\right)}{(N-1)N^2(N+1)^2(N+2)} \\
& - \frac{32(N^2+N+2)^2 S_{1,1,1,1}\left(2, \frac{1}{2}, 1, 1, N\right)}{(N-1)N^2(N+1)^2(N+2)} - \frac{32(N^2+N+2)^2 S_{1,1,1,1}\left(2, 1, \frac{1}{2}, 1, N\right)}{(N-1)N^2(N+1)^2(N+2)} + \cdots \Bigg\} + \cdots
\end{aligned}
$$

Conclusions

- 2009: 10-14 Mellin Moments for all massive 3-loop OMEs, WC.
  2010: Wilson Coefficients $L_q^{(3),\mathrm{PS}}(N)$, $L_g^{(3),\mathrm{S}}(N)$.
- All logarithmic contributions to the Wilson Coefficients were calculated (including those of the OMEs).
- 2013: Ladder, V-Graph and Benz-topologies for graphs, with no singularities in ε can be systematically calculated for general N.
- Here new functions occur (including a larger number of root-letters in iterated integrals)
- $L_q^{\mathrm{NS},(3)}$, $H_q^{\mathrm{PS},(3)}$, $A_{gq,Q}^{\mathrm{S},(3)}$, and $A_{qq,Q}^{\mathrm{NS,TR}(3)}$ have been completed.
- The corresponding 3-loop anomalous dimensions were computed, those for transversity for the first time ab initio.
- Different new Computer-algebra and mathematical technologies were developed.
- Last Sunday: $O(T_F^2 C_{F,A})$ corrections to $A_{gg,Q}^{(3)}$ completed.