The $O(\alpha^2)$ Initial State Radiation to e^+e^- Annihilation into a Neutral Vector Boson Revisited

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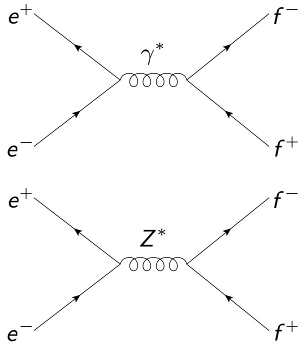
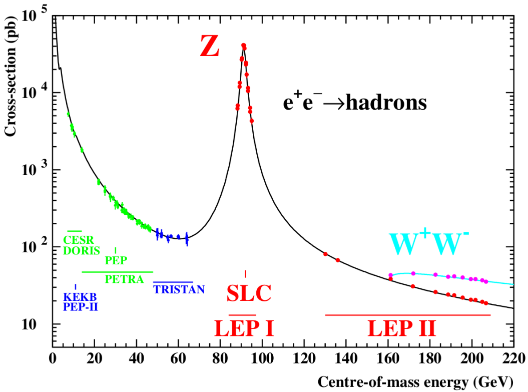
DESY, Zeuthen & JKU Linz

RADCOR 2019, Avignon, France, September 2019

based on:

J. Blümlein, A. De Freitas, C. Raab and K. Schönwald, Phys.Lett. B791 (2019) 206-209; DESY 18–196.
J. Blümlein, A. De Freitas and W.L. van Neerven, Nucl. Phys. B855 (2012) 508-560.





- We revisit the initial state corrections to $e^+ e^-$ annihilation to a neutral (virtual) vector boson.
- These corrections are important for the prediction of the Z-boson peak and for $t\,\bar{t}$ production at LEP, ILC and FCC-ee, and at Higgs factories through $e^+ e^- \rightarrow Z^* H^0$ and scanning the $t\bar{t}$-threshold.

We calculate the complete $O(\alpha^2)$ corrections to this process.

Why do we have to revisit these matters ?

ISR corrections have been calculated up to $O(\alpha^2)$ in:

Nuclear Physics B297 (1988) 429–478
North-Holland, Amsterdam

HIGHER ORDER RADIATIVE CORRECTIONS AT LEP ENERGIES

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Received 21 August 1987

A complete two-loop $O(\alpha^2)$ initial state radiative correction to the Z-resonance shape is presented. The correction is compared with those expressions where only the soft-photon effects are resummed in all orders of perturbation theory. Our result shows that the soft-photon part constitutes the bulk of the radiative correction near the top of the Z-peak. The effect of non-photonic QED processes on the Z-resonance is found to be very small. The above results have been obtained by means of a standard Feynman diagram calculation. In addition we have also computed the cross sections by using the renormalization group method, where besides the leading logs $\ln(s/m^2)$, the next-to-leading ones also have been taken into account.

1. Introduction

Electron-positron colliding beam experiments which will be carried out in the near future at SLC and LEP will provide us with a wealth of information about the standard model of the electroweak interactions. A vast amount of literature [1] exists about the subjects one wants to investigate. We just mention topics such as the search for new particles like the Higgs boson, the top quark and maybe some supersymmetric partners of the particles in the standard model. Furthermore, one wants to make precise determinations of the electroweak parameters among which the most interesting are the mass and width of the Z boson. For the study of new physics effects, it is of the utmost importance to measure these quantities with a very high degree of accuracy. As has been extensively discussed in refs. [2, 3], the determination of the mass and width of the Z boson will be greatly affected by radiative corrections. In particular the pure QED part of the standard model leads to a distortion and a shift of the resonance peak. Up to now complete one-loop radiative corrections of the process $e^+e^- \to \mu^+\mu^-$ in which the Z is produced have been carried out [4]. The calculations reveal that the bulk of the corrections can be attributed to photonic contributions from the initial electron-positron state. Exponentiation of the lowest-order soft photon contribution shows [2, 3] that higher-order

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HADRONIC CONTRIBUTIONS TO $O(\alpha^2)$ RADIATIVE CORRECTIONS IN e^+e^- ANNIHILATION

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The complete hadronic contribution to $O(\alpha^2)$ real and virtual corrections from initial state radiation in e^+e^- annihilation is calculated using experimental information on the imaginary part of the hadronic vacuum polarization $\mathrm{Im}\,\Pi(q^2)$. For high energies both real and virtual corrections are expressed by four moments of $\mathrm{Im}\,\Pi(q^2)$ which characterize its behaviour in the low- and high-q^2 region. The formalism is applied to study the influence of hadronic radiation on the Z line shape and on the cross section in the continuum.

1. Introduction

Measurements of the total cross section for e^+e^- annihilation have reached a level of precision [1] where the influence of higher order radiative corrections is no longer negligible. Also the determination of the Z mass and width through the resonance line shape at future e^+e^- colliders will be influenced by radiative corrections and again the treatment to $O(\alpha)$ is insufficient [2].

In these reactions radiative corrections are dominantly due to initial-state radiation. To $O(\alpha^2)$ purely photonic contributions as well as those from real and virtual leptonic and hadronic states are relevant. For the leptonic and soft photonic case terms that are enhanced by powers of $\ln s/m^2$ have been calculated in ref. [3] and the remaining constant terms – typically of order 10^{-5} – are given in refs. [4,5].

Hadronic corrections are known to contribute approximately 50% to the large logarithms that appear in the vacuum polarization $\Pi(q^2)$ for large q^2. They are therefore expected to be as important in the high energy region for $O(\alpha^2)$ vertex corrections as the aforementioned large leptonic terms. Just as for $\Pi(q^2)$ these $O(\alpha^2)$ hadronic corrections are also determined by the quantity $R(s) \equiv \sigma_{had}/\sigma_{point}$ measured in lower energy e^+e^- collisions, assuming that $R(s)$ approaches a constant value for large s.

In the following we derive an expression for the hadronic contribution to the virtual $O(\alpha^2)$ corrections which is valid for arbitrary q^2. It becomes particularly simple in the large-q^2 region. The information contained in $R(s)$ can be condensed in its asymptotic behaviour together with three moments which fix the coefficients of the $\ln^n q^2/m^2$ terms. A similar approach will be developed for real soft and hard hadron radiation which in the high energy region depends on the same moments. The formalism can be easily applied to the special case of lepton radiation and reproduces earlier results for virtual radiation [3,4] and the logarithmically enhanced terms from real radiation [3]. However, in the latter case, we disagree with ref. [5] in subleading terms.

A special situation arises close to the Z peak. The Born cross section changes rapidly when the energy varies by $\Gamma_Z/2 \approx 1.3$ GeV. On the other hand, hadron production in e^+e^- annihilation has its threshold at $2m_\pi$, peaks at about 800 MeV and shows up drastic variations up to 4 GeV. The "soft" approximation that can be justified

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[2] Supported in part by Ministry of Science and Higher Education, Research Problem CPBP 01.03.

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(North-Holland Physics Publishing Division) 337

Berends et al.: Complete $O(\alpha^2)$ ISR, (386 citations)
Kniehl et al.: so-called $O(\alpha^2)$ NS-corrections (process II of BBN).

Are these widely used results correct ?

Why do we have to revisit these matters ?

ISR corrections have been calculated up to $O(\alpha^2)$ in the asymptotic limit $m_e^2/s \ll 1$ with two different techniques:

1. Berends, Burgers, van Neerven (Nucl. Phys. B297 (1988))
   - full calculation with massive electrons in the limit $m_e^2 \ll s$ calculation in $d = 4$ with soft-hard separation, including soft and virtual photons, hard bremsstrahlung, as well as fermion pair production
   - Calculation Technique:

     o direct integration over the phase space in $d = 4$ with soft-hard photon separator

     o expansion in $m_e^2 \ll s$ on integrand level
2. Blümlein, De Freitas, van Neerven (Nucl. Phys. B855 (2012))
   - direct calculation of the asymptotic limit using operator matrix elements
   - technique based on asymptotic factorization
   - already used in Berends et al. to check the logarithmically enhanced terms only

Factorization in the Asymptotic Region

In the asymptotic region the cross section factorizes

$$\frac{d\sigma_{ij}(s')}{ds'} = \frac{\sigma^{(0)}(s')}{s} \sum_{l,k} \Gamma_{l,i}\left(z, \frac{\mu^2}{m_e^2}\right) \otimes \tilde{\sigma}_{lk}\left(z, \frac{s'}{\mu^2}\right) \otimes \Gamma_{k,j}\left(z, \frac{\mu^2}{m_e^2}\right)$$

into

- massless cross sections $\tilde{\sigma}_{ij}\left(z, \frac{s'}{\mu^2}\right)$

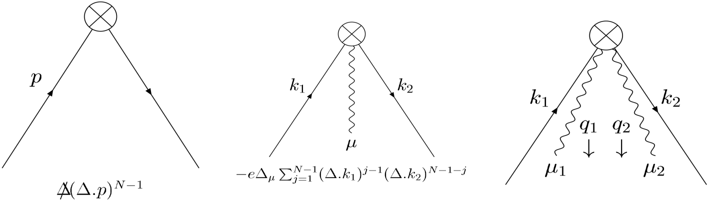
  Hamberg, van Neerven, Matsuura (Nucl. Phys. B 359 (1991))

  Harlander, Kilgore (Phys. Rev. Lett. 88 (2002))
- massive operator matrix elements $\Gamma_{ij}\left(z, \frac{\mu^2}{m_e^2}\right)$, which carry all mass dependence

  Blümlein, De Freitas, van Neerven (Nucl.Phys. B855 (2012))

$\sigma^{(0)}(s')$ is the Born cross section and the convolution $\otimes$ is given by

$$f(z) \otimes g(z) = \int_0^1 dz_1 \int_0^1 dz_2\, f(z_1) g(z_2) \delta(z - z_1 z_2).$$



$$\Gamma_{e^+e^+} = \Gamma_{e^-e^-} = \langle e|\, O_F^{\mathrm{NS,S}}\, |e\rangle\,,$$
$$\Gamma_{e^+\gamma} = \Gamma_{e^-\gamma} = \langle \gamma|\, O_F^{\mathrm{S}}\, |\gamma\rangle\,,$$
$$\Gamma_{\gamma e^+} = \Gamma_{\gamma e^-} = \langle e|\, O_V^{\mathrm{S}}\, |e\rangle\,,$$

$$O_{F;\mu_1,\ldots,\mu_N}^{\mathrm{NS,S}} = i^{N-1} \mathrm{S}\left[\bar{\psi}\gamma_{\mu_1} D_{\mu_2}\ldots D_{\mu_N}\psi\right] - \text{traces},$$
$$O_{V;\mu_1,\ldots,\mu_N}^{\mathrm{S}} = 2 i^{N-2} \mathrm{S}\left[F_{\mu_1\alpha} D_{\mu_2}\ldots D_{\mu_{N-1}} F_{\mu_N}^{\alpha}\right] - \text{traces}$$

- technique has been used to derive deep-inelastic scattering (DIS) structure functions in the asymptotic limit $Q^2 \gg m^2$ up to $O(\alpha_s^3)$
- in the context of DIS proven to work at α_s^2 in the
  - non-singlet process
    Buza, Matiounine, Smith, van Neerven (Nucl.Phys. B485 (1997))
    Blümlein, Falcioni, De Freitas (Nucl.Phys. B910 (2016))
  - pure-singlet process
    Blümlein, De Freitas, Raab, Schönwald (Nucl. Phys B (2019), in print, arXiv:1903.06155)

  through analytic calculations

The comparison between both calculations shows:

- the one-loop, i.e. $O(\alpha)$, agrees between both calculations
- the logarithmically enhanced terms at two-loops ($O(\alpha^2)$) agree between both calculations
- the constant terms do not agree

$\Rightarrow$ breakdown of asymptotic factorization or errors?

The solutions of these renormalization group equations are:

$$\Gamma_{ee}\left(N,a,\frac{\mu^2}{m^2}\right) = 1 + a\left[-\frac{1}{2}\gamma_{ee}^{(0)}L + \Gamma_{ee}^{(0)}\right] + a^2\left[\left\{\frac{1}{8}\gamma_{ee}^{(0)}\left(\gamma_{ee}^{(0)} - 2\beta_0\right) + \frac{1}{8}\gamma_{e\gamma}^{(0)}\gamma_{\gamma e}^{(0)}\right\}L^2 + \frac{1}{2}\left\{-\gamma_{ee}^{(1)} + 2\beta_0\Gamma_{ee}^{(0)} - \gamma_{ee}^{(0)}\Gamma_{ee}^{(0)} - \gamma_{e\gamma}^{(0)}\Gamma_{\gamma e}^{(0)}\right\}L + \Gamma_{ee}^{(1)}\right] + O(a^3),$$

$$\tilde{\sigma}_{ee}\left(N,a,\frac{s'}{\mu^2}\right) = 1 + a\left[-\frac{1}{2}\gamma_{ee}^{(0)}\lambda + \tilde{\sigma}_{ee}^{(0)}\right] + a^2\left[\left\{\frac{1}{2}\gamma_{ee}^{(0)}\left(\gamma_{ee}^{(0)} + \beta_0\right) + \frac{1}{4}\gamma_{e\gamma}^{(0)}\gamma_{\gamma e}^{(0)}\right\}\lambda^2 + 1 + \left\{-\gamma_{ee}^{(1)} - \beta_0\tilde{\sigma}_{ee}^{(0)} - \gamma_{ee}^{(0)}\tilde{\sigma}_{ee}^{(0)} - \gamma_{\gamma e}^{(0)}\tilde{\sigma}_{e\gamma}^{(0)}\right\}\lambda + \tilde{\sigma}_{ee}^{(1)}\right] + O(a^3),$$

$$\Gamma_{\gamma e}\left(N,a,\frac{\mu^2}{m^2}\right) = a\left[-\frac{1}{2}\gamma_{\gamma e}^{(0)}L + \Gamma_{\gamma e}^{(0)}\right] + O(a^2)$$

$$\tilde{\sigma}_{e\gamma}\left(N,a,\frac{\mu^2}{m^2}\right) = a\left[-\frac{1}{2}\gamma_{e\gamma}^{(0)}\lambda + \tilde{\sigma}_{e\gamma}^{(0)}\right] + O(a^2)\ ,$$

with the logarithms $L = \ln\left(\frac{\mu^2}{m^2}\right)$ and $\lambda = \ln\left(\frac{s'}{\mu^2}\right)$

Different ingredients to the calculation :

- Splitting functions P_{ij} to $O(\alpha^2)$

E.G. Floratos, D.A. Ross and C.T. Sachrajda, Nucl. Phys. B **129** (1977) 66 [Erratum-ibid. B **139** (1978) 545];
Nucl. Phys. B **152** (1979) 493;
A. Gonzalez-Arroyo, C. Lopez and F.J. Yndurain, Nucl. Phys. B **153** (1979) 161;
A. Gonzalez-Arroyo and C. Lopez, Nucl. Phys. B **166** (1980) 429;
E.G. Floratos, C. Kounnas and R. Lacaze, Nucl. Phys. B **192** (1981) 417;
G. Curci, W. Furmanski and R. Petronzio, Nucl. Phys. B **175** (1980) 27;
W. Furmanski and R. Petronzio, Phys. Lett. B **97** (1980) 437;
R. Hamberg and W.L. van Neerven, Nucl. Phys. B **379** (1992) 143;
R.K. Ellis and W. Vogelsang, arXiv:hep-ph/9602356;
S. Moch and J.A.M. Vermaseren, Nucl. Phys. B **573** (2000) 853;
J. Ablinger *et al.*, Nucl. Phys. B **882** (2014) 263; Nucl. Phys. B **886** (2014) 733; Nucl. Phys. B **890** (2014) 48;
Nucl. Phys. B **922** (2017) 1.

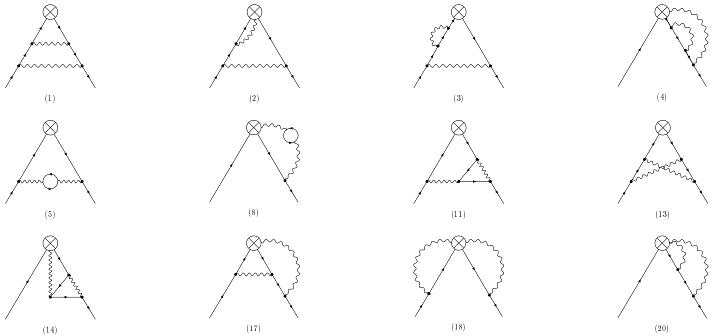
- massless Drell-Yan Cross Section $\tilde{\sigma}_{ij}$ to $O(\alpha^2)$

R. Hamberg, W.L. van Neerven and T. Matsuura, Nucl. Phys. B **359** (1991) 343 [E: B **644** (2002) 403];
R.V. Harlander and W.B. Kilgore, Phys. Rev. Lett. **88** (2002) 201801.

- massive OMEs Γ_{ij} to $O(\alpha^2)$ $\Longrightarrow$ our 2011 paper.

(Some errors at $O(\alpha)$ in earlier work corrected.)

The contributing processes

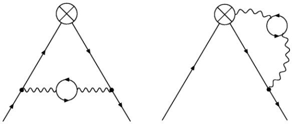


Two-loop diagrams contributing to the massive operator matrix element $A_{ee}(N, \alpha)$.
The antisymmetric diagrams count twice.

The result for the the matrix element $\hat{\Gamma}_{ee}^{(1),\mathrm{I}}$ is [corresponding to BBN: I+IV]

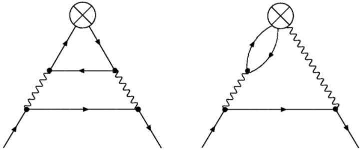
$$\frac{1+3x^2}{1-x}\left[6\zeta_2\ln(x)-8\ln(x)\mathrm{Li}_2(1-x)-4\ln^2(x)\ln(1-x)\right]+\left(\frac{122}{3}x+22+\frac{32}{1-x}\right)\zeta_2+(8-112\zeta_2)\,\mathcal{D}_1(x)$$
$$+16\frac{1+x^2}{1-x}\left[2\mathrm{Li}_3(-x)-\ln(x)\mathrm{Li}_2(-x)\right]+\frac{80}{3(1-x)}+56(1+x)\zeta_2\ln(1-x)+(16-52\zeta_2+128\zeta_3)\,\mathcal{D}_0(x)$$
$$+\left(\frac{22}{3}x+32+\frac{64}{3(1-x)^2}-\frac{51}{1-x}-\frac{16}{3(1-x)^3}\right)\ln^2(x)-(92+20x)\ln^2(1-x)+14(x-2)\ln(1-x)+120\mathcal{D}_2(x)$$
$$+\left(\frac{178}{3}-36x+\frac{64}{3(1-x)^2}-\frac{140}{3(1-x)}-\frac{48}{1+x}\right)\ln(x)-\frac{1}{3}(1+x)\ln^3(x)+4\frac{x^2-8x-6}{1-x}\ln(x)\ln(1-x)$$
$$-2\frac{1+17x^2}{1-x}\ln(x)\ln^2(1-x)-\frac{112}{3}(1+x)\ln^3(1-x)+32\frac{1+x}{1-x}\left[\ln(x)\ln(1+x)+\mathrm{Li}_2(-x)\right]-22x-\frac{62}{3}$$
$$-4\frac{13x^2+9}{1-x}\mathrm{S}_{1,2}(1-x)+4\frac{5-11x^2}{1-x}\left[\ln(1-x)\mathrm{Li}_2(1-x)-\mathrm{Li}_3(1-x)-2\zeta_3\right]+\frac{4(16x^2-10x-27)}{3(1-x)}\mathrm{Li}_2(1-x)$$
$$+\frac{224}{3}\mathcal{D}_3(x)+\left[\frac{433}{8}-\frac{67}{45}\pi^4+\left(\frac{37}{2}-48\ln(2)\right)\zeta_2+58\zeta_3\right]\delta(1-x)+(-1)^n\left\{\frac{2(1-x)(45x^2+74x+45)}{3(1+x)^2}\right.$$
$$+\frac{2(9+12x+30x^2-20x^3-15x^4)}{3(1+x)^3}\ln(x)+\frac{4(x^2+10x-3)}{3(1+x)}\left(\zeta_2+2\mathrm{Li}_2(-x)+2\ln(x)\ln(1+x)\right)$$
$$+\frac{1+x^2}{1+x}\left[36\zeta_3-24\zeta_2\ln(1+x)+8\zeta_2\ln(x)-\frac{2}{3}\ln^3(x)+40\mathrm{Li}_3(-x)-4\ln^2(x)\ln(1+x)-24\ln(x)\ln^2(1+x)\right.$$
$$\left.-24\ln(x)\mathrm{Li}_2(-x)-48\ln(1+x)\mathrm{Li}_2(-x)-8\ln(x)\mathrm{Li}_2(1-x)-16\mathrm{S}_{1,2}(1-x)-48\mathrm{S}_{1,2}(-x)\right]$$
$$\left.-\frac{16(x^4+12x^3+12x^2+8x+3)}{3(1+x)^3}\mathrm{Li}_2(1-x)+4x\frac{1-x-5x^2+x^3}{(1+x)^3}\ln^2(x)\right\}$$

Process II



The result for $\hat{\Gamma}_{ee}^{(1),\mathrm{II}}$ is

$$\begin{aligned}\hat{\Gamma}_{ee}^{(1),\mathrm{II}} \quad = \quad & \frac{76}{27}x - \frac{572}{27} - \left(12x + \frac{4}{3} + \frac{8}{1-x}\right)\ln(x) + \frac{128}{9(1-x)^2} + \frac{80}{27(1-x)} - \frac{64}{9(1-x)^3} \\ & - \frac{32}{9}\left(\frac{1}{(1-x)^2} - \frac{5}{(1-x)^3} + \frac{2}{(1-x)^4}\right)\ln(x) + \frac{16}{3}(1+x)\left(\ln(1-x) + \ln^2(1-x)\right) \\ & - \frac{2(1+x^2)}{3(1-x)}\ln^2(x) + \left(\frac{224}{27} - \frac{8}{3}\zeta_2\right)\mathcal{D}_0(x) + \frac{4}{3}(1+x)\zeta_2 - \frac{32}{3}\left(\mathcal{D}_1(x) + \mathcal{D}_2(x)\right) \\ & + \left(\frac{8}{3}\zeta_3 + 10\zeta_2 - \frac{1411}{162}\right)\delta(1-x)\end{aligned}$$



(a) (b)

The result for $\hat{\Gamma}_{ee}^{(1),\mathrm{III}}$ is

$$
\begin{aligned}
\hat{\Gamma}_{ee}^{(1),\mathrm{III}} = \; & \frac{2}{x}(1-x)(4x^2+13x+4)\zeta_2 + \frac{1}{3x}(8x^3+135x^2+75x+32)\ln^2(x) \\
& + \left[\frac{304}{9x} - \frac{80}{9}x^2 - \frac{32}{3}x + 108 - \frac{32}{1+x} - \frac{64(1+2x)}{3(1+x)^3}\right]\ln(x) - \frac{224}{27}x^2 \\
& + 16\frac{1-x}{3x}(x^2+4x+1)\left[2\ln(x)\ln(1+x) - \mathrm{Li}_2(1-x) + 2\mathrm{Li}_2(-x)\right] \\
& + (1+x)\Bigg[4\zeta_2\ln(x) + \frac{14}{3}\ln^3(x) - 32\ln(x)\mathrm{Li}_2(-x) - 16\ln(x)\mathrm{Li}_2(x) + 64\mathrm{Li}_3(-x) \\
& + 32\mathrm{Li}_3(x) + 16\zeta_3\Bigg] - \frac{182}{3}x + 50 - \frac{32}{1+x} + \frac{800}{27x} + \frac{64}{3(1+x)^2}
\end{aligned}
$$

The first moment vanishes for all three contributions. .

$\rightarrow$ Fermion number conservation is satisfied.

The 2–loop corrections to the process $e^+e^- \to Z^0$ can be organized in the following form :

$$\frac{d\sigma_{e^+e^-}}{ds'} = \frac{1}{s}\sigma^{(0)}(s)\left\{1 + a_0\left[T_{11}\hat{\mathbf{L}} + T_{10}\right] + a_0^2\left[T_{22}\hat{\mathbf{L}}^2 + T_{21}\hat{\mathbf{L}} + T_{20}\right]\right\}$$

- Universal Corrections : $T_{ii}(z)$ $\Longrightarrow$ depend on LO splitting functions and β_0

$$\begin{aligned}
T_{11} &= 8\mathcal{D}_0(z) - 4(1+z) + 6\delta(1-z) = 4\left[\frac{1+z^2}{1-z}\right]_+ ,\\
T_{22} &= \Bigg\{64\mathcal{D}_1(z) + 48D_0(z) + (18 - 32\zeta_2)\delta(1-z)\\
&\quad -32\frac{\ln(z)}{1-z} - 32(1+z)\ln(1-z) + 24(1+z)\ln(z) - 8(5+z)\Bigg\}_{\mathrm{I}}\\
&\quad +\frac{2}{3}\Big\{8\mathcal{D}_0(z) - 4(1+z) + 6\delta(1-z)\Big\}_{\mathrm{II}}\\
&\quad +16\left\{\frac{1}{2}(1-z)\ln(z) + \frac{1}{4}(1-z) + \frac{1}{3}\frac{1}{3z}(1-z^3)\right\}_{\mathrm{III}} .
\end{aligned}$$

Assembling the differential scattering cross section

- $O(\alpha)$ Term : $T_{10}(z)$ $\Longrightarrow$ depend on LO OME + LO DY

$$T_{10} = -4\left[\frac{1+z^2}{1-z}\right]_+ + 2(4\zeta_2 - 1)\delta(1-z)$$

$$T_{11}\hat{\mathbf{L}} + T_{10} = P_{ee}^{(0)}(z)\left[\hat{\mathbf{L}} - 1\right] + 2(4\zeta_2 - 1)\delta(1-z)\ .$$

Complete 1-Loop Result.

- $O(\alpha^2\hat{\mathbf{L}})$ Terms : $T_{21}(z)$ $\Longrightarrow$ depend on LO,NLO splitting fcts., LO OME + LO DY

Contributions to the three main processes I-III :

$$T_{21}^{\mathrm{I}} = 16\Bigg\{-8\mathcal{D}_1(z) - (7-4\zeta_2)\mathcal{D}_0(z) + \left(-\frac{45}{16} + \frac{11}{2}\zeta_2 + 3\zeta_3\right)\delta(1-z)$$
$$+\left(\frac{1+z^2}{1-z}\right)\left[\ln(z)\ln(1-z) - \ln^2(z) + \frac{11}{4}\ln(z)\right]$$
$$+(1+z)\left[4\ln(1-z) + \frac{1}{4}\ln^2(z) - \frac{7}{4}\ln(z) - 2\zeta_2\right] - \ln(z) + 3 + 4z\Bigg\}$$

$$
\begin{aligned}
T_{21}^{\mathrm{II}} &= 16\Bigg\{\frac{4}{3}\mathcal{D}_1(z) - \frac{10}{9}\mathcal{D}_0(z) - \frac{17}{12}\delta(1-z) \\
&\quad - \frac{2}{3}\frac{\ln(z)}{1-z} - \frac{1}{3}(1+z)\left[2\ln(1-z) - \ln(z)\right] - \frac{1}{9} + \frac{11}{9}z\Bigg\} \\
T_{21}^{\mathrm{III}} &= 16\Bigg\{(1+z)\left[2\mathrm{Li}_2(1-z) - \ln^2(z) + 2\ln(z)\ln(1-z)\right] \\
&\quad + \left(\frac{4}{3}\frac{1}{z} + 1 - z - \frac{4}{3}z^2\right)\ln(1-z) - \left(\frac{2}{3}\frac{1}{z} + 1 - \frac{1}{2}z - \frac{4}{3}z^2\right)\ln(z) \\
&\quad - \frac{8}{9}\frac{1}{z} - \frac{8}{3} + \frac{8}{3}z + \frac{8}{9}z^2\Bigg\}
\end{aligned}
$$

Up to this point, we find agreement with Berends et al. (1988).

Our Approach to the recalculation:

- full integration over the phase space in $d = 4$, i.e. no a-priori expansion in the mass
  - this is a four-fold integration, three integrals can be performed using standard techniques
  - integrand of the last integral contains rational, logarithmic and polylogarithmic expressions with involved argument structures
  - the last integration is performed in terms of iterated integrals after determining the minimal set of contributing letters
  - the final result is expressed as iterated integrals over involved square root valued letters.
  - One obtains very voluminous expressions (several Mb) including roots of roots etc to be dealt with.

$\Rightarrow$ the analytic results can be expanded in the electron mass

In the following we use $r = \rho = m_e^2/s; \quad z = s'/s$ and t denotes an integration variable.

$O(\alpha^2)$: New Letters

$$
\begin{aligned}
f_{d_1} &= \frac{1}{\sqrt{(1-t)(16r^2-8r(1+z)t+(1-z)^2t^2)}} \\
f_{d_2} &= \frac{t}{\sqrt{(1-t)(16r^2-8r(1+z)t+(1-z)^2t^2)}} \\
f_{d_3} &= \frac{1}{t\sqrt{(1-t)(16r^2-8r(1+z)t+(1-z)^2t^2)}} \\
f_{d_4} &= \frac{1}{(16r^2+(4z-8r(1+z))t+(1-z)^2t^2)\sqrt{(1-t)(16r^2-8r(1+z)t+(1-z)^2t^2)}} \\
f_{d_5} &= \frac{t}{(16r^2+(4z-8r(1+z))t+(1-z)^2t^2)\sqrt{(1-t)(16r^2-8r(1+z)t+(1-z)^2t^2)}} \\
f_{d_6} &= \frac{1}{(16r^2+(4z-8r(1+z))t+(1-z)^2t^2)\sqrt{16r^2-8r(1+z)t+(1-z)^2t^2}} \\
f_{d_7} &= \frac{t}{(16r^2+(4z-8r(1+z))t+(1-z)^2t^2)\sqrt{16r^2-8r(1+z)t+(1-z)^2t^2}} \\
f_{d_8} &= \frac{1-z}{(4r-(1-z)t)\sqrt{1-t}} \\
f_{w_3} &= \frac{1}{t\sqrt{1-t}}\,.
\end{aligned}
$$

.... in total 37 letters of this kind;

+ HPL letters

Iterative integrals:

$$
\mathrm{H}^*_{a,\vec{b}}(t) = \int_t^1 dy f_a(y) \mathrm{H}^*_{\vec{b}}(y); \qquad r = \rho = m^2/s, \quad z = s'/s
$$

$O(\alpha^2)$ process I

2 photon emission:

- $T_2^{\mathrm{S}_2}$: both emitted photons are soft $\surd$
- $T_2^{\mathrm{V}_2}$: both photons are virtual $\surd$
- $T_2^{\mathrm{S}_1\mathrm{V}_1}$: one photon is soft, one is virtual $\surd$
- $T_2^{\mathrm{S}_1\mathrm{H}_1}$: one photon is soft, one is hard $\surd$
- $T_2^{\mathrm{V}_1\mathrm{H}_1}$: one photon is virtual, one is hard: disagreement

$$\frac{d\sigma^{(2),\mathrm{I,\ BBN}}}{ds'} - \frac{d\sigma^{(2),\mathrm{I}}}{ds'} = \frac{\sigma^{(0)}(s')}{s}\left(\frac{\alpha}{4\pi}\right)^2\left\{8 - \frac{100}{33}z\mathrm{H}_0^2 - 8\mathrm{H}_1 - 4(1-z)\mathrm{H}_1^2 + \frac{8(2-z)z}{1-z}\mathrm{H}_{0,1} - \frac{8\left(2-2z+z^2\right)}{1-z}\zeta_2\right\}$$

- $T_2^{\mathrm{H}_2}$: both emitted photons are hard, $\surd$

Here and in the following we only report the vector-case.
There are differences in the axial-vector case (not clear from Berends et al.)
Since we can work in 4-dimensions (only Abelian couplings) we can treat γ_5 without a further finite renormalization.

$O(\alpha^2)$ Process II (full form)

$$
\begin{aligned}
\frac{d\sigma^{(2),\mathrm{II}}(z,\rho)}{ds'} = \frac{\sigma^{(0)}(s')}{s}\, a^2 \Bigg\{ & \frac{64}{3} z(1-z)(1+z-4\rho)\mathrm{H}^*_{w_3,d_7} + \frac{256}{3} z\rho(1+z-4\rho)\mathrm{H}^*_{w_3,d_6} \\
&+ \frac{128z(1-4\rho^2)(1-z+2\rho)(1-z-4\rho)}{3(1-z)^2}\mathrm{H}^*_{d_8,d_7} \\
&+ \frac{512z\rho(1-4\rho^2)(1-z+2\rho)(1-z-4\rho)}{3(1-z)^3}\mathrm{H}^*_{d_8,d_6} \\
&+ \frac{16}{9(1-z)^2}\Big[(1+z)^2\big(4-9z+4z^2\big) + 2\big(9-16z+13z^2-2z^3\big)\rho + 32\rho^2\Big]\mathrm{H}^*_{d_2} \\
&+ \frac{512z\rho}{9(1-z)^4}\Big[3(1-z)^4 z - (1-z)^3\big(4+z^2\big)\rho - 2\big(9-29z+38z^2-17z^3+3z^4\big)\rho^2 \\
&- 4(2-z)\big(3+6z-5z^2\big)\rho^3 + 16\big(7-8z+9z^2\big)\rho^4 + 128(3-z)\rho^5\Big]\mathrm{H}^*_{d_4} \\
&- \frac{16}{9(1-z)^4}\Big[3-34z+129z^2-212z^3+129z^4-34z^5+3z^6+8\big(2-16z+9z^2 \\
&+ 4z^3-5z^4+2z^5\big)\rho + 16z\big(12-13z+18z^2-z^3\big)\rho^2 + 32\big(1+22z-7z^2\big)\rho^3\Big]\mathrm{H}^*_{d_1} \\
&- \frac{128z}{9(1-z)^4}\Big[1+7z-47z^2+86z^3-47z^4+7z^5+z^6-2\big(7-55z+54z^2 \\
&+ 16z^3-17z^4+3z^5\big)\rho - 4\big(39-16z+16z^2+4z^3+5z^4\big)\rho^2 \\
&+ 16\big(8-23z+22z^2+9z^3\big)\rho^3 + 128\big(7+2z-z^2\big)\rho^4\Big]\mathrm{H}^*_{d_5} - \frac{64}{3}(2z+(1-z)\rho)\mathrm{H}^*_{d_3} \\
&+ \left[\frac{16}{3\sqrt{1-4\rho}}(1+z-4\rho)\mathrm{H}^*_{w_3} + \frac{32(1-4\rho^2)(1-z+2\rho)(1-z-4\rho)}{3(1-z)^3\sqrt{1-4\rho}}\mathrm{H}^*_{d_8}\right] \\
&\times \ln\left(\frac{1-z-4\rho-\sqrt{1-4\rho}\sqrt{(1-z)^2-8(1+z)\rho+16\rho^2}}{1-z-4\rho+\sqrt{1-4\rho}\sqrt{(1-z)^2-8(1+z)\rho+16\rho^2}}\right)\Bigg\}
\end{aligned}
$$

$O(\alpha^2)$ Process II ($s \gg m^2$)

$$
\begin{aligned}
\frac{d\sigma^{(2),\mathrm{II}}(z,\rho)}{ds'} = \frac{\sigma^{(0)}(s')}{s} \left(\frac{\alpha}{4\pi}\right)^2 \Biggl\{ & \frac{8}{3}\frac{1+z^2}{1-z}L^2 - \Biggl[\frac{16}{9}\frac{11-12z+11z^2}{1-z} + \frac{16}{3}\frac{1+z^2}{1-z}\mathrm{H}_0 \\
& + \frac{32}{3}\frac{1+z^2}{1-z}\mathrm{H}_1\Biggr]L + \frac{32}{9(1-z)^3}\bigl(7-13z+8z^2-13z^3+7z^4\bigr) - \frac{16z}{9(1-z)^4}\bigl(3-36z \\
& + 94z^2-72z^3+19z^4\bigr)\mathrm{H}_0 - \frac{8z^2}{3(1-z)}\mathrm{H}_0^2 + \left(\frac{32}{9}\frac{11-12z+11z^2}{1-z} + \frac{16}{3}\frac{2+z^2}{1-z}\mathrm{H}_0\right)\mathrm{H}_1 \\
& + \frac{32}{3}\frac{1+z^2}{1-z}\mathrm{H}_1^2 + \frac{16z^2}{3(1-z)}\mathrm{H}_{0,1} - \frac{16\bigl(2+3z^2\bigr)}{3(1-x)}\zeta_2\Biggr\} + \mathcal{O}\left(\frac{m^2}{s}L^2\right).
\end{aligned}
$$

equal mass case: differs from Berends et al. and from Kniehl et al.

$\mu^+\mu^-$-production: agrees with Berends et al. and Kniehl et al.

$O(\alpha^2)$ Process III ($s \gg m^2$)

$$
\begin{aligned}
\frac{d\sigma^{(2),\mathrm{III}}_{\mathrm{dir}}}{ds'} = \frac{\sigma^{(0)}(s')}{s}\left(\frac{\alpha}{4\pi}\right)^2 \Biggl\{ & \left[\frac{4(1-z)\left(4+7z+4z^2\right)}{3z} + 8(1+z)\mathrm{H}_0\right] L^2 - \left[\frac{128(1-z)\left(1+4z+z^2\right)}{9z}\right.\\
& + \frac{8\left(4+6z-3z^2-8z^3\right)}{3z}\mathrm{H}_0 + 16(1+z)\mathrm{H}_0^2 + \frac{16(1-z)\left(4+7z+4z^2\right)}{3z}\mathrm{H}_1 + 32(1+z)\mathrm{H}_{0,1}\\
& \left. - 32(1+z)\zeta_2\right] L - \frac{2(1-z)}{27z(1+z)^2}\left(80 - 2463z - 5041z^2 - 2949z^3 - 163z^4\right)\\
& - \left[\frac{4}{9z(1+z)^3}\left(40 + 3z - 345z^2 - 445z^3 + 213z^4 + 318z^5 + 64z^6\right)\right.\\
& \left. - \frac{64(1-z)\left(1+4z+z^2\right)}{3z}\mathrm{H}_{-1}\right]\mathrm{H}_0 - \frac{4\left(12+21z-27z^2-4z^3\right)}{3z}\mathrm{H}_0^2 - 8(1+z)\mathrm{H}_0^3\\
& + \left[\frac{256(1-z)\left(1+4z+z^2\right)}{9z} + \frac{8(1-z)\left(4+7z+4z^2\right)}{3z}\mathrm{H}_0\right]\mathrm{H}_1 + \frac{16(1-z)\left(4+7z+4z^2\right)}{3z}\mathrm{H}_1^2\\
& + \left[\frac{8\left(4+9z-3z^2-12z^3\right)}{3z} + 16(1+z)\mathrm{H}_0\right]\mathrm{H}_{0,1} - \left[\frac{64(1-z)\left(1+4z+z^2\right)}{3z}\right.\\
& \left. - 64(1+z)\mathrm{H}_0\right]\mathrm{H}_{0,-1} + 32(1+z)\mathrm{H}_{0,0,1} - 128(1+z)\mathrm{H}_{0,0,-1} + 64(1+z)\mathrm{H}_{0,1,1}\\
& - \left[\frac{8\left(8+3z+3z^2-16z^3\right)}{3z} + 48(1+z)\mathrm{H}_0\right]\zeta_2 \Biggr\} + \mathcal{O}\left(\frac{m^2}{s}L^2\right).
\end{aligned}
$$

differs from Berends et al.

$O(\alpha^2)$ Process III ($s \gg m^2$)

$$
\begin{aligned}
\frac{d\sigma^{(2),\mathrm{III}}_{\mathrm{interf}}}{ds'} = \frac{\sigma^{(0)}(s')}{s}\left(\frac{\alpha}{4\pi}\right)^2 \Bigg\{ & -160(1-z) - \left[16(5+4z) - 80(1+z)\mathrm{H}_{-1} + \frac{48\left(2+2z+z^2\right)}{z}\mathrm{H}_{-1}^2\right]\mathrm{H}_0 \\
& - \left[52z - \frac{40\left(2+2z+z^2\right)}{z}\mathrm{H}_{-1}\right]\mathrm{H}_0^2 - \frac{16}{3}z\mathrm{H}_0^3 + \left[8(5-4z)\mathrm{H}_0 - \frac{8\left(4-6z+3z^2\right)}{z}\mathrm{H}_0^2\right]\mathrm{H}_1 \\
& - \frac{4\left(4-6z+3z^2\right)}{z}\mathrm{H}_0\mathrm{H}_1^2 - \left[8(5-4z) - \frac{8\left(8-2z+5z^2\right)}{z}\mathrm{H}_0 - \frac{8\left(4-6z+3z^2\right)}{z}\mathrm{H}_1\right]\mathrm{H}_{0,1} \\
& - \left[80(1+z) + \frac{32\left(5+2z^2\right)}{z}\mathrm{H}_0 - \frac{96\left(2+2z+z^2\right)}{z}\mathrm{H}_{-1}\right]\mathrm{H}_{0,-1} - \frac{32\left(2+2z+z^2\right)}{z}\mathrm{H}_{0,0,1} \\
& + \frac{16\left(10-10z+3z^2\right)}{z}\mathrm{H}_{0,0,-1} - \frac{8\left(4-6z+3z^2\right)}{z}\mathrm{H}_{0,1,1} - \frac{96\left(2+2z+z^2\right)}{z}\mathrm{H}_{0,-1,-1} \\
& + \left[8(10+z) + 160\mathrm{H}_0 - \frac{8\left(4-6z+3z^2\right)}{z}\mathrm{H}_1 - \frac{48\left(2+2z+z^2\right)}{z}\mathrm{H}_{-1}\right]\zeta_2 + 32(5+z)\zeta_3 \Bigg\} \\
& + \mathcal{O}\left(\frac{m^2}{s}\right)
\end{aligned}
$$

also calculated by A.N. Schellekens (Thesis, Nijmegen, 1981)

$O(\alpha^2)$ Process IV ($s \gg m^2$)

$$
\begin{aligned}
\frac{d\sigma^{(2),\mathrm{IV}}}{ds'} = \frac{\sigma^{(0)}(s')}{s}\left(\frac{\alpha}{4\pi}\right)^2 \Bigg\{ & -\Bigg[8(8-7z) + \frac{8\left(5-2z^2\right)}{1-z}\mathrm{H}_0 + \frac{8\left(1+z^2\right)}{1-z}\left(\mathrm{H}_0^2 + 2\mathrm{H}_0\mathrm{H}_1\right. \\
& \left. - 2\mathrm{H}_{0,1} + 2\zeta_2\right)\Bigg] L + \frac{8\left(27-42z+23z^2\right)}{1-z} + \Bigg[\frac{8}{(1-z)^2(1+z)}\left(3+10z-11z^2+22z^3-8z^4\right) \\
& + \frac{64(1+z)}{1-z}\mathrm{H}_{-1}\Bigg]\mathrm{H}_0 - \frac{8(1+z)^2}{1-z}\mathrm{H}_0^2 - \frac{8\left(1+2z^2\right)}{3(1-z)}\mathrm{H}_0^3 + \Bigg[16(8-7z) - \frac{8\left(3-2z-2z^2\right)}{1-z}\mathrm{H}_0 \\
& + \frac{16\left(2+z^2\right)}{1-z}\mathrm{H}_0^2\Bigg]\mathrm{H}_1 + \frac{16}{1-z}\mathrm{H}_0\mathrm{H}_1^2 + \Bigg[\frac{8\left(13-2z-6z^2\right)}{1-z} - \frac{16\left(5+4z^2\right)}{1-z}\mathrm{H}_0 + \frac{32z^2}{1-z}\mathrm{H}_1\Bigg]\mathrm{H}_{0,1} \\
& - \Bigg[\frac{64(1+z)}{1-z} - \frac{32\left(1+z^2\right)}{1-z}\mathrm{H}_0\Bigg]\mathrm{H}_{0,-1} + \frac{128\left(1+z^2\right)}{1-z}\mathrm{H}_{0,0,1} - \frac{64\left(1+z^2\right)}{1-z}\mathrm{H}_{0,0,-1} \\
& - \frac{32\left(1+2z^2\right)}{1-z}\mathrm{H}_{0,1,1} - \Bigg[\frac{24\left(3-2z-2z^2\right)}{1-z} + \frac{16\left(2+3z^2\right)}{1-z}\mathrm{H}_0 + \frac{32z^2}{1-z}\mathrm{H}_1\Bigg]\zeta_2 \\
& - \frac{16\left(3+z^2\right)}{1-z}\zeta_3\Bigg\} + \mathcal{O}\left(\frac{m^2}{s}L\right).
\end{aligned}
$$

differs from Berends et al.

$O(\alpha^2)$ Drell-Yan Contributions

$$\frac{d\sigma^{(2),\mathrm{BB}}}{ds'} = \frac{\sigma^{(0)}(s')}{s}\left(\frac{\alpha}{4\pi}\right)^2\Bigg\{\frac{40}{3}(1-z^2) + \Bigg[\frac{8}{3}\big(3+4z+3z^2\big) - \frac{32}{3}(1+z)^2\mathrm{H}_{-1}\Bigg]\mathrm{H}_0 + \frac{8}{3}(1+z)^2\mathrm{H}_0^2 + \frac{32}{3}(1+z)^2\mathrm{H}_{0,-1} - \frac{16}{3}(1+z)^2\zeta_2\Bigg\} + \mathcal{O}\left(\frac{m^2}{s}\right)$$

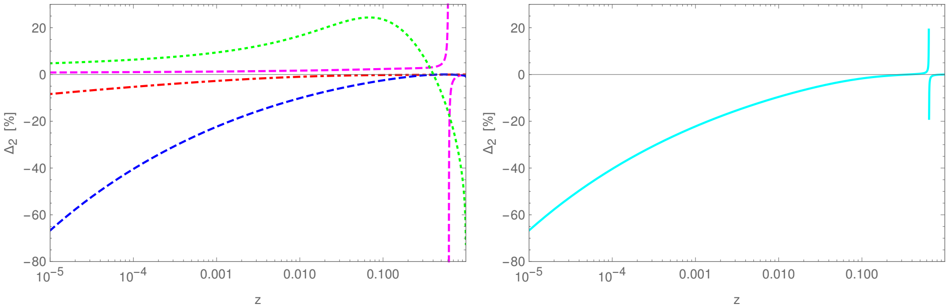
$$\frac{d\sigma^{(2),\mathrm{BC}}}{ds'} = \frac{\sigma^{(0)}(s')}{s}\left(\frac{\alpha}{4\pi}\right)^2\Bigg\{2(1-z)(27+13z) + \Bigg[4(9+11z) + 24(1+z)^2\mathrm{H}_{-1} - 24(1+z)^2\mathrm{H}_{-1}^2\Bigg]\mathrm{H}_0 + \Bigg[2\big(6-8z-15z^2\big) + 20(1+z)^2\mathrm{H}_{-1}\Bigg]\mathrm{H}_0^2 + \frac{4}{3}\big(1+4z+z^2\big)\mathrm{H}_0^3 + 36(1-z^2)\mathrm{H}_0\mathrm{H}_1 - \Bigg[36(1-z^2) - 16\big(1+3z+z^2\big)\mathrm{H}_0\Bigg]\mathrm{H}_{0,1} - \Bigg[24(1+z)^2 + 24(1+z)^2\mathrm{H}_0 - 48(1+z)^2\mathrm{H}_{-1}\Bigg]\mathrm{H}_{0,-1} - 32\big(1+3z+z^2\big)\mathrm{H}_{0,0,1} + 8(1+z)^2\mathrm{H}_{0,0,-1} - 48(1+z)^2\mathrm{H}_{0,-1,-1} + \Bigg[24(2-z)(1+z) + 8\big(3+8z+3z^2\big)\mathrm{H}_0 - 24(1+z)^2\mathrm{H}_{-1}\Bigg]\zeta_2 + 32\big(1+3z+z^2\big)\zeta_3\Bigg\} + \mathcal{O}\left(\frac{m^2}{s}\right)$$

agreement with Hamberg, van Neerven, Matsuura (Nucl. Phys. B 359 (1991))

not contained in Berends, Burgers, van Neerven (Nucl. Phys. B297 (1988))

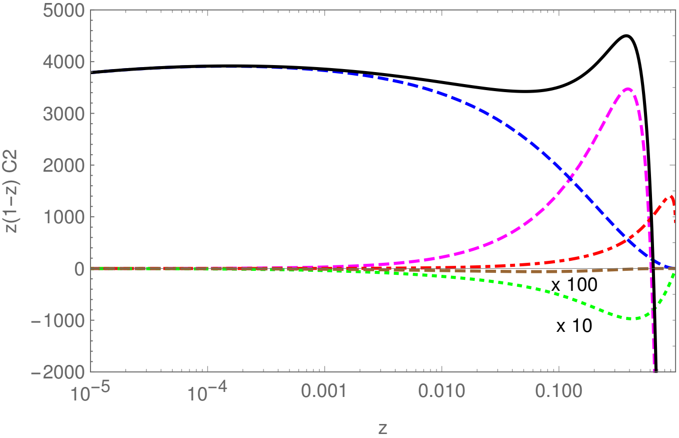
The ratios [in %] of the difference between the present result and the BBN result for $x < 1$ to the full result:

$$\Delta_{2,i} = \frac{d\sigma_i^{(2)} - d\sigma_i^{(2),BBN}}{d\sigma_i^{(2)}}$$



$\gamma\gamma$; NS; PS; Process IV; forgotten processes are included in the normalization
cyan line: the total result.

The Radiator Functions



$\gamma\gamma$; NS; PS; Process IV; further DY-contribution (process B); the total result.

Conclusions

- We have recalculated the process $e^+e^- \to \gamma^*/Z^*$ up to $O(\alpha^2)$ maintaining all mass terms until the last integral and performed then the expansion in m_e^2/s.
- Processes not dealt with in the work by Berends, Burgers and van Neerven (1987) have been added.
- Both the vector and axial-vector contributions, which give partly different corrections, were considered.
- We found differences to Berends et al. in all their processes I–IV, partly as large as the logarithmic terms for a scale $s = M_Z^2$ in individual processes.
- Our results do fully agree with the results of JB, De Freitas and van Neerven (2001); this also verifies the factorization of the Drell-Yan process in massive environments and in particular with external massive external states.
- The contributing integrals are iterated integrals of square-root valued letters containing real parameters and lead in parts to incomplete elliptic integrals and associated higher functions.
- The $O(\alpha^2)$ corrections to $e^+e^- \to \gamma^*/Z^*$ are now fully understood, which is important for the physical measurements at future high-luminosity e^+e^- collider.