$O(\alpha_s^3)$ Contributions to the Heavy Flavor DIS Wilson Coefficients at general Values of N

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[Nuclear Physics B 844 (2011), pp. 26-54, arXiv:1008.3347]
[Journ. Math. Phys., arXiv:1105.6063 [math-ph]], and in preparation

RADCOR2011, Mamallapuram, India, September 2011

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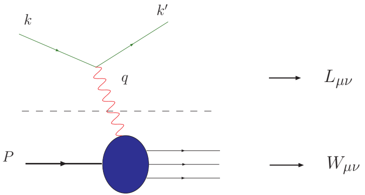
Introduction



- kinematic quantities: $Q^2 := -q^2, \quad x := \frac{Q^2}{2pq}, \quad \nu := \frac{Pq}{M}$
- differential cross-section: $\frac{d\sigma}{dQ^2\,dx} \sim W_{\mu\nu} L^{\mu\nu}$

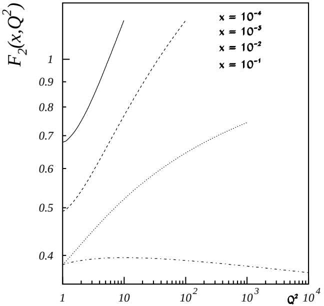
$$W_{\mu\nu}(q,P,s) = \frac{1}{4\pi}\int d^4\xi \exp(iq\xi)\langle P,s \mid [J_\mu^{em}(\xi), J_\nu^{em}(0)] \mid P,s\rangle$$

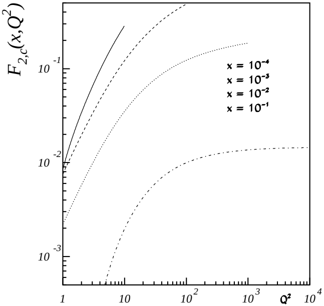
$$\text{unpol.}\left\{ = \frac{1}{2x}\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) F_L(x,Q^2) + \frac{2x}{Q^2}\left(P_\mu P_\nu + \frac{q_\mu P_\nu + q_\nu P_\mu}{2x} - \frac{Q^2}{4x^2} g_{\mu\nu}\right) F_2(x,Q^2)\right.$$

$$\text{pol.}\left\{ -\frac{M}{2Pq}\varepsilon_{\mu\nu\alpha\beta} q^\alpha \left[s^\beta g_1(x,Q^2) + \left(s^\beta - \frac{sq}{Pq}p^\beta\right) g_2(x,Q^2)\right]\right. .$$

Structure Functions: $F_{2,L}$
contain light and heavy quark contributions

Heavy flavor contributions to F_2





LO charm contributions: PDFs from [Alekhin, Melnikov, Petriello, 2006.]
$\rightarrow$ different scaling violations,
$\rightarrow$ massive contributions at lower values of x are of order 20%-35%.
Hence for the prediction of cross sections at the LHC the precise knowledge of all PDFs and the exact value of $\alpha_s(M_Z^2)$ is needed. $\alpha_s(M_Z^2) = 0.1135 \pm 0.0014$

Representation for F_2 at $Q^2 > 10m^2$

- in the asymptotic region F_L is known for general values of N to NNLO
  [Blümlein, De Freitas, van Neerven, Klein, 2006.]
- F_2 for N_F massless and one heavy quark flavor:
  [Bierenbaum, Blümlein, Klein, 2009.]

$$
\begin{aligned}
F_{(2,L)}^{Q\overline{Q}}(x, N_F+1, Q^2, m^2) &= \sum_{k=1}^{N_F} e_k^2 \Bigg\{ L_{q,(2,L)}^{\mathrm{NS}}\left(x, N_F+1, \frac{Q^2}{m^2}, \frac{m^2}{\mu^2}\right) \otimes \left[f_k(x,\mu^2,N_F) + f_{\overline{k}}(x,\mu^2,N_F)\right] \\
&+ \frac{1}{N_F}\left[L_{q,(2,L)}^{\mathrm{PS}}\left(x, N_F+1, \frac{Q^2}{m^2}, \frac{m^2}{\mu^2}\right) \otimes \Sigma(x,\mu^2,N_F) + L_{g,(2,L)}^{\mathrm{S}}\left(x, N_F+1, \frac{Q^2}{m^2}, \frac{m^2}{\mu^2}\right) \otimes G(x,\mu^2,N_F)\right]\Bigg\} \\
&+ e_Q^2 \left[H_{q,(2,L)}^{\mathrm{PS}}\left(x, N_F+1, \frac{Q^2}{m^2}, \frac{m^2}{\mu^2}\right) \otimes \Sigma(x,\mu^2,N_F) + H_{g,(2,L)}^{\mathrm{S}}\left(x, N_F+1, \frac{Q^2}{m^2}, \frac{m^2}{\mu^2}\right) \otimes G(x,\mu^2,N_F)\right]
\end{aligned}
$$

- $\otimes$ denotes the Mellin convolution

$$
[A \otimes B](x) = \int_0^1 \int_0^1 dx_1 dx_2 \; \delta(x - x_1 x_2) A(x_1) B(x_2) \; ,
$$

- The asymptotic representation for $F_2(x, Q^2)$ becomes effective at $Q^2 \geq 10 \cdot m^2$

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Heavy flavor Wilson Coefficients

- In this limit the massive Wilson coefficients up to $O(a_s^3)$ read

$$
\begin{aligned}
L_{q,(2,L)}^{\mathsf{NS}}(N_F+1) &= a_s^2\Big[A_{qq,Q}^{(2),\mathsf{NS}}(N_F+1)\,\delta_2+\hat{C}_{q,(2,L)}^{(2),\mathsf{NS}}(N_F)\Big]\\
&+ a_s^3\Big[A_{qq,Q}^{(3),\mathsf{NS}}(N_F+1)\,\delta_2+A_{qq,Q}^{(2),\mathsf{NS}}(N_F+1)C_{q,(2,L)}^{(1),\mathsf{NS}}(N_F+1)+\hat{C}_{q,(2,L)}^{(3),\mathsf{NS}}(N_F)\Big]\\
L_{q,(2,L)}^{\mathsf{PS}}(N_F+1) &= a_s^3\Big[\ A_{qq,Q}^{(3),\mathsf{PS}}(N_F+1)\,\delta_2+A_{gq,Q}^{(2)}(N_F)\ \ N_F\tilde{C}_{g,(2,L)}^{(1)}(N_F+1)+N_F\hat{\tilde{C}}_{q,(2,L)}^{(3),\mathsf{PS}}(N_F)\Big]\\
L_{g,(2,L)}^{\mathsf{S}}(N_F+1) &= a_s^2A_{gg,Q}^{(1)}(N_F+1)N_F\tilde{C}_{g,(2,L)}^{(1)}(N_F+1)+a_s^3\Big[\ A_{qg,Q}^{(3)}(N_F+1)\ \delta_2\\
&+A_{gg,Q}^{(1)}(N_F+1)\ \ N_F\tilde{C}_{g,(2,L)}^{(2)}(N_F+1)+A_{gg,Q}^{(2)}(N_F+1)\ \ N_F\tilde{C}_{g,(2,L)}^{(1)}(N_F+1)\\
&+\ A_{Qg}^{(1)}(N_F+1)\ \ N_F\tilde{C}_{q,(2,L)}^{(2),\mathsf{PS}}(N_F+1)+N_F\hat{\tilde{C}}_{g,(2,L)}^{(3)}(N_F)\Big]\ ,\\
H_{q,(2,L)}^{\mathsf{PS}}(N_F+1) &= a_s^2\Big[\ A_{Qq}^{(2),\mathsf{PS}}(N_F+1)\ \delta_2+\ \tilde{C}_{q,(2,L)}^{(2),\mathsf{PS}}(N_F+1)\Big]+a_s^3\Big[\ A_{Qq}^{(3),\mathsf{PS}}(N_F+1)\ \delta_2\\
&+\tilde{C}_{q,(2,L)}^{(3),\mathsf{PS}}(N_F+1)+A_{gq,Q}^{(2)}(N_F+1)\ \tilde{C}_{g,(2,L)}^{(1)}(N_F+1)\\
&+A_{Qq}^{(2),\mathsf{PS}}(N_F+1)\ C_{q,(2,L)}^{(1),\mathsf{NS}}(N_F+1)\Big]\ ,\\
H_{g,(2,L)}^{\mathsf{S}}(N_F+1) &= a_s\Big[\ A_{Qg}^{(1)}(N_F+1)\ \delta_2+\ \tilde{C}_{g,(2,L)}^{(1)}(N_F+1)\Big]+a_s^2\Big[\ A_{Qg}^{(2)}(N_F+1)\ \delta_2\\
&+\ A_{Qg}^{(1)}(N_F+1)\ C_{q,(2,L)}^{(1),\mathsf{NS}}(N_F+1)+\ A_{gg,Q}^{(1)}(N_F+1)\ \tilde{C}_{g,(2,L)}^{(1)}(N_F+1)\\
&+\ \tilde{C}_{g,(2,L)}^{(2)}(N_F+1)\Big]+\ a_s^3\Big[\ A_{Qg}^{(3)}(N_F+1)\ \delta_2+\ A_{Qg}^{(2)}(N_F+1)\ C_{q,(2,L)}^{(1),\mathsf{NS}}(N_F+1)\\
&+\ A_{gg,Q}^{(2)}(N_F+1)\ \tilde{C}_{g,(2,L)}^{(1)}(N_F+1)+\ A_{Qg}^{(1)}(N_F+1)\Big\{C_{q,(2,L)}^{(2),\mathsf{NS}}(N_F+1)\\
&+\ \tilde{C}_{q,(2,L)}^{(2),\mathsf{PS}}(N_F+1)\Big\}+\ A_{gg,Q}^{(1)}(N_F+1)\ \tilde{C}_{g,(2,L)}^{(2)}(N_F+1)+\ \tilde{C}_{g,(2,L)}^{(3)}(N_F+1)\Big]
\end{aligned}
$$

- Fixed moments $N = 2 \ldots 10$ $(12, 14)$ are known [Bierenbaum, Blümlein, Klein, 2009]
- The renormalization of this problem has been worked out by [Bierenbaum, Blümlein, Klein, 2009]
- Through the renormalization the general structure of the unrenormalized OME's is known
- Example:

$$
\begin{aligned}
\hat{A}_{Qg}^{(3)} = {} & \left(\frac{\hat{m}^2}{\mu^2}\right)^{3\varepsilon/2} \Bigg[\frac{\hat{\gamma}_{qg}^{(0)}}{6\varepsilon^3} \Bigg((N_F+1)\gamma_{gq}^{(0)}\hat{\gamma}_{qg}^{(0)} + \gamma_{qq}^{(0)}\Big[\gamma_{qq}^{(0)} - 2\gamma_{gg}^{(0)} - 6\beta_0 - 8\beta_{0,Q}\Big] + 8\beta_0^2 \\
& +28\beta_{0,Q}\beta_0 + 24\beta_{0,Q}^2 + \gamma_{gg}^{(0)}\Big[\gamma_{gg}^{(0)} + 6\beta_0 + 14\beta_{0,Q}\Big]\Bigg) + \frac{1}{6\varepsilon^2}\Bigg(\hat{\gamma}_{qg}^{(1)}\Big[2\gamma_{qq}^{(0)} - 2\gamma_{gg}^{(0)} \\
& -8\beta_0 - 10\beta_{0,Q}\Big] + \hat{\gamma}_{qg}^{(0)}\Big[\hat{\gamma}_{qq}^{(1),\mathrm{PS}}\{1 - 2N_F\} + \gamma_{qq}^{(1),\mathrm{NS}} + \hat{\gamma}_{qq}^{(1),\mathrm{NS}} + 2\hat{\gamma}_{gg}^{(1)} - \gamma_{gg}^{(1)} - 2\beta_1 \\
& -2\beta_{1,Q}\Big] + 6\delta m_1^{(-1)}\hat{\gamma}_{qg}^{(0)}\Big[\gamma_{gg}^{(0)} - \gamma_{qq}^{(0)} + 3\beta_0 + 5\beta_{0,Q}\Big]\Bigg) + \frac{1}{\varepsilon}\Bigg(\frac{\hat{\gamma}_{qg}^{(2)}}{3} - N_F\frac{\hat{\tilde{\gamma}}_{qg}^{(2)}}{3} \\
& +\hat{\gamma}_{qg}^{(0)}\Big[a_{gg,Q}^{(2)} - N_F a_{Qq}^{(2),\mathrm{PS}}\Big] + a_{Qg}^{(2)}\Big[\gamma_{qq}^{(0)} - \gamma_{gg}^{(0)} - 4\beta_0 - 4\beta_{0,Q}\Big] + \frac{\hat{\gamma}_{qg}^{(0)}\zeta_2}{16}\Big[\gamma_{gg}^{(0)}\Big\{2\gamma_{qq}^{(0)} \\
& -\gamma_{gg}^{(0)} - 6\beta_0 + 2\beta_{0,Q}\Big\} - (N_F+1)\gamma_{gq}^{(0)}\hat{\gamma}_{qg}^{(0)} + \gamma_{qq}^{(0)}\Big\{-\gamma_{qq}^{(0)} + 6\beta_0\Big\} - 8\beta_0^2 \\
& +4\beta_{0,Q}\beta_0 + 24\beta_{0,Q}^2\Big] + \frac{\delta m_1^{(-1)}}{2}\Big[-2\hat{\gamma}_{qg}^{(1)} + 3\delta m_1^{(-1)}\hat{\gamma}_{qg}^{(0)} + 2\delta m_1^{(0)}\hat{\gamma}_{qg}^{(0)}\Big] \\
& +\delta m_1^{(0)}\hat{\gamma}_{qg}^{(0)}\Big[\gamma_{gg}^{(0)} - \gamma_{qq}^{(0)} + 2\beta_0 + 4\beta_{0,Q}\Big] - \delta m_2^{(-1)}\hat{\gamma}_{qg}^{(0)}\Bigg) + a_{Qg}^{(3)}\Bigg]
\end{aligned}
$$

The 3-loop logarithmic contributions

- Generalized structure of renormalized OMEs:

$$A_{ij}^{(3)}\left(\frac{m^2}{Q^2}\right) = a_{ij}^{(3),3}\ln^3\left(\frac{m^2}{Q^2}\right) + a_{ij}^{(3),2}\ln^2\left(\frac{m^2}{Q^2}\right) + a_{ij}^{(3),1}\ln\left(\frac{m^2}{Q^2}\right) + a_{ij}^{(3),0} \quad (1)$$

- all logarithmic contributions are known [Ablinger,Bierenbaum, Blümlein, Klein, Wißbrock 2011] (explicit N- and $x-$space representation & analytic continuation $N \in \mathbb{C}$)
- but: in the relevant kinematic region there is no logarithmic dominance
- terms $a_{ij}^{(3),0}$ are needed to describe the correct behaviour of the structure functions:

$$\begin{aligned}
a_{Qg}^{(3),3} = {} & \frac{8(N^2+N+2)T_F}{9N(N+1)(N+2)}\Bigg[T_F N_F\Bigg(C_F\Bigg(\frac{P_1}{(N-1)N^2(N+1)^2(N+2)} - 4S_1\Bigg)\\
& +C_A\Bigg(4S_1 - \frac{8(N^2+N+1)}{(N-1)N(N+1)(N+2)}\Bigg)\Bigg) - 8T_F^2 + C_A^2\Bigg(-\frac{(11N^4+22N^3-59N^2-70N-48)S_1}{(N-1)N(N+1)(N+2)}\\
& -12S_1^2 + \frac{2(N^2+N+1)(11N^4+22N^3-35N^2-46N-24)}{(N-1)^2N^2(N+1)^2(N+2)^2}\Bigg) + C_A T_F\Bigg(-\frac{56(N^2+N+1)}{(N-1)N(N+1)(N+2)}\\
& +28S_1\Bigg) + C_F^2\Bigg(-3\frac{(3N^2+3N+2)^2}{4N^2(N+1)^2} + \frac{6S_1(3N^2+3N+2)}{N(N+1)} - 12S_1^2\Bigg) + C_F T_F\Bigg(-16S_1\\
& +\frac{2P_2}{(N-1)N^2(N+1)^2(N+2)}\Bigg) + C_A C_F\Bigg(24S_1^2 - \frac{(N^2+N+6)(7N^2+7N+4)S_1}{(N-1)N(N+1)(N+2)}\\
& -\frac{(3N^2+3N+2)(11N^4+22N^3-59N^2-70N-48)}{4(N-1)N^2(N+1)^2(N+2)}\Bigg)\Bigg]
\end{aligned}$$

$O\left(\alpha_s^3 T_F^2 N_F C_{A,F}\right)$: Contributing diagrams I

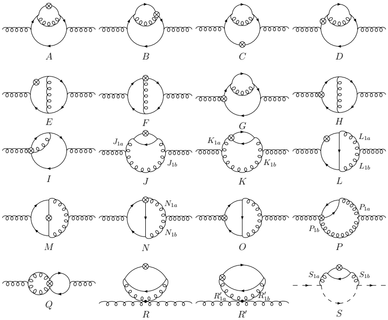


Figure: Generating 2-loop diagrams

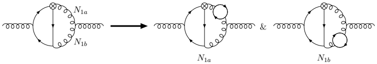


Figure: Gluons are decorated with quark bubbles

Contributing diagrams II

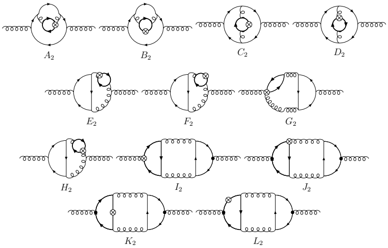


Figure: Further diagrams

- 289 Diagrams $\propto N_F T_F^2$ contribute
- 167 Diagrams $\propto T_F^2$ contribute
- due to symmetry some diagrams are identical

Evaluation of Feynman integrals

- Typical Feynman parameter integral after momentum integration

$$I_1 = \int_0^1 \int_0^1 \int_0^1 \int_0^1 dx_1 dx_2 dx_4 dx_5 x_1^{2+\varepsilon} x_2^{1-\varepsilon/2} x_5^{1-\varepsilon} (1-x_1)^{\varepsilon/2} (1-x_5)^2 (x_4 - x_5 x_4 + x_2 x_5)^N \times \left(1 - x_5\left(1 - \frac{1}{1-x_1}\right)\right)^{3/2\varepsilon}$$

- Performing the integral yields a linear combination of sums over B-functions and Hypergeometric ${}_PF_Q$s

$$I_1 = \frac{\Gamma(1-\varepsilon)\,\Gamma(3+\varepsilon)}{6(N+1)} \left\{ \sum_{j=1}^{N+1} \binom{1+N}{j} (-1)^j\, B(2-\varepsilon+j,2)\, B(1+j,2-\varepsilon/2)\; {}_3F_2\left[\begin{matrix} -3/2\,\varepsilon, 2, 3+\varepsilon \\ 4+j-\varepsilon, 4 \end{matrix}, 1\right] + B(3+N-\varepsilon,2)\, B(1,3+N-\varepsilon/2)\; {}_3F_2\left[\begin{matrix} -3/2\varepsilon, 2, 3+\varepsilon \\ 5-\varepsilon, 4 \end{matrix}; 1\right] \right\}$$

- where the ${}_PF_Q$ is defined by

$${}_PF_Q\left[\begin{matrix} a_1, \ldots, a_P \\ b_1, \ldots, b_Q \end{matrix}; z\right] = \sum_{i=0}^{\infty} \frac{(a_1)_i \ldots (a_P)_i}{(b_1)_i \ldots (b_Q)_i} \frac{z^i}{\Gamma(i+1)} .$$

Mathematical structures

- Now: perform a series expansion in ε and evaluate the remaining sums
- Up to 4 (in)finite sums occur, which are computed using modern summation methods encoded in SIGMA [C. Schneider, 2007]
- results are given in terms of ζ_2, ζ_3 and harmonic Sums $S_{\vec{a}}(N)$

$$S_{a_1,\ldots,a_m}(N) = \sum_{n_1=1}^{N} \sum_{n_2=1}^{n_1} \cdots \sum_{n_m=1}^{n_{m-1}} \frac{(\mathrm{sign}(a_1))^{n_1}}{n_1^{|a_1|}} \frac{(\mathrm{sign}(a_2))^{n_2}}{n_2^{|a_2|}} \cdots \frac{(\mathrm{sign}(a_m))^{n_m}}{n_m^{|a_m|}}$$

- in intermediary steps also generalized harmonic Sums occur

$$\widetilde{S}_{m_1,\ldots}(x_1,\ldots;N) = \sum_{i_1}^{N} \frac{x_1^{i_1}}{i_1^{m_1}} \sum_{i_2=1}^{i_1-1} \frac{x_2^{i_2}}{i_2^{m_2}} \widetilde{S}_{m_3,\ldots}(x_3,\ldots;i_2) + \widetilde{S}_{m_1+m_2,m_3,\ldots}(x_1 \cdot x_2, x_3,\ldots;N)$$

[Moch, Uwer, Weinzierl, 2002]

- algebraic and structural relations for these sums have been worked out [Ablinger, Blümlein, Schneider, 2011]

Results for the contributions $\propto N_F T_F^2 C_{F,A}$

$$
\begin{aligned}
\hat{a}_{Qg}^{(3),0} = \; & N_F T_F^2 C_A \Biggl\{ \frac{16(N^2+N+2)}{27N(N+1)(N+2)} \Bigl[108 S_{-2,1,1} - 78 S_{2,1,1} - 90 S_{-3,1} + 72 S_{2,-2} - 6 S_{3,1} \\
& - 108 S_{-2,1} S_1 + 42 S_{2,1} S_1 - 6 S_{-4} + 90 S_{-3} S_1 + 118 S_3 S_1 + 120 S_4 + 18 S_{-2} S_2 + 54 S_{-2} S_1^2 \\
& + 33 S_2 S_1^2 + 15 S_2^2 + 2 S_1^4 + 18 S_{-2}\zeta_2 + 9 S_2 \zeta_2 + 9 S_1^2 \zeta_2 - 42 S_1 \zeta_3 \Bigr] \\
& + 32 \frac{5N^4+14N^3+53N^2+82N+20}{27N(N+1)^2(N+2)^2} \Bigl[6 S_{-2,1} - 5 S_{-3} - 6 S_{-2} S_1 \Bigr] \\
& - \frac{64(5N^4+11N^3+50N^2+85N+20)}{27N(N+1)^2(N+2)^2} S_{2,1} - \frac{16(40N^4+151N^3+544N^2+779N+214)}{27N(N+1)^2(N+2)^2} S_2 S_1 \\
& - \frac{32(65N^6+429N^5+1155N^4+725N^3+370N^2+496N+648)}{81(N-1)N^2(N+1)^2(N+2)^2} S_3 \\
& - \frac{16(20N^4+107N^3+344N^2+439N+134)}{81N(N+1)^2(N+2)^2} S_1^3 + \frac{Q_1(N)}{81(N-1)N^3(N+1)^3(N+2)^3} S_2 \\
& + \frac{32(47N^6+278N^5+1257N^4+2552N^3+1794N^2+284N+448)}{81N(N+1)^3(N+2)^3} S_{-2} \\
& + \frac{8(22N^6+271N^5+2355N^4+6430N^3+6816N^2+3172N+1256)}{81N(N+1)^3(N+2)^3} S_1^2 \\
& + \frac{Q_2(N)}{243(N-1)N^2(N+1)^4(N+2)^4} S_1 + \frac{448(N^2+N+1)(N^2+N+2)}{9(N-1)N^2(N+1)^2(N+2)^2} \zeta_3 \\
& - \frac{16(5N^4+20N^3+59N^2+76N+20)}{9N(N+1)^2(N+2)^2} S_1 \zeta_2 - \frac{Q_3(N)}{9(N-1)N^3(N+1)^3(N+2)^3} \zeta_2 \\
& - \frac{Q_4(N)}{243(N-1)N^5(N+1)^5(N+2)^5} \Biggr\}
\end{aligned}
$$

$$+N_F T_F^2 C_F \Biggl\{ \frac{16(N^2+N+2)}{27N(N+1)(N+2)} \Bigl[144 S_{2,1,1} - 72 S_{3,1} - 72 S_{2,1} S_1 + 48 S_4 - 16 S_3 S_1$$
$$-24 S_2^2 - 12 S_2 S_1^2 - 2 S_1^4 - 9 S_1^2 \zeta_2 + 42 S_1 \zeta_3 \Bigr] + 32 \frac{10N^3+49N^2+83N+24}{81N^2(N+1)(N+2)} \Bigl[3 S_2 S_1 + S_1^3\Bigr]$$
$$-\frac{128(N^2-3N-2)}{3N^2(N+1)(N+2)} S_{2,1} - \frac{Q_5(N)}{81(N-1)N^3(N+1)^3(N+2)^2} S_3$$
$$+\frac{Q_6(N)}{27(N-1)N^4(N+1)^4(N+2)^3} S_2 - \frac{32(10N^4+185N^3+789N^2+521N+141)}{81N^2(N+1)^2(N+2)} S_1^2$$
$$-\frac{16(230N^5-924N^4-5165N^3-7454N^2-10217N-2670)}{243N^2(N+1)^3(N+2)} S_1$$
$$+\frac{16(5N^3+11N^2+28N+12)}{9N^2(N+1)(N+2)} S_1 \zeta_2 - \frac{Q_7(N)}{9(N-1)N^3(N+1)^3(N+2)^2} \zeta_3$$
$$+\frac{Q_8(N)}{9(N-1)N^4(N+1)^4(N+2)^3} \zeta_2 + \frac{Q_9(N)}{243(N-1)N^6(N+1)^6(N+2)^5} \Biggr\}$$

$L_{2,q}^{\rm PS}$

$$
\begin{aligned}
a_{qq,Q}^{(3),PS} = \; & C_F T_F^2 N_F \Biggl\{ \frac{128}{27} \frac{\left(N^2+N+2\right)^2}{(-1+N)\,N^2\,(1+N)^2\,(2+N)} S_1^3 - \frac{64}{27} \frac{\left(266\,N^4+181\,N^5+269\,N^3+230\,N^2+74\,N^6+16\,N^7+44\,N-24\right)}{N^3\,(-1+N)\,(2+N)^2\,(1+N)^3} S_1^2 \\
& + \frac{128}{9} \frac{\left(N^2+N+2\right)^2}{(-1+N)\,N^2\,(1+N)^2\,(2+N)} S_1 S_2 + \frac{64}{81} \frac{P_1(N)}{(-1+N)\,N^4\,(1+N)^4\,(2+N)^3} + \frac{32}{3} \frac{\left(N^2+N+2\right)^2}{(-1+N)\,N^2\,(1+N)^2\,(2+N)} \zeta_2 S_1 \\
& - \frac{64}{27} \frac{\left(266\,N^4+181\,N^5+269\,N^3+230\,N^2+74\,N^6+16\,N^7+44\,N-24\right)}{N^3\,(-1+N)\,(2+N)^2\,(1+N)^3} S_2 + \frac{256}{27} \frac{\left(N^2+N+2\right)^2}{(-1+N)\,N^2\,(1+N)^2\,(2+N)} S_3 \\
& - \frac{32}{243} \frac{P_2(N)}{N^5\,(-1+N)\,(2+N)^4\,(1+N)^5} - \frac{16}{9} \frac{P_3(N)}{N^3\,(-1+N)\,(2+N)^2\,(1+N)^3} \zeta_2 + \frac{224}{9} \frac{\left(N^2+N+2\right)^2}{(-1+N)\,N^2\,(1+N)^2\,(2+N)} \zeta_3 \Biggr\}
\end{aligned}
$$

$$
\begin{aligned}
P_1(N) &= 10101\,N^7+4737\,N^8+14923\,N^6+17085\,N^5+14133\,N^4+5944\,N^3+144-48\,N+568\,N^2+1352\,N^9+181\,N^{10} \\
P_2(N) &= (311482\,N^{10}+105173\,N^{11}+636490\,N^9+966828\,N^8+1126568\,N^7+968818\,N^6+550813\,N^5+169250\,N^4-864-1008\,N \\
& \quad +12104\,N^3-3408\,N^2+21728\,N^{12}+2074\,N^{13} \\
P_3(N) &= 266\,N^4+181\,N^5+269\,N^3+230\,N^2+74\,N^6+16\,N^7+44\,N-24
\end{aligned}
$$

(complete OME)

L_{2g}^{S}

$$
\begin{aligned}
a_{qg,Q}^{(3),0} \;=\; & N_F T_F^2 \Biggl\{ C_F \Biggl[\frac{N^2+N+2}{N(N+1)(N+2)} \Biggl[-\frac{56}{9} S_4 + \frac{32}{27} S_3 S_1 + \frac{8}{9} S_2 S_1^2 + \frac{4}{9} S_2^2 + \frac{4}{27} S_1^4 + \frac{256}{9} S_1 \zeta_3 \Biggr] \\
& - \frac{16(10N^3+13N^2+29N+6)}{81N^2(1+N)(2+N)} \left[S_1^3 + 3 S_2 S_1 \right] + \frac{32(5N^3-16N^2+N-6)}{81N^2(1+N)(2+N)} S_3 \\
& + \frac{8(109N^4+291N^3+478N^2+324N+40)}{27N^2(1+N)^2(2+N)} S_2 \\
& + \frac{8(215N^4+481N^3+930N^2+748N+120)}{81N^2(1+N)^2(2+N)} S_1^2 - \frac{R_4(N)}{243N^2(1+N)^3(2+N)} S_1 \\
& - \frac{64(N^2+N+2)R_5(N)}{9(N-1)N^3(1+N)^3(2+N)^2} \zeta_3 + \frac{R_6(N)}{243(N-1)N^6(1+N)^6(2+N)^5} \Biggr] \\
& + C_A \Biggl[\frac{N^2+N+2}{N(N+1)(N+2)} \Biggl[-\frac{56}{9} S_4 - \frac{128}{9} S_{-4} + \frac{160}{27} S_3 S_1 - \frac{4}{9} S_2^2 + \frac{8}{9} S_2 S_1^2 \\
& - \frac{4}{27} S_1^4 - \frac{64}{9} S_{2,1} S_1 - \frac{128}{9} S_{3,1} + \frac{64}{9} S_{2,1,1} - \frac{256}{9} \zeta_3 S_1 \Biggr] \\
& + \frac{32(5N^4+20N^3+41N^2+49N+20)}{81N(1+N)^2(2+N)^2} \left[S_1^3 + 12 S_{2,1} - 3 S_2 S_1 \right] \\
& + \frac{64}{81} \frac{(5N^4+38N^3+59N^2+31N+20)}{N(1+N)^2(2+N)^2} S_3 + \frac{128}{27} \frac{(5N^2+8N+10)}{N(1+N)(2+N)} S_{-3} \\
& + \frac{512}{9} \frac{(N^2+N+1)(N^2+N+2)}{(N-1)N^2(1+N)^2(2+N)^2} \zeta_3 - \frac{16 R_7(N)}{81N(1+N)^3(2+N)^3} S_2 \\
& - \frac{32(121N^3+293N^2+414N+224)}{81N(1+N)^2(2+N)} S_{-2} - \frac{R_8(N)}{81N(1+N)^3(2+N)^3} S_1^2 \\
& + \frac{16 R_9(N)}{243(N-1)N^2(1+N)^4(2+N)^4} S_1 + \frac{8 R_{10}(N)}{243(N-1)N^5(1+N)^5(2+N)^5} \Biggr] \Biggr\}
\end{aligned}
$$

(complete OME)

$$
\begin{aligned}
\gamma_{qg}^{(2)} \; = \; & \frac{N_F^2 T_F^2}{(N+1)(N+2)} \Biggl\{ C_A \Biggl[(N^2+N+2)\Biggl(\frac{128}{3N} S_{2,1} + \frac{128}{3N} S_{-3} + \frac{64}{9N} S_3 + \frac{32}{9N} S_1^3 \\
& - \frac{32}{3N} S_2 S_1 \Biggr) - \frac{128(5N^2+8N+10)}{9N} S_{-2} - \frac{64(5N^4+26N^3+47N^2+43N+20)}{9N(N+1)(N+2)} S_2 \\
& - \frac{64(5N^4+20N^3+41N^2+49N+20)}{9N(N+1)(N+2)} S_1^2 + \frac{64 P_1(N)}{27N(N+1)^2(N+2)^2} S_1 \\
& + \frac{16 P_2(N)}{27(N-1)N^4(N+1)^3(N+2)^3} \Biggr] \\
& + C_F \Biggl[\frac{32}{9} \frac{N^2+N+2}{N} \left\{ 10 S_3 - S_1^3 - 3 S_1 S_2 \right\} \\
& + \frac{32(5N^2+3N+2)}{3N^2} S_2 + \frac{32(10N^3+13N^2+29N+6)}{9N^2} S_1^2 \\
& - \frac{32(47N^4+145N^3+426N^2+412N+120)}{27N^2(N+1)} S_1 + \frac{4 P_3(N)}{27(N-1)N^5(N+1)^4(N+2)^3} \Biggr] \Biggr\}
\end{aligned}
$$

in agreement with [Moch, Vermaseren, Vogt 2004]

- furthermore the $N_F T_F^2 C_F$ and $N_F T_F^2 C_A$-contributions to the OMEs $A_{qq,Q}^{NS,(3)}$, $A_{Qq}^{PS,(3)}$, and $A_{qq,Q}^{NS,Trans.,(3)}$ have been computed
- this holds also for contributions to the 3-loop anomalous dimensions γ_{qq}^{PS}, γ_{qq}^{NS} and $\gamma_{qq}^{NS,Trans}$

Cyclotomic Harmonic Sums and Polylogarithms

- The usual harmonic polylogarithms are extended allowing for more letters in the alphabet :

$$\frac{1}{x}, \frac{1}{1-x}, \frac{1}{1+x}, \frac{1}{1+x+x^2}, \frac{1}{1+x^2}, \frac{1}{1+x+x^2+x^3+x^4}, \frac{1}{1-x+x^2}, \ldots$$

- Their iterated integrals are the cyclotomic harmonic polylogarithms (CHPL).
- The Mellin transforms of the CHPLs are spanned by the cyclotomic harmonic sums (CHS).
- The cyclotomic harmonic numbers are the CHSs at $N \to \infty$.
- They are related to the MZVs of higher roots of unity and the motivic numbers.
- Quantities of this kind emerge in massive Feynman diagram calculations (simplest example: sums containing 6th root of unity numerators).
- All formalism is again incorporated into the package `HarmonicSums`

J. Ablinger, J.B., C. Schneider, Journ. Math. Phys., in print arXiv:1105.6063 [math-ph]].

First contributions $\propto T_F^2 C_{A,F}$, $m_1 = m_2$

- Feynman integrals could not be mapped directly onto higher functions
- $\rightarrow$ Mellin-Barnes representation is introduced
- this yields Meijer G-functions which can be expanded into hypergeometric functions

The flavor non-singlet contributions

$$\begin{aligned}
\hat{a}_{qq,Q}^{(3),NS} &= T_F^2 C_F \Bigg\{ \frac{128}{27} S_4 - \frac{1024}{27}\zeta_3 S_1 + \frac{64}{9}\zeta_2 S_2 + \frac{256\left(3N^2+3N+2\right)}{27N(N+1)}\zeta_3 - \frac{320}{27}\zeta_2 S_1 - \frac{640}{81} S_3 \\
&+ \frac{8\left(3N^4+6N^3+47N^2+20N-12\right)}{27N^2(N+1)^2}\zeta_2 + \frac{1856}{81} S_2 - \frac{19424}{729} S_1 \\
&- \frac{4\left(417N^8+1668N^7-4822N^6-12384N^5-6507N^4+740N^3+216N^2+144N+432\right)}{729N^4(N+1)^4} \Bigg\} \\
\hat{\gamma}_{qq}^{(2),NS} &= C_F T_F^2 \left(\frac{128 S_3}{9} - \frac{640 S_2}{27} - \frac{128 S_1}{27} + \frac{8\left(51N^6+153N^5+57N^4+35N^3+96N^2+16N-24\right)}{27N^3(N+1)^3} \right)
\end{aligned}$$

in agreement with [Moch, Vermaseren, Vogt 2004]

$$\begin{aligned}
a_{qq,Q}^{(3),TR} &= C_F T_F^2 \Bigg\{ \frac{128}{27} S_4 - \frac{1024}{27}\zeta_3 S_1 + \frac{64}{9} S_2\zeta_2 + \frac{256}{9}\zeta_3 - \frac{320}{27} S_1\zeta_2 - \frac{640}{81} S_3 \\
&+ \frac{8}{9}\zeta_2 + \frac{1856}{81} S_2 - \frac{19424}{729} S_1 - \frac{4\left(139N^4+278N^3-101N^2+48N+144\right)}{243N^2(N+1)^2} \Bigg\} \\
\hat{\gamma}_{qq}^{(2),TR} &= C_F T_F^2 \left\{ \frac{128 S_3}{9} - \frac{640 S_2}{27} - \frac{128 S_1}{27} + \frac{8\left(17N^2+17N-8\right)}{9N(N+1)} \right\}
\end{aligned}$$

in agreement with [Moch, Vermaseren, Vogt 2004]

The flavor pure-singlet contributions

$$\hat{a}_{Qq}^{(3),PS} = \frac{T_F^2 C_F}{(N-1)N^2(N+1)^2(2+N)}\Bigg\{\left(N^2+N+2\right)^2\left(\frac{32}{27}S_1^3-\frac{512}{27}S_3+\frac{128}{3}S_{2,1}-\frac{1024}{9}\zeta_3-\frac{160}{9}S_2S_1\right.$$
$$\left.+\frac{32}{3}\zeta_2 S_1\right)-\frac{32P_1(N)}{9N(N+2)}\zeta_2+\frac{32P_2(N)}{27N(N+2)(N+3)(N+4)(N+5)}S_2-\frac{32P_3(N)}{27N(N+1)(N+2)(N+3)(N+4)(N+5)}S_1^2$$
$$+\frac{64P_4(N)}{81N^2(N+1)^2(N+2)^2(N+3)(N+4)(N+5)}S_1-\frac{64P_5(N)}{243N^3(N+1)^2(N+2)^3(N+3)(N+4)(N+5)}\Bigg\}\,.$$

$$\hat{\gamma}_{qq}^{(3),PS} = \frac{T_F^2 C_F}{(N-1)N^2(N+1)^2(2+N)}\Bigg\{-\frac{32}{3}\left(N^2+N+2\right)^2\left(S_1^2+S_2\right)$$
$$+\frac{64P_6(N)}{9N(N+1)(N+2)}S_1-\frac{64P_7(N)}{27N^2(N+1)^2(N+2)^2}\Bigg\}\,,$$

with

$$P_6(N) = 8N^7+37N^6+68N^5-11N^4-86N^3-56N^2-104N-48\,,$$
$$P_7(N) = 52N^{10}+392N^9+1200N^8+1353N^7-317N^6-1689N^5-2103N^4$$
$$-2672N^3-1496N^2-48N+144\,.$$

$m_1 \neq m_2$

- $m_c/m_b \simeq 1.3\text{GeV}/4.2\text{GeV} \rightarrow x^3 := (m_c/m_b)^6 \simeq 0.0001 \rightarrow$ expand in masses
- for fixed values of N the diagrams can be mapped onto tadpole diagrams by projection operators [Bierenbaum, Blümlein, Klein 2009.]
- e.g. $N = 2$

$$\Pi_{\mu\nu} = \frac{1}{d-1}\left(\frac{-g_{\mu\nu}}{p^2} + d\,\frac{p(\mu)p(\nu)}{p^4}\right)$$

- more complex structures occur for higher Moments
- expansion in masses was performed using EXP [Harlander, Seidensticker, Steinhauser 1998, Seidensticker 1999]
- 120 out of 256 diagrams have been computed for general values of N
- These diagrams cannot be re-interpreted in the VFNS either for c- or b-distributions.
- Ergo: the VFNS ends at 2 loops.

$N = 2$

$$
\begin{aligned}
a_{Qg}^{(3)} = {} & T_F^2 C_A \Biggl\{ \frac{156458}{2187} - \frac{1696}{81}\zeta_3 - \frac{148}{81}\zeta_2 + \frac{512608}{10125}x + \frac{3130072}{496125}x^2 + \frac{112173472}{843908625}x^3 \\
& + \ln\left(\frac{m_2^2}{\mu^2}\right)\left[-\frac{10}{3} + \frac{280}{9}\zeta_2 - \frac{14368}{675}x - \frac{12016}{4725}x^2 + \frac{328928}{2679075}x^3\right] \\
& + \ln^2\left(\frac{m_2^2}{\mu^2}\right)\left[-\frac{70}{81} - \frac{16}{45}x - \frac{16}{45}x^2 - \frac{5104}{8505}x^3\right] + \ln^3\left(\frac{m_2^2}{\mu^2}\right)\frac{1192}{81} + \ln^2\left(\frac{m_2^2}{\mu^2}\right)\ln\left(\frac{m_1^2}{\mu^2}\right)\frac{560}{27} \\
& + \ln\left(\frac{m_2^2}{\mu^2}\right)\ln\left(\frac{m_1^2}{\mu^2}\right)\left[-\frac{304}{81} + \frac{32}{45}x + \frac{32}{45}x^2 + \frac{10208}{8505}x^3\right] + \ln\left(\frac{m_2^2}{\mu^2}\right)\ln^2\left(\frac{m_1^2}{\mu^2}\right)\frac{208}{27} \\
& + \ln\left(\frac{m_1^2}{\mu^2}\right)\left[\frac{18710}{243} + \frac{280}{9}\zeta_2 + \frac{14368}{675}x + \frac{12016}{4725}x^2 - \frac{328928}{2679075}x^3\right] \\
& + \ln^2\left(\frac{m_1^2}{\mu^2}\right)\left[-\frac{70}{81} - \frac{16}{45}x - \frac{16}{45}x^2 - \frac{5104}{8505}x^3\right] + \ln^3\left(\frac{m_1^2}{\mu^2}\right)\frac{1544}{81} \Biggr\} \\
& + T_F^2 C_F \Biggl\{ + \frac{128}{243} + \frac{3584}{81}\zeta_3 + \frac{640}{27}\zeta_2 - \frac{1517888}{30375}x + \frac{339785728}{10418625}x^2 + \frac{1653611968}{843908625}x^3 \\
& + \ln\left(\frac{m_2^2}{\mu^2}\right)\left[\frac{13504}{243} - \frac{128}{9}\zeta_2 - \frac{45952}{2025}x - \frac{2056384}{99225}x^2 - \frac{11786368}{2679075}x^3\right] \\
& + \ln^2\left(\frac{m_2^2}{\mu^2}\right)\left[\frac{1936}{81} + \frac{896}{135}x + \frac{9536}{945}x^2 + \frac{47744}{8505}x^3\right] - \ln^3\left(\frac{m_2^2}{\mu^2}\right)\frac{896}{81} - \ln^2\left(\frac{m_2^2}{\mu^2}\right)\ln\left(\frac{m_1^2}{\mu^2}\right)\frac{256}{27} \\
& + \ln\left(\frac{m_2^2}{\mu^2}\right)\ln\left(\frac{m_1^2}{\mu^2}\right)\left[\frac{1888}{81} - \frac{1792}{135}x - \frac{19072}{945}x^2 - \frac{95488}{8505}x^3\right] + \ln\left(\frac{m_2^2}{\mu^2}\right)\ln^2\left(\frac{m_1^2}{\mu^2}\right)\frac{256}{27} \\
& + \ln\left(\frac{m_1^2}{\mu^2}\right)\left[\frac{256}{81} - \frac{128}{9}\zeta_2 + \frac{45952}{2025}x + \frac{2056384}{99225}x^2 + \frac{11786368}{2679075}x^3\right] \\
& + \ln^2\left(\frac{m_1^2}{\mu^2}\right)\left[\frac{1936}{81} + \frac{896}{135}x + \frac{9536}{945}x^2 + \frac{47744}{8505}x^3\right] - \ln^3\left(\frac{m_1^2}{\mu^2}\right)\frac{1408}{81} \Biggr\} + O\left(x^4\ln(x)^3\right)
\end{aligned}
$$

$N = 4$

$$
\begin{aligned}
a_{Qg}^{(3)} = \; & T_F^2 C_A \Bigg\{ \frac{4887988511}{24300000} - \frac{47146}{2025}\zeta_3 + \frac{5807}{180}\zeta_2 + \frac{496855133}{7441875}x + \frac{2510388298}{468838125}x^2 + \frac{250077164867}{5616211899375}x^3 \\
& + \ln\left(\frac{m_2^2}{\mu^2}\right)\left[\frac{47956573}{810000} + \frac{17963}{450}\zeta_2 - \frac{1877399}{70875}x - \frac{3269548}{1488375}x^2 + \frac{156082853}{1620840375}x^3\right] \\
& + \ln^2\left(\frac{m_2^2}{\mu^2}\right)\left[\frac{532373}{16200} + \frac{707}{1350}x - \frac{284}{525}x^2 - \frac{744283}{935550}x^3\right] + \ln^2\left(\frac{m_2^2}{\mu^2}\right)\frac{74657}{4050} + \ln^2\left(\frac{m_2^2}{\mu^2}\right)\ln\left(\frac{m_1^2}{\mu^2}\right)\frac{17963}{675} \\
& + \ln\left(\frac{m_2^2}{\mu^2}\right)\ln\left(\frac{m_1^2}{\mu^2}\right)\left[\frac{62893}{2025} - \frac{707}{675}x + \frac{568}{525}x^2 + \frac{744283}{467775}x^3\right] + \ln\left(\frac{m_2^2}{\mu^2}\right)\ln^2\left(\frac{m_1^2}{\mu^2}\right)\frac{7579}{675} \\
& + \ln\left(\frac{m_1^2}{\mu^2}\right)\left[\frac{384762007}{2430000}\frac{17963}{450}\zeta_2 + \frac{1877399}{70875}x + \frac{3269548}{1488375}x^2 - \frac{156082853}{1620840375}x^3\right] \\
& + \ln^2\left(\frac{m_1^2}{\mu^2}\right)\left[\frac{532373}{16200} + \frac{707}{1350}x - \frac{284}{525}x^2 - \frac{744283}{935550}x^3\right] + \ln^3\left(\frac{m_1^2}{\mu^2}\right)\frac{3817}{162}\Bigg\} \\
& + T_F^2 C_F \Bigg\{\left[-\frac{33406758667}{1093500000} + \frac{260414}{10125}\zeta_3 + \frac{1473641}{202500}\zeta_2 - \frac{119314474}{4134375}x + \frac{582667691}{37507050}x^2 + \frac{46049137562}{44929695195}x^3\right] \\
& + \ln\left(\frac{m_2^2}{\mu^2}\right)\left[\frac{76621423}{4050000} - \frac{18601}{2250}\zeta_2 - \frac{368428}{39375}x - \frac{2876423}{297675}x^2 - \frac{570093292}{324168075}x^3\right] \\
& + \ln^2\left(\frac{m_2^2}{\mu^2}\right)\left[\frac{530371}{81000} + \frac{4456}{1125}x + \frac{27101}{4725}x^2 + \frac{1759616}{467775}x^3\right] - \ln^3\left(\frac{m_2^2}{\mu^2}\right)\frac{130207}{20250} - \ln^2\left(\frac{m_2^2}{\mu^2}\right)\ln\left(\frac{m_1^2}{\mu^2}\right)\frac{18601}{3375} \\
& + \ln\left(\frac{m_2^2}{\mu^2}\right)\ln\left(\frac{m_1^2}{\mu^2}\right)\left[\frac{442267}{50625} - \frac{8912}{1125}x - \frac{54202}{4725}x^2 - \frac{3519232}{467775}x^3\right] + \ln\left(\frac{m_2^2}{\mu^2}\right)\ln^2\left(\frac{m_1^2}{\mu^2}\right)\frac{18601}{3375} \\
& + \ln\left(\frac{m_1^2}{\mu^2}\right)\left[-\frac{37307959}{2430000} - \frac{18601}{2250}\zeta_2 + \frac{368428}{39375}x + \frac{2876423}{297675}x^2 + \frac{570093292}{324168075}x^3\right] \\
& + \ln^2\left(\frac{m_1^2}{\mu^2}\right)\left[\frac{530371}{81000} + \frac{4456}{1125}x + \frac{27101}{4725}x^2 + \frac{1759616}{467775}x^3\right] - \ln^3\left(\frac{m_1^2}{\mu^2}\right)\frac{204611}{20250}\Bigg\} + O\left(x^4 \ln^3(x)\right)
\end{aligned}
$$

Flavor non-singlet, $m_1 \neq m_2$

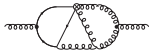
With $x = (m_1/m_2)^2$ we obtain

$$
\begin{aligned}
a_{qq,Q}^{(3),\mathrm{NS}} \; = \; & C_F T_F^2 \Biggl\{ -\frac{8\left(10459N^8+41836N^7+52418N^6+18748N^5-5501N^4-2272N^3+216N^2+144N+432\right)}{729N^4(N+1)^4} + \frac{87040}{729}S_1 \\
& -\frac{3712}{81}S_2 + \frac{1280}{81}S_3 - \frac{256}{27}S_4 + \zeta_2\left[-\frac{16\left(29N^4+58N^3-15N^2-20N+12\right)}{27N^2(N+1)^2} + \frac{640}{27}S_1 - \frac{128}{9}S_2\right] \\
& +\zeta_3\left[\frac{64\left(5N^2+5N-2\right)}{27N(N+1)} - \frac{256}{27}S_1\right] + \ln(x)\left[\frac{8\left(115N^6+345N^5+881N^4+659N^3-40N^2-48N+72\right)}{81N^3(N+1)^3} - \frac{16(N-1)(N+2)}{3N(N+1)}\zeta_2 - \frac{1984}{81}S_1\right] \\
& +\ln^2(x)\left[-\frac{16\left(17N^4+34N^3-27N^2-20N+12\right)}{27N^2(N+1)^2} + \frac{160}{27}S_1 - \frac{32}{9}S_2\right] + \ln^3(x)\left[-\frac{32\left(N^2+N-4\right)}{27N(N+1)} - \frac{32S_1}{27}\right] \\
& +x\left[-\frac{1504\left(5N^2+5N-2\right)}{225N(N+1)} + \frac{6016}{225}S_1 + \ln(x)\left[\frac{64\left(5N^2+5N-2\right)}{15N(N+1)} - \frac{256}{15}S_1\right]\right] \\
& +x^2\left[-\frac{897044\left(5N^2+5N-2\right)}{385875N(N+1)} + \frac{3588176}{385875}S_1 + \ln(x)\left[\frac{4232\left(5N^2+5N-2\right)}{3675N(N+1)} - \frac{16928}{3675}S_1\right] + \ln^2(x)\left[\frac{32}{35}S_1 - \frac{8\left(5N^2+5N-2\right)}{35N(N+1)}\right]\right] \\
& +x^3\Biggl[-\frac{23812576\left(5N^2+5N-2\right)}{281302875N(N+1)} + \frac{95250304}{281302875}S_1 + \ln(x)\left[\frac{93376\left(5N^2+5N-2\right)}{893025N(N+1)} - \frac{373504}{893025}S_1\right] \\
& +\ln^2(x)\left[\frac{512}{2835}S_1 - \frac{128\left(5N^2+5N-2\right)}{2835N(N+1)}\right]\Biggr]\Biggr\}
\end{aligned}
$$

Example: Diagram D_{872}

$$
\begin{aligned}
D_{872} = {} & C_A T_F^2 \Bigg[\Bigg\{ \delta(N-2)\,[\cdots] + \delta(N-3)\,[\cdots] + \delta(N-4)\,[\cdots] \\
& + \theta(N-5) \Bigg[\frac{1}{\varepsilon^3} \frac{256(N-3)(2N^2+N-4)}{9N^2(N+1)^2} - \frac{1}{\varepsilon^2} \Bigg\{ \frac{64(2N^5+33N^4-13N^3+87N^2-283N+354)}{(N-1)N^2(N+1)^2(N+2)} \\
& + \frac{64(N-3)(2N^2+N-4)}{3N^2(N+1)^2} \ln(x) \Bigg\} + \frac{1}{\varepsilon} \Bigg\{ \frac{32 Q_1}{81(N-1)^2 N^4 (N+1)^4 (2+N)^2} \\
& + \frac{32(N-3)(2N^2+N-4)}{3N^2(N+1)^2} \left(\zeta_2 + S_2 + \ln^2(x) \right) \Bigg\} + \frac{8 Q_2}{243(N-1)^3 N^4 (N+1)^5 (N+2)^3} \\
& - \frac{8(2N^5+33N^4-13N^3+87N^2-283N+354)}{9(N-1)N^2(N+1)^2(N+2)} \zeta_2 \\
& - \frac{8(N-3)(2N^2+N-4)}{9N^2(N+1)^2} \left(4\zeta_3 + 9\zeta_2 + 6S_2 + 12S_3 \right) \\
& + \frac{4 Q_3}{225(N-2)(N-1)^2 N^3 (N+1)^3 (N+2)(N+3)^2 (N+4)^2} x \\
& + \frac{Q_4}{7350(N-3)^2(N-2)^2(N-1)N(N+1)^3(N+2)^3(N+3)^3(N+4)^3(N+5)^3(N+6)^3} x^2 \\
& + \frac{2 Q_5}{893025(N-4)^2(N-3)(N-2)(N-1)N(N+1)^2(N+2)^2(N+3)^3(N+4)^3(N+5)^3(N+6)^3(N+7)^3(N+8)^3} x^3 \\
& - \frac{8(2N^5+33N^4-13N^3+87N^2-283N+354)}{9(N-1)N^2(N+1)^2(N+2)} \ln^2(x) \\
& + \frac{4 Q_6}{(N+1)(N+2)(N+3)(N+4)(N+5)(N+6)} x^2 \ln^2(x) + \frac{32 Q_7}{9(N+3)(N+4)(N+5)(N+6)(N+7)(N+8)} x^3 \ln^2(x) \\
& + \frac{32(N-3)(2N^2+N-4)}{9N^2(N+1)^2} \ln^3(x) - \frac{8 Q_8}{9(N-1)N^3(N+1)^3(N+2)} S_2 \\
& + \frac{8 Q_6}{(N+1)(N+2)(N+3)(N+4)(N+5)(N+6)} S_2 x^2 + \frac{64 Q_7}{9(N+3)(N+4)(N+5)(N+6)(N+7)(N+8)} S_2 x^3 \\
& + \frac{8 Q_9}{27(N-1)^2 N^4 (N+1)^4 (N+2)^2} \ln(x) + \frac{8 Q_{10}}{15(N-1)N^2(N+1)^2(N+3)(N+4)} x \ln(x) \\
& - \frac{Q_{11}}{35(N-3)(N-2)(N-1)N(N+1)^2(N+2)^2(N+3)^2(N+4)^2(N+5)^2(N+6)^2} x^2 \ln(x) \\
& + \frac{8 Q_{12}}{2835(N-4)(N-3)(N-2)(N-1)N(N+1)^2(N+2)^2(N+3)^2(N+4)^2(N+5)^2(N+6)^2(N+7)^2(N+8)^2} x^3 \ln(x) \Bigg]
\end{aligned}
$$

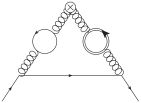
3-loop ladder diagrams: up to 6 massive propagators



$$
\begin{aligned}
\hat{I}_{10a} = {} & \frac{1}{(N+3)(N+4)}\Biggl\{\Biggl[-\frac{4\left(N^3+3N^2-N-5\right)}{(N+1)(N+2)(N+3)}S_1+2S_1^2+\frac{4(-1)^N}{N+3}S_1+4S_{-2} \\
& +2(2N+5)S_2+\frac{4(-1)^N\left(2N^3+7N^2+4N-3\right)}{(N+1)^2(N+2)^2(N+3)}+\frac{4\left(6N^3+34N^2+63N+39\right)}{(N+1)^2(N+2)^2(N+3)}\Biggr]\frac{1}{\varepsilon^2} \\
& +\Biggl[\frac{\left(-4N^4-25N^3-30N^2+49N+76\right)}{(N+1)(N+2)(N+3)(N+4)}S_1^2-\frac{4\left(2N^4+14N^3+27N^2+5N-16\right)}{(N+1)(N+2)(N+3)(N+4)}S_{-2} \\
& +\frac{\left(10N^4+73N^3+158N^2+73N-52\right)}{(N+1)(N+2)(N+3)(N+4)}S_2 \\
& +\frac{2(-1)^N\left(12N^5+127N^4+538N^3+1177N^2+1354N+648\right)}{(N+1)^2(N+2)^2(N+3)^2(N+4)}S_1 \\
& -\frac{2\left(8N^6+51N^5-72N^4-1330N^3-4062N^2-5151N-2436\right)}{(N+1)^2(N+2)^2(N+3)^2(N+4)}S_1 \\
& +S_1^3+\frac{(-1)^N}{N+3}\left(S_1^2-S_2\right)+4S_{-2}S_1-5S_2S_1+2(4N+15)S_{-3}+2(N-1)S_3 \\
& -12S_{-2,1}+8(N+4)S_{2,1} \\
& +\frac{2(-1)^N\left(11N^6+60N^5-160N^4-1837N^3-5005N^2-5801N-2508\right)}{(N+1)^3(N+2)^3(N+3)^2(N+4)} \\
& +\frac{2\left(70N^6+893N^5+4640N^4+12626N^3+19074N^2+15269N+5100\right)}{(N+1)^3(N+2)^3(N+3)^2(N+4)}\Biggr]\frac{1}{\varepsilon} \\
& +\frac{7}{24}S_1^4+\frac{\left(-10N^4-61N^3-68N^2+129N+188\right)}{6(N+1)(N+2)(N+3)(N+4)}S_1^3 \\
& +\frac{(-1)^N\left(12N^5+127N^4+538N^3+1177N^2+1354N+648\right)}{2(N+1)^2(N+2)^2(N+3)^2(N+4)}S_1^2 \\
& +\frac{P_{22}}{2(N+1)^2(N+2)^2(N+3)^2(N+4)^2}S_1^2+\frac{3}{4}\zeta_2S_1^2-4S_{-2}S_1^2-\frac{13}{4}S_2S_1^2 \\
& +\frac{(-1)^NP_{23}}{(N+1)^3(N+2)^3(N+3)^3(N+4)^2}S_1+\frac{P_{24}}{(N+1)^3(N+2)^3(N+3)^3(N+4)^2}S_1 \\
& -\frac{3\left(N^3+3N^2-N-5\right)}{2(N+1)(N+2)(N+3)}\zeta_2S_1-2S_{-3}S_1 \\
& -\frac{4\left(4N^4+41N^3+155N^2+254N+148\right)}{(N+1)(N+2)(N+3)(N+4)}S_{-2}S_1 \\
& +\frac{(-1)^N}{N+3}\left(-4S_{-2}S_1+\frac{9}{2}S_2S_1+\frac{3}{2}\zeta_2S_1+\frac{1}{6}S_1^3-2S_{-3}+\frac{10}{3}S_3+2S_{2,1}+12S_{-2,1}\right) \\
& +\frac{\left(-14N^4-201N^3-936N^2-1715N-1044\right)}{2(N+1)(N+2)(N+3)(N+4)}S_2S_1-\frac{119}{3}S_3S_1 \\
& -12S_{-2,1}S_1+22S_{2,1}S_1-2S_{-2}^2+\frac{1}{8}(32N+119)S_2^2 \\
& +\frac{(-1)^NP_{25}}{(N+1)^4(N+2)^4(N+3)^3(N+4)^2}+\frac{P_{26}}{(N+1)^4(N+2)^4(N+3)^3(N+4)^2} \\
& +\frac{3(-1)^N\left(2N^3+7N^2+4N-3\right)}{2(N+1)^2(N+2)^2(N+3)}\zeta_2+\frac{3\left(6N^3+34N^2+63N+39\right)}{2(N+1)^2(N+2)^2(N+3)}\zeta_2 \\
& +(8N+39)S_{-4}+\frac{2\left(8N^5+108N^4+558N^3+1365N^2+1553N+640\right)}{(N+1)(N+2)(N+3)(N+4)}S_{-3} \\
& -\frac{4(-1)^N\left(2N^3+7N^2+4N-3\right)}{(N+1)^2(N+2)^2(N+3)}S_{-2}-\frac{4P_{27}}{(N+1)^2(N+2)^2(N+3)^2(N+4)^2}S_{-2} \\
& +\frac{3}{2}\zeta_2S_{-2}+\frac{(-1)^N\left(8N^5+79N^4+186N^3-279N^2-1426N-1224\right)}{2(N+1)^2(N+2)^2(N+3)^2(N+4)}S_2 \\
& +\frac{P_{28}}{2(N+1)^2(N+2)^2(N+3)^2(N+4)^2}S_2+\frac{3}{4}(2N+5)\zeta_2S_2+8S_{-2}S_2 \\
& +\frac{\left(-18N^5-229N^4-1498N^3-5558N^2-10017N-6460\right)}{3(N+1)(N+2)(N+3)(N+4)}S_3 \\
& +\frac{1}{4}(20N-29)S_4-14S_{-3,1}+\frac{4\left(4N^4+22N^3+11N^2-85N-96\right)}{(N+1)(N+2)(N+3)(N+4)}S_{-2,1} \\
& -14S_{-2,2}+\frac{2\left(11N^4+107N^3+397N^2+640N+361\right)}{(N+1)(N+2)(N+3)}S_{2,1}+2(N+36)S_{3,1} \\
& +28S_{-2,1,1}+2(2N-7)S_{2,1,1}\Biggr\}+O(\varepsilon)\ , \qquad (1)
\end{aligned}
$$

$$
\begin{aligned}
P_{22} &= -6N^8-164N^7-1613N^6-7762N^5-19526N^4-22888N^3-2137N^2 \\
&\quad +19968N+13264\ , \qquad (2) \\
P_{23} &= 119N^8+2250N^7+18755N^6+90365N^5+275464N^4+542281N^3+668958N^2 \\
&\quad +469072N+142112\ , \qquad (3) \\
P_{24} &= 16N^{11}+448N^{10}+5568N^9+41171N^8+204092N^7+720291N^6+1858328N^5 \\
&\quad +3504939N^4+4712624N^3+4272331N^2+2335952N+581072\ , \qquad (4) \\
P_{25} &= 78N^9+937N^8+2466N^7-17638N^6-155141N^5-538674N^4-1047495N^3 \\
&\quad -1197445N^2-757472N-206256\ , \qquad (5) \\
P_{26} &= 568N^9+11297N^8+98332N^7+492027N^6+1561688N^5+3266831N^4 \\
&\quad +4516420N^3+3994885N^2+2061840N+475824\ , \qquad (6) \\
P_{27} &= 4N^8+96N^7+942N^6+4995N^5+15753N^4+30351N^3+34903N^2 \\
&\quad +21844N+5648\ , \qquad (7) \\
P_{28} &= -32N^9-730N^8-7180N^7-40057N^6-139918N^5-317434N^4-466820N^3 \\
&\quad -426421N^2-216416N-45040\ , \qquad (8)
\end{aligned}
$$

First 3 loop contributions VFNS OMEs: $A_{gq,Q}^{(3)}(N)$



$$
\begin{aligned}
A_{gq,Q,C_F T_F^2 n_f}^{(3),\overline{\mathsf{MS}}} = {} & C_F n_f T_F^2 \frac{1+(-1)^N}{2} \Bigg\{ \frac{88\,(N^2+N+2)}{27(N-1)N(N+1)} \log^3\left(\frac{m^2}{\mu^2}\right) \\
& + \left(\frac{8\,(N^2+N+2)}{9(N-1)N(N+1)} S_1 - \frac{8\,(8N^3+13N^2+27N+16)}{27(N-1)N(N+1)^2} \right) \log^2\left(\frac{m^2}{\mu^2}\right) \\
& + \Bigg[\frac{16\,(N^2+N+2)}{9(N-1)N(N+1)} \left(S_1^2 + S_2 - \frac{1}{2}\zeta_2 \right) - \frac{32\,(8N^3+13N^2+27N+16)}{27(N-1)N(N+1)^2} S_1 \\
& + \frac{16\,(179N^4+489N^3+913N^2+925N+358)}{81(N-1)N(N+1)^3} \Bigg] \log\left(\frac{m^2}{\mu^2}\right) \\
& + \Bigg[\frac{16(N^2+N+2)}{9(N-1)N(N+1)} \left(S_1^3 + 3S_2S_1 + 2S_3 + 5S_1\zeta_2 + \frac{11}{2}\zeta_3 \right) \\
& - \frac{16(8N^3+13N^2+27N+16)}{9(N-1)N(N+1)^2} \left(S_1^2 + S_2 + \frac{5}{3}\zeta_2 \right) \\
& + \frac{32(160N^4+408N^3+827N^2+845N+320)}{81(N-1)N(N+1)^3} S_1 \\
& - \frac{32(1189N^5+4276N^4+9248N^3+12289N^2+8668N+2378)}{243(N-1)N(N+1)^4} \Bigg] \Bigg\}
\end{aligned}
$$

Conclusion

- The present status of the calculation of the $O(\alpha_s^3)$ massive Wilson Coefficients in DIS for general values of the Mellin variable N has been presented.
- All logarithmic contributions $O(\alpha_s^3 \ln^k(Q^2/m^2)),\ k = 1, 2, 3$ to all the OMEs have been computed.
- The $O\left(\alpha_s^3 N_F T_F^2 C_{A,F}\right)$ contributions to all the OMEs A_{ij} which contribute to the nucleonic structure function $F_2(x, Q^2)$ and transversity for general values of the Mellin variable N were calculated.
- These results constitute first complete expressions for two color factors contributing to the heavy flavor Wilson Coefficients for $F_2(x, Q^2)$ at $O(a_s^3)$ for the Wilson Coefficients $L_{qq,Q}^{\mathrm{PS}}$ and $L_{qg,Q}^{\mathrm{S}}$.
- Along with the computation of the massive OMEs we obtained the corresponding parts of the 3-loop anomalous dimensions and confirmed results given in the literature analytically, partly for the first time.
- Results have been obtained for the $O(\alpha_s^3 T_F^2 C_{A,F})$ terms of the NS and PS OMEs resulting from the graphs with two massive lines with equal and non-equal masses.
- For the OME A_{Qg} fixed moments have been generated for the case of two non-equal masses. Many diagrams have already been computed for general values of N.

Conclusion

- The ladder graphs can be calculated by the technologies being available to us.
- The major charged current DIS heavy flavor Wilson Coefficients to $O(\alpha_s^2)$ were calculated.
- Along with the calculation a lot of new technologies to perform analytic massless and massive 3-loop computations are developed and are improved significantly.
- Summation technologies in product and difference fields are further refined along with other discrete algorithms.
- New higher transcendental function spaces are coined and explored (Generalized harmonic sums, cyclotomic harmonic sums and polylogarithms, hyperlogarithms, higher hypergeometric functions).