The Polarized Three-Loop Anomalous Dimensions from On-Shell Massive Operator Matrix Elements

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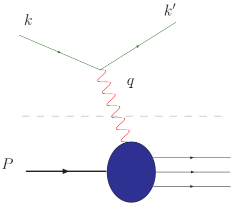
A. Behring et al., DESY 19–118, Nucl. Phys. B (2019) in print.

Introduction

Polarized Deep–Inelastic Scattering (DIS):



$\longrightarrow L_{\mu\nu}$

$\longrightarrow W_{\mu\nu}$

$$Q^2 := -q^2, \quad x := \frac{Q^2}{2P.q}$$ Bjorken–x

$$\frac{d\sigma}{dQ^2\,dx} \sim W_{\mu\nu} L^{\mu\nu}$$

$$W_{\mu\nu}(q,P,s) = \frac{1}{4\pi}\int d^4\xi \exp(iq\xi)\langle P,s \mid [J_\mu^{em}(\xi), J_\nu^{em}(0)] \mid P,s\rangle$$
$$= i\varepsilon_{\mu\nu\lambda\sigma}\frac{q^\lambda s^\sigma}{P.q} g_1(x,Q^2) + i\varepsilon_{\mu\nu\lambda\sigma}\frac{q^\lambda(P.qs^\sigma - s.qP^\sigma)}{(P.q)^2} g_2(x,Q^2)\ .$$

Structure Functions: $g_{1,(2)}$ contain light and heavy quark contributions. At 3-Loop order also graphs with two heavy quarks of different mass contribute.

$\Longrightarrow$ Single and 2-mass contributions: c and b quarks in one graph.

Introduction

Why is the precision study of scaling violations important ?

- Extract concise polarized parton distributions
- Precise 3-loop corrections are needed to determine $\alpha_s(M_Z)$, m_c and perhaps m_b
- Input for high energy colliders like RHIC and EIC

NNLO: Present status in the unpolarized case :

S. Alekhin, J. Blümlein, S. Moch and R. Placakyte, Phys. Rev. D **96** (2017) no.1, 014011 [arXiv:1701.05838 [hep-ph]].

$$\alpha_s(M_Z^2) = 0.1147 \pm 0.0008$$

$$m_c(m_c) = 1.252 \pm 0.018(exp)\ {}^{+0.03}_{-0.02}\ (scale)\ {}^{+0.00}_{-0.07}\ (thy)\mathrm{GeV},$$

$$m_b(m_b) = 3.84 \pm 0.12\mathrm{GeV}$$

$$m_t(m_t) = 160.9 \pm 1.1\mathrm{GeV}$$ [all in $\overline{\mathrm{MS}}$ scheme.]

Yet approximate NNLO treatment H. Kawamura et al. Nucl. Phys. B **864** (2012) 399 [arXiv:1205.5727].

NS & PS corrections are exact J. Ablinger et al. Nucl. Phys. B **886** (2014) 733 [arXiv:1406.4654 [hep-ph]]; Nucl. Phys. B **890** (2014) 48 [arXiv:1409.1135 [hep-ph]].

Introduction

Status of the calculations: [first calculations.]

- LO anomalous dimensions:

  K. Sasaki, Prog. Theor. Phys. **54** (1975) 1816; M.A. Ahmed and G.G. Ross, Phys. Lett. B **56** (1975) 385.

- NLO anomalous dimensions:

  R. Mertig and W.L. van Neerven, Z. Phys. C **70** (1996) 637; W. Vogelsang, Phys. Rev. D **54** (1996) 2023; Nucl. Phys. B **475** (1996) 47.

- N^2LO non-singlet anomalous dimension:

  S. Moch, J.A.M. Vermaseren and A. Vogt, Nucl. Phys. B **688** (2004) 101.

- N^2LO non-singlet anomalous dimension $\propto T_F$:

  J. Ablinger et al. Nucl. Phys. B **886** (2014) 733.

- N^2LO anomalous dimensions in the M-scheme:

  S. Moch, J.A.M. Vermaseren and A. Vogt, Nucl. Phys. B **889** (2014) 351.

- Present calculation: first independent recalculation of all contributions of $\propto T_F$
  For $\gamma_{qq}^{(2),\mathrm{PS}}$ and $\gamma_{qg}^{(2)}$ these are the complete results.

The polarized massive operator matrix elements

Example: $A_{Qg}^{(3)}$

$$
\begin{aligned}
\hat{\hat{A}}_{Qg}^{(3)} = & \left(\frac{\hat{m}^2}{\mu^2}\right)^{3\varepsilon/2}\Biggl[\frac{\hat{\gamma}_{qg}^{(0)}}{6\varepsilon^3}\Biggl((N_F+1)\gamma_{gq}^{(0)}\hat{\gamma}_{qg}^{(0)}+\gamma_{qq}^{(0)}\Bigl[\gamma_{qq}^{(0)}-2\gamma_{gg}^{(0)}-6\beta_0-8\beta_{0,Q}\Bigr]+8\beta_0^2 \\
& +28\beta_{0,Q}\beta_0+24\beta_{0,Q}^2+\gamma_{gg}^{(0)}\Bigl[\gamma_{gg}^{(0)}+6\beta_0+14\beta_{0,Q}\Bigr]\Biggr)+\frac{1}{6\varepsilon^2}\Biggl(\hat{\gamma}_{qg}^{(1)}\Bigl[2\gamma_{qq}^{(0)}-2\gamma_{gg}^{(0)}-8\beta_0 \\
& -10\beta_{0,Q}\Bigr]+\hat{\gamma}_{qg}^{(0)}\Bigl[\hat{\gamma}_{qq}^{(1),\mathsf{PS}}\{1-2N_F\}+\gamma_{qq}^{(1),\mathsf{NS}}+\hat{\gamma}_{qq}^{(1),\mathsf{NS}}+2\hat{\gamma}_{gg}^{(1)}-\gamma_{gg}^{(1)}-2\beta_1-2\beta_{1,Q}\Bigr] \\
& +6\delta m_1^{(-1)}\hat{\gamma}_{qg}^{(0)}\Bigl[\gamma_{gg}^{(0)}-\gamma_{qq}^{(0)}+3\beta_0+5\beta_{0,Q}\Bigr]\Biggr)+\frac{1}{\varepsilon}\Biggl(\frac{\hat{\gamma}_{qg}^{(2)}}{3}-N_F\frac{\hat{\tilde{\gamma}}_{qg}^{(2)}}{3}+\hat{\gamma}_{qg}^{(0)}\Bigl[a_{gg,Q}^{(2)} \\
& -N_F a_{Qq}^{(2),\mathsf{PS}}\Bigr]+a_{Qg}^{(2)}\Bigl[\gamma_{qq}^{(0)}-\gamma_{gg}^{(0)}-4\beta_0-4\beta_{0,Q}\Bigr]+\frac{\hat{\gamma}_{qg}^{(0)}\zeta_2}{16}\Bigl[\gamma_{gg}^{(0)}\Bigl\{2\gamma_{qq}^{(0)}-\gamma_{gg}^{(0)}-6\beta_0 \\
& +2\beta_{0,Q}\Bigr\}-(N_F+1)\gamma_{gq}^{(0)}\hat{\gamma}_{qg}^{(0)}+\gamma_{qq}^{(0)}\Bigl\{-\gamma_{qq}^{(0)}+6\beta_0\Bigr\}-8\beta_0^2+4\beta_{0,Q}\beta_0+24\beta_{0,Q}^2\Bigr] \\
& +\frac{\delta m_1^{(-1)}}{2}\Bigl[-2\gamma_{qg}^{(1)}+3\delta m_1^{(-1)}\hat{\gamma}_{qg}^{(0)}+2\delta m_1^{(0)}\hat{\gamma}_{qg}^{(0)}\Bigr]+\delta m_1^{(0)}\hat{\gamma}_{qg}^{(0)}\Bigl[\gamma_{gg}^{(0)}-\gamma_{qq}^{(0)}+2\beta_0+4\beta_{0,Q}\Bigr] \\
& -\delta m_2^{(-1)}\hat{\gamma}_{qg}^{(0)}\Biggr)+a_{Qg}^{(3)}\Biggr].
\end{aligned}
$$

$\gamma_{ik}^{(k)}$ – anomalous dimensions; $a_{ij}^{(k)}$ – constant parts of lower order massive OMEs

$\hat{\gamma}_{qg}^{(2)} = \gamma_{qg}^{(2)}/N_F$.

In total: 7 massive OMEs.

Feynman rules corrected

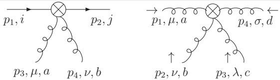


Figure 1: The four-leg polarized local operator vertices.

$$
\begin{aligned}
O^{\mu\nu}_{ab}(p_1,p_2,p_3,p_4) &= g^2\Delta^\mu\Delta^\nu \not{\Delta}\gamma_5 \sum_{j=0}^{N-3}\sum_{l=j+1}^{N-2}(\Delta.p_2)^j(\Delta.p_1)^{N-l-2} \\
&\times\left[(t_a t_b)_{kl}(\Delta.p_1+\Delta.p_4)^{l-j-1}+(t_b t_a)_{kl}(\Delta.p_1+\Delta.p_3)^{l-j-1}\right],\ N\geq 3.
\end{aligned}
$$

$$
\begin{aligned}
O^{\mu\nu\rho\sigma}_{abcd}(p_1,p_2,p_3,p_4) &= ig^2[1-(-1)^N][f_{abe}f_{cde}O^{\mu\nu\rho\sigma}(p_1,p_2,p_3,p_4) \\
&\quad +f_{ace}f_{bde}O^{\mu\rho\nu\sigma}(p_1,p_3,p_2,p_4)-f_{ade}f_{bce}O^{\rho\nu\mu\sigma}(p_3,p_2,p_1,p_4)] \\
O^{\mu\nu\rho\sigma}(p,q,r,s) &= (\varepsilon^{\Delta\nu\rho\sigma}\Delta^\mu-\varepsilon^{\Delta\mu\rho\sigma}\Delta^\nu)[\Delta.r+\Delta.s]^{N-2} \\
&\quad -\Delta^\rho(\varepsilon^{\nu\sigma\Delta s}\Delta^\mu-\varepsilon^{\mu\sigma\Delta s}\Delta^\nu)\sum_{i=1}^{N-3}[\Delta.r+\Delta.s]^i(\Delta.s)^{N-i-3} \\
&\quad +\Delta^\sigma(\varepsilon^{\rho\nu\tau\Delta}\Delta^\mu-\varepsilon^{\rho\mu r\Delta}\Delta^\nu)\sum_{i=0}^{N-3}[\Delta.r+\Delta.s]^{N-i-3}(\Delta.r)^i \\
&\quad -\Delta^\nu(\varepsilon^{\mu\sigma\Delta p}\Delta^\rho-\varepsilon^{\mu\rho\Delta p}\Delta^\sigma)\sum_{i=0}^{N-3}[\Delta.r+\Delta.s]^{N-i-3}(-\Delta.p)^i \\
&\quad +\Delta^\mu(\varepsilon^{\nu\sigma\Delta q}\Delta^\rho-\varepsilon^{\nu\rho\Delta q}\Delta^\sigma)\sum_{i=0}^{N-3}[\Delta.r+\Delta.s]^{N-i-3}(-\Delta.q)^i \\
&\quad +\Delta^\nu\Delta^\rho(\varepsilon^{\Delta\sigma ps}\Delta^\mu+\varepsilon^{\mu\sigma\Delta s}\Delta.p)\sum_{j=0}^{N-4}\sum_{i=0}^{j}(\Delta.p)^{N-j-4}[\Delta.p+\Delta.q)^{j-i}(-\Delta.s)^i \\
&\quad -\Delta^\mu\Delta^\rho(\varepsilon^{\Delta\sigma qs}\Delta^\nu+\varepsilon^{\nu\sigma\Delta s}\Delta.q)\sum_{j=0}^{N-4}\sum_{i=0}^{j}(\Delta.q)^{N-j-4}[\Delta.p+\Delta.q]^{j-i}(-\Delta.s)^i \\
&\quad -\Delta^\nu\Delta^\sigma(\varepsilon^{\Delta\mu pr}\Delta^\rho+\varepsilon^{\mu\rho\Delta p}\Delta.r)\sum_{j=0}^{N-4}\sum_{i=0}^{j}(\Delta.p)^{N-j-4}[\Delta.p+\Delta.q]^{j-i}(-\Delta.r)^i \\
&\quad +\Delta^\mu\Delta^\sigma(\varepsilon^{\Delta\nu qr}\Delta^\rho+\varepsilon^{\nu\rho\Delta p}\Delta.r)\sum_{j=0}^{N-4}\sum_{i=0}^{j}(\Delta.q)^{m-j-4}(\Delta.p+\Delta.q)^{j-i}(-\Delta.r)^i.
\end{aligned}
$$

R. Mertig and W.L. van Neerven, Z. Phys. C **70** (1996) 637, to be corrected.

Agreement with FORM-files by J. Smith.

The Larin Scheme

The following D-dimensional treatment of γ_5 is used

S. Larin, Phys. Lett. B **303** (1993) 113

$$\gamma^5 = \frac{i}{24}\varepsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma,$$

$$\not{\Delta}\gamma^5 = \frac{i}{6}\varepsilon_{\mu\nu\rho\sigma}\Delta^\mu\gamma^\nu\gamma^\rho\gamma^\sigma.$$

$$\varepsilon_{\mu\nu\rho\sigma}\varepsilon^{\alpha\lambda\tau\gamma} = -\mathrm{Det}[g^\beta_\omega], \qquad \beta = \alpha, \lambda, \tau, \gamma; \quad \omega = \mu, \nu, \rho, \sigma.$$

For the external massless gluon and quark lines one has to use the projectors:

$$P_g \hat{G}^{ab}_{\mu\nu} = \frac{\delta^{ab}}{N_C^2 - 1}\frac{1}{(D-2)(D-3)}(\Delta p)^{-N-1}\varepsilon^{\mu\nu\rho\sigma}\Delta_\rho p_\sigma \hat{G}^{ab}_{\mu\nu}.$$

$$P_q \hat{G}^{ij}_l = -\delta_{ij}\frac{i(\Delta.p)^{-N-1}}{4N_C(D-2)(D-3)}\varepsilon_{\mu\nu p\Delta}\mathrm{tr}\left[\not{p}\gamma^\mu\gamma^\nu\hat{G}^{ij}_l\right]$$

Transition to the M scheme

- The use of any consistent scheme for γ_5 will lead to violations of Ward- and Slavnov-Taylor identities, which have to be restored.
- This applies in particular to the HVBM scheme G. 't Hooft and M.J.G. Veltman, Nucl. Phys. B **44** (1972) 189; D.A. Akyeampong and R. Delbourgo, Nuovo Cim. A **17** (1973) 578; A **18** (1973) 94; A **19** (1974) 219; P. Breitenlohner and D. Maison, Commun. Math. Phys. **52** (1977) 39; 55. and the Larin scheme S. Larin, Phys. Lett. B **303** (1993) 113
- Finally, one would like to present the results in the $\overline{\mathrm{MS}}$ scheme.
- The NLO calculations by Mertig and van Neerven and Vogelsang are believed to be in the $\overline{\mathrm{MS}}$ scheme.
- Van Neerven et al. in Y. Matiounine, J. Smith and W.L. van Neerven, Phys. Rev. D **58** (1998) 076002 have formulated corresponding criteria at NLO, now called the M scheme.
- They also apply to the $1/\varepsilon$ term at NNLO.
- However, a rigorous explicit proof that all the Ward- and Slavnov-Taylor identities are fulfill from NLO on, has still to be performed.

The calculation methods

For most of the OMEs we use standard integration methods like:

- Generation of Feynman diagrams by using `QGRAF` by P. Nogueira.
- Dirac and color algebra by using `FORM` and `Color` by J. Vermaseren et al.
- Reduction to Master Integrals by using the package `Reduce2` by A. von Manteuffel.
- ${}_pF_q$ methods
- Mellin-Barnes integrals
- Ordinary differential equations mapped to difference equations [decoupling using the package `OreSys`]
- The creation of recurrences using the (multivariate) Almkvist-Zeilberger algorithm
- Solution of the recurrences using difference-field theory as encoded in the packages `Sigma, EvaluateMultiSum, Sumproduction` by C. Schneider.
- Special functions and Sums are dealt with the package `HarmonicSums` by J. Ablinger.

For detailed references on the integration methods see e.g.: JB, C. Schneider, Int. J. Mod. Phys. A **33** (2018) no.17, 1830015

The calculation methods

- These methods do thoroughly work for $A_{qq,Q}^{\mathrm{NS},(3)}, A_{Qq}^{\mathrm{PS},(3)}, A_{qq,Q}^{\mathrm{PS},(3)}, A_{qg,Q}^{(3)}, A_{gq,Q}^{(3)}$ and $A_{gg,Q}^{(3)}$ since the corresponding master integrals have first order factorizing representations.
- In the case of $A_{Qg}^{(3)}$ this is not the case, due to the shift ε. Here elliptic terms contribute in the master integrals.
- These terms do not contribute to the anomalous dimensions.
- Way out: calculate, using the IBP relations, a sufficiently high number of moments using the associated recursions and form the corresponding massive OME.
- All pole terms are free of elliptic contributions.
- Use the method of guessing [M. Kauers, *Guessing Handbook*, JKU Linz, Technical Report RISC 09-07] to determine the corresponding recurrences, which are solved by the package `Sigma`.
- All these recurrences factorize at first order.

The calculation methods

color/ζ	order	degree
$C_F T_F^2$	7	68
$C_F T_F^2 \zeta_2$	3	17
$C_F T_F^2 N_F$	7	68
$C_F T_F^2 N_F \zeta_2$	3	17
$C_F^2 T_F$	22	283
$C_F^2 T_F \zeta_2$	6	32
$C_F^2 T_F \zeta_3$	2	10
$C_A T_F^2$	10	85
$C_A T_F^2 \zeta_2$	3	12
$C_A T_F^2 N_F$	14	131
$C_A T_F^2 N_F \zeta_2$	4	16
$C_F C_A T_F$	30	484
$C_F C_A T_F \zeta_2$	8	46
$C_F C_A T_F \zeta_3$	3	19
$C_A^2 T_F$	30	472
$C_A^2 T_F \zeta_2$	10	57
$C_A^2 T_F \zeta_3$	4	19

A survey on the different contributing recurrences for $A_{Qg}^{(3)}$ at $O(1/\varepsilon)$.

The calculation methods

The anomalous dimensions, as correct also for all pole terms of the massive OMEs, can be expressed in terms of nested harmonic sums

J.A.M. Vermaseren, Int. J. Mod. Phys. A **14** (1999) 2037; J. Blümlein and S. Kurth, Phys. Rev. D **60** (1999) 014018.

$$S_{b,\vec{a}}(N) = \sum_{k=1}^{N} \frac{(\mathrm{sign}(b))^k}{k^{|b|}} S_{\vec{a}}(k), \quad S_{\emptyset} = 1 \quad , b, a_i \in \mathbb{Z}\backslash\{0\}.$$

The splitting functions are corresponding written in terms of harmonic polylogarithms E. Remiddi and J.A.M. Vermaseren, Int. J. Mod. Phys. A **15** (2000) 725.

$$\mathrm{H}_{b,\vec{a}}(z) = \int_0^z dx f_b(x) \mathrm{H}_{\vec{a}}(x), \qquad \mathrm{H}_{\emptyset} = 1, b, a_i \in \{0, 1, -1\},$$

over the alphabet

$$f_c(z) \in \left\{ \frac{1}{z}, \frac{1}{1-z}, \frac{1}{1+z} \right\}.$$

The two-loo massive OMEs

Example:

$$a_{qq,Q}^{(2),\mathrm{NS}} = C_F T_F \Bigg\{ \frac{R_1}{54N^3(N+1)^3} + \left(\frac{2(2+3N+3N^2)}{3N(N+1)} - \frac{8}{3} S_1 \right) \zeta_2 - \frac{224}{27} S_1 + \frac{40}{9} S_2 - \frac{8}{3} S_3 \Bigg\}$$

$$\overline{a}_{qq,Q}^{(2),\mathrm{NS}} = C_F T_F \Bigg\{ \frac{R_2}{648N^4(1+N)^4} + \left(\frac{2(2+3N+3N^2)}{9N(N+1)} - \frac{8}{9} S_1 \right) \zeta_3 + \left(\frac{R_3}{18N^2(N+1)^2} - \frac{20}{9} S_1 + \frac{4}{3} S_2 \right) \zeta_2 - \frac{656}{81} S_1 + \frac{112}{27} S_2 - \frac{20}{9} S_3 + \frac{4}{3} S_4 \Bigg\},$$

$$R_1 = 72 + 240N + 344N^2 + 379N^3 + 713N^4 + 657N^5 + 219N^6,$$

$$R_2 = -432 - 1872N - 3504N^2 - 3280N^3 + 1407N^4 + 7500N^5 + 9962N^6 + 6204N^7 + 1551N^8,$$

$$R_3 = -12 - 28N - N^2 + 6N^3 + 3N^4.$$

[Larin scheme].

The finite renormalization from the Larin to the M-scheme

The anomalous dimensions have the following representation:

$$\gamma_{qq}^{\mathrm{NS,M}} = \sum_{k=0}^{\infty} a_s^{k+1} \gamma_{qq}^{(k),\mathrm{NS,M}}$$

$$\gamma_{ij}^{\mathrm{M}} = \sum_{k=0}^{\infty} a_s^{k+1} \gamma_{ij}^{(k),\mathrm{M}}, \quad i,j \in \{q,g\}.$$

$$\begin{aligned}
\gamma_{qq}^{(1),\mathrm{NS,M}} &= \gamma_{qq}^{(1),\mathrm{NS,L}} + 2\beta_0 z_{qq}^{(1)}, \\
\gamma_{qq}^{(1),\mathrm{PS,M}} &= \gamma_{qq}^{(1),\mathrm{PS,L}}, \\
\gamma_{qg}^{(1),\mathrm{M}} &= \gamma_{qg}^{(1),\mathrm{L}} + \gamma_{qg}^{(0)} z_{qq}^{(1)}, \\
\gamma_{gq}^{(1),\mathrm{M}} &= \gamma_{gq}^{(1),\mathrm{L}} - \gamma_{gq}^{(0)} z_{qq}^{(1)}, \\
\gamma_{gg}^{(1),\mathrm{M}} &= \gamma_{gg}^{(1),\mathrm{L}}. \\
\gamma_{qq}^{(2),\mathrm{NS,M}} &= \gamma_{qq}^{(2),\mathrm{NS,L}} - 2\beta_0 \left(\left(z_{qq}^{(1)}\right)^2 - 2 z_{qq}^{(2),\mathrm{NS}} \right) + 2\beta_1 z_{qq}^{(1)}, \\
\gamma_{qq}^{(2),\mathrm{PS,M}} &= \gamma_{qq}^{(2),\mathrm{PS,L}} + 4\beta_0 z_{qq}^{(2),\mathrm{PS}}, \\
\gamma_{qg}^{(2),\mathrm{M}} &= \gamma_{qg}^{(2),\mathrm{L}} + \gamma_{qg}^{(1),\mathrm{M}} z_{qq}^{(1)} + \gamma_{qg}^{(0)} \left(z_{qq}^{(2)} - \left(z_{qq}^{(1)}\right)^2 \right),
\end{aligned}$$

The finite renormalization from the Larin to the M-scheme

$$\gamma_{gq}^{(2),\mathrm{M}} = \gamma_{gq}^{(2),\mathrm{L}} - \gamma_{gq}^{(1),\mathrm{M}} z_{qq}^{(1)} - \gamma_{gq}^{(0)} z_{qq}^{(2)},$$
$$\gamma_{gg}^{(2),\mathrm{M}} = \gamma_{gg}^{(2),\mathrm{L}},$$

The Z-factors are given by: Y. Matiounine, J. Smith and W.L. van Neerven, Phys. Rev. D **58** (1998) 076002; S. Moch, J.A.M. Vermaseren and A. Vogt, Nucl. Phys. B **889** (2014) 351.

$$z_{qq}^{(1)} = -\frac{8C_F}{N(N+1)},$$

$$z_{qq}^{(2),\mathrm{NS}} = C_F T_F N_F \frac{16\left(-3-N+5N^2\right)}{9N^2(1+N)^2} + C_A C_F \left\{-\frac{4Q_1}{9N^3(1+N)^3} - \frac{16}{N(1+N)} S_{-2}\right\}$$
$$+C_F^2 \left\{\frac{8\left(2+5N+8N^2+N^3+2N^4\right)}{N^3(1+N)^3} + \frac{16(1+2N)}{N^2(1+N)^2} S_1 + \frac{16}{N(1+N)} S_2 + \frac{32}{N(1+N)} S_{-2}\right\},$$

$$z_{qq}^{(2),\mathrm{PS}} = 8C_F T_F N_F \frac{(N+2)(1+N-N^2)}{N^3(N+1)^3},$$

$$z_{qq}^{(2)} = z_{qq}^{(2),\mathrm{NS}} + z_{qq}^{(2),\mathrm{PS}}$$

The polarized anomalous dimensions up to twoloop order

The LO anomalous dimensions:

$$\gamma_{qq}^{(0)} = C_F\left\{-\frac{2(2+3N+3N^2)}{N(N+1)}+8S_1\right\}$$
$$\gamma_{qg}^{(0)} = -T_FN_F\frac{8(N-1)}{N(N+1)}$$
$$\gamma_{gq}^{(0)} = -C_F\frac{4(2+N)}{N(N+1)}$$
$$\gamma_{gg}^{(0)} = T_FN_F\frac{8}{3}+C_A\left\{-\frac{2(24+11N+11N^2)}{3N(1+N)}+8S_1\right\}$$

The NLO anomalous dimensions:

$$\begin{aligned}
\gamma_{qq}^{(1),\mathrm{NS}} = {} & C_F\Bigg\{T_FN_F\left[\frac{4P_1}{9N^2(1+N)^2}-\frac{160}{9}S_1+\frac{32}{3}S_2\right]+C_A\Bigg[\frac{P_2}{9N^3(1+N)^3}+\frac{536}{9}S_1-\frac{88}{3}S_2\\
&+16S_3+\left(-\frac{16}{N(1+N)}+32S_1\right)S_{-2}+16S_{-3}-32S_{-2,1}\Bigg]\Bigg\}\\
&+C_F^2\Bigg\{\frac{P_3}{N^3(1+N)^3}+\left(\frac{16(1+2N)}{N^2(1+N)^2}-32S_2\right)S_1+\frac{8\big(2+3N+3N^2\big)}{N(1+N)}S_2\\
&-32S_3+\left(\frac{32}{N(1+N)}-64S_1\right)S_{-2}-32S_{-3}+64S_{-2,1}\Bigg\},\\
\gamma_{qq}^{(1),\mathrm{PS}} = {} & C_FT_FN_F\frac{16(2+N)\big(1+2N+N^3\big)}{N^3(1+N)^3},
\end{aligned}$$

The polarized anomalous dimensions up to twoloop order

The NLO anomalous dimensions:

$$\begin{aligned}
\gamma_{qg}^{(1)} &= C_F T_F N_F \Bigg\{ -\frac{8(N-1)\big(2-N+10N^3+5N^4\big)}{N^3(N+1)^3} + \frac{32(N-1)}{N^2(N+1)} S_1 - \frac{16(N-1)}{N(N+1)} S_1^2 \\
&\quad + \frac{16(N-1)}{N(N+1)} S_2 \Bigg\} + C_A T_F N_F \Bigg\{ -\frac{16 P_4}{N^3(N+1)^3} - \frac{64}{N(N+1)^2} S_1 + \frac{16(N-1)}{N(1+N)} S_1^2 \\
&\quad + \frac{16(N-1)}{N(1+N)} S_2 + \frac{32(N-1)}{N(1+N)} S_{-2} \Bigg\}, \\
\gamma_{gq}^{(1)} &= C_F \Bigg\{ T_F N_F \Bigg[\frac{32(2+N)(2+5N)}{9N(N+1)^2} - \frac{32(2+N)}{3N(N+1)} S_1 \Bigg] + C_A \Bigg[-\frac{8P_5}{9N^3(N+1)^3} \\
&\quad + \frac{8\big(12+22N+11N^2\big)}{3N^2(N+1)} S_1 - \frac{8(2+N)}{N(N+1)} S_1^2 + \frac{8(2+N)}{N(N+1)} S_2 + \frac{16(2+N)}{N(N+1)} S_{-2} \Bigg] \Bigg\} \\
&\quad + C_F^2 \Bigg\{ \frac{4(2+N)(1+3N)\big(-2-N+3N^2+3N^3\big)}{N^3(N+1)^3} - \frac{8(2+N)(1+3N)}{N(N+1)^2} S_1 \\
&\quad + \frac{8(2+N)}{N(N+1)} S_1^2 + \frac{8(2+N)}{N(N+1)} S_2 \Bigg\}, \\
\gamma_{gg}^{(1)} &= C_F T_F N_F \frac{8P_8}{N^3(1+N)^3} + C_A T_F N_F \Bigg\{ \frac{32 P_6}{9N^2(1+N)^2} - \frac{160}{9} S_1 \Bigg\} + C_A^2 \Bigg\{ -\frac{4P_9}{9N^3(1+N)^3} \\
&\quad + \Bigg(\frac{8P_7}{9N^2(1+N)^2} - 32 S_2 \Bigg) S_1 + \frac{64}{N(1+N)} S_2 - 16 S_3 + \Bigg(\frac{64}{N(1+N)} - 32 S_1 \Bigg) S_{-2} \\
&\quad - 16 S_{-3} + 32 S_{-2,1} \Bigg\}
\end{aligned}$$

The contributions to the polarized three-loop anomalous dimensions $\propto T_F$

$$
\begin{aligned}
\gamma_{qq}^{(2),\mathrm{PS}} = {} & C_F^2 T_F N_F \Biggl\{ -\frac{16(2+N)P_{16}}{N^5(1+N)^5} + \left[\frac{16(2+N)P_{13}}{N^4(1+N)^4} - \frac{32(N-1)(2+N)}{N^2(1+N)^2} S_2\right] S_1 \\
& -\frac{8(N-1)(2+N)\bigl(2+3N+3N^2\bigr)}{N^3(1+N)^3} S_1^2 + \frac{32(N-1)(2+N)}{3N^2(1+N)^2} S_1^3 \\
& -\frac{8(2+N)\bigl(14+23N+11N^3\bigr)}{N^3(1+N)^3} S_2 - \frac{224(N-1)(2+N)}{3N^2(1+N)^2} S_3 \\
& +\frac{64(N-1)(2+N)}{N^2(1+N)^2} S_{2,1} + \frac{192(N-1)(2+N)}{N^2(1+N)^2} \zeta_3 \Biggr\} + C_F T_F^2 N_F^2 \Biggl\{ -\frac{64(2+N)P_{14}}{27N^4(1+N)^4} \\
& +\frac{64(2+N)\bigl(6+10N-3N^2+11N^3\bigr)}{9N^3(1+N)^3} S_1 - \frac{32(N-1)(2+N)}{3N^2(1+N)^2}[S_1^2+S_2] \Biggr\} \\
& +C_A C_F T_F N_F \Biggl\{ \frac{8P_{11}}{3N^3(1+N)^3} S_1^2 + \frac{8P_{12}}{3N^3(1+N)^3} S_2 + \frac{16P_{17}}{27N^5(1+N)^5} \\
& +\left[-\frac{16P_{15}}{9N^4(1+N)^4} + \frac{32(N-1)(2+N)}{N^2(1+N)^2} S_2\right] S_1 - \frac{32(-1+N)(2+N)}{3N^2(1+N)^2} S_1^3 \\
& +\frac{16\bigl(-58+23N+23N^2\bigr)}{3N^2(1+N)^2} S_3 + \left[-\frac{32P_{10}}{N^3(1+N)^3} + \frac{64(N-1)(2+N)}{N^2(1+N)^2} S_1\right] S_{-2} \\
& +\frac{32\bigl(-10+7N+7N^2\bigr)}{N^2(1+N)^2} S_{-3} - \frac{64(N-1)(2+N)}{N^2(1+N)^2} S_{2,1} - \frac{64\bigl(-2+3N+3N^2\bigr)}{N^2(1+N)^2} S_{-2,1} \\
& -\frac{192(N-1)(2+N)}{N^2(1+N)^2} \zeta_3 \Biggr\},
\end{aligned}
$$

The contributions to the polarized threeloop anomalous dimensions $\propto T_F$

$$
\begin{aligned}
\gamma_{qg}^{(2)} = \;& C_A T_F^2 N_F^2 \Big\{ \frac{16P_{28}}{27N^4(1+N)^4} + \Big[\frac{64\big(23+50N+10N^2+19N^3\big)}{27N(1+N)^3} - \frac{32(N-1)}{3N(1+N)} S_2 \Big] S_1 \\
& -\frac{64\big(-2+5N^2\big)}{9N(1+N)^2} S_1^2 + \frac{32(N-1)}{9N(1+N)} S_1^3 - \frac{64\big(-2+6N+5N^2\big)}{9N(1+N)^2} S_2 + \frac{64(N-1)}{9N(1+N)} S_3 \\
& -\frac{128(5N-2)}{9N(1+N)} S_{-2} + \frac{128(N-1)}{3N(1+N)} S_{-3} + \frac{128(N-1)}{3N(1+N)} S_{2,1} \Big\} \\
& + C_A^2 T_F N_F \Big\{ \frac{16P_{25}}{9N^3(1+N)^3} S_2 - \frac{8P_{33}}{27N^5(1+N)^5(2+N)} + \Big[-\frac{8P_{29}}{27N^4(1+N)^4} \\
& + \frac{8\big(-72+181N-48N^2+11N^3\big)}{3N^2(1+N)^2} S_2 - \frac{704(N-1)}{3N(1+N)} S_3 + \frac{128(N-1)}{N(1+N)} S_{2,1} \\
& + \frac{512(N-1)}{N(1+N)} S_{-2,1} + \frac{192(N-1)}{N(1+N)} \zeta_3 \Big] S_1 + \Big[\frac{16P_{24}}{9N^3(1+N)^3} - \frac{160(N-1)}{N(1+N)} S_2 \Big] S_1^2 \\
& + \frac{8\big(24+59N-11N^3\big)}{9N^2(1+N)^2} S_1^3 - \frac{16(N-1)}{3N(1+N)} S_1^4 - \frac{16(N-1)}{N(1+N)} S_2^2 - \frac{32(N-1)}{N(1+N)} S_4 \\
& - \frac{16\big(345-428N+11N^3\big)}{9N^2(1+N)^2} S_3 + \Big[\frac{32P_{26}}{9N^3(1+N)^3(2+N)} - \frac{64(N-5)(2N-1)}{N^2(1+N)^2} S_1 \\
& - \frac{192(N-1)}{N(1+N)} S_1^2 - \frac{128(N-1)}{N(1+N)} S_2 \Big] S_{-2} - \frac{96(N-1)}{N(1+N)} S_{-2}^2 - \Big[\frac{512(N-1)}{N(1+N)} S_1 \\
& + \frac{32\big(69-92N+11N^3\big)}{3N^2(1+N)^2} \Big] S_{-3} - \frac{352(N-1)}{N(1+N)} S_{-4} - \frac{128(N-1)}{N(1+N)} S_{3,1} \\
& - \frac{32(N-1)\big(24+11N+11N^2\big)}{3N^2(1+N)^2} S_{2,1} - \frac{64(11N-7)}{N^2(1+N)^2} S_{-2,1} + \frac{448(N-1)}{N(1+N)} S_{-2,2} \\
& + \frac{512(N-1)}{N(1+N)} S_{-3,1} - \frac{768(N-1)}{N(1+N)} S_{-2,1,1} + \frac{96(N-1)\big(-8+3N+3N^2\big)}{N^2(1+N)^2} \zeta_3 \Big\} \\
& + C_F^2 T_F N_F \Big\{ -\frac{8P_{21}}{N^3(1+N)^3} S_1^2 + \frac{8P_{22}}{N^3(1+N)^3} S_2 + \frac{P_{31}}{N^4(1+N)^5(2+N)} \\
& + \Big[-\frac{8P_{27}}{N^4(1+N)^4} - \frac{8\big(-6+7N+28N^2+3N^3\big)}{N^2(1+N)^2} S_2 - \frac{704(N-1)}{3N(1+N)} S_3 \\
& + \frac{256(N-1)}{N(1+N)} S_{2,1} \Big] S_1 - \frac{8(N-1)\big(-10-9N+3N^2\big)}{3N^2(1+N)^2} S_1^3 - \frac{16(N-1)}{3N(1+N)} S_1^4 \\
& - \frac{48(N-1)}{N(1+N)} S_2^2 - \frac{16(N-1)\big(-22+27N+3N^2\big)}{3N^2(1+N)^2} S_3 - \frac{160(N-1)}{N(1+N)} S_4 \\
& + \Big[\frac{64P_{18}}{N^2(1+N)^3(2+N)} - \frac{256(N-1)}{N(1+N)^2} S_1 - \frac{128(N-1)}{N(1+N)} S_2 \Big] S_{-2} \\
& - \frac{64(N-1)}{N(1+N)} S_{-2}^2 + \Big[-\frac{128(N-1)^2}{N^2(1+N)^2} - \frac{256(N-1)}{N(1+N)} S_1 \Big] S_{-3} - \frac{320(N-1)}{N(1+N)} S_{-4} \\
& - \frac{128(N-1)}{N^2(1+N)^2} S_{2,1} + \frac{64(N-1)}{N(1+N)} S_{3,1} + \frac{256(N-1)}{N(1+N)^2} S_{-2,1} + \frac{128(N-1)}{N(1+N)} S_{-2,2} \\
& + \frac{256(N-1)}{N(1+N)} S_{-3,1} - \frac{192(N-1)}{N(1+N)} S_{2,1,1} + \frac{96(N-1)\big(-2+3N+3N^2\big)}{N^2(1+N)^2} \zeta_3 \Big\} \\
& + C_F T_F^2 N_F^2 \Big\{ \frac{4P_{32}}{27N^5(1+N)^5} + \Big[-\frac{32\big(-24+4N+47N^2\big)}{27N^2(1+N)} - \frac{32(N-1)}{3N(1+N)} S_2 \Big] S_1 \\
& + \frac{32(N-1)(3+10N)}{9N^2(1+N)} S_1^2 - \frac{32(N-1)}{9N(1+N)} S_1^3 + \frac{32(5N-1)}{3N^2(1+N)} S_2 + \frac{320(N-1)}{9N(1+N)} S_3 \Big\} \\
& + C_A C_F T_F N_F \Big\{ \frac{8P_{23}}{3N^3(1+N)^3} S_2 + \frac{P_{34}}{27N^5(1+N)^5(2+N)^4} + \Big[\frac{640(N-1)}{3N(1+N)} S_3 \\
& + \frac{16P_{30}}{27N^4(1+N)^4(2+N)} + \frac{16\big(75+14N+18N^2+N^3\big)}{3N^2(1+N)^2} S_2 - \frac{384(N-1)}{N(1+N)} S_{2,1} \\
& - \frac{192(N-1)}{N(1+N)} \zeta_3 \Big] S_1 + \Big[-\frac{8P_{20}}{9N^3(1+N)^3} + \frac{160(N-1)}{N(1+N)} S_2 \Big] S_1^2 + \frac{32(N-1)}{3N(1+N)} S_1^4 \\
& + \frac{16\big(3-31N-18N^2+10N^3\big)}{9N^2(1+N)^2} S_1^3 - \frac{16(N-1)\big(240-17N+19N^2\big)}{9N^2(1+N)^2} S_3 \\
& - \frac{64(N-1)}{N(1+N)} S_2^2 + \Big[-\frac{32P_{19}}{N^3(1+N)^3(2+N)} - \frac{128(N-1)\big(4+N-N^2\big)}{N^2(1+N)^2(2+N)} S_1 \\
& + \frac{192(N-1)}{N(1+N)} S_1^2 \Big] S_{-2} + \frac{96(N-1)}{N(1+N)} S_{-2}^2 + \frac{32(N-1)(2+N)(-1+3N)}{N^2(1+N)^2} S_{-3} \\
& + \frac{160(N-1)}{N(1+N)} S_{-4} + \frac{96(N-1)\big(4+N+N^2\big)}{N^2(1+N)^2} S_{2,1} + \frac{64(N-1)}{N(1+N)} S_{3,1} \\
& - \frac{128(N-1)^2}{N^2(1+N)^2} S_{-2,1} + \frac{64(N-1)}{N(1+N)} S_{-2,2} + \frac{192(N-1)}{N(1+N)} S_{2,1,1} - \frac{256(N-1)}{N(1+N)} S_{-2,1,1} \\
& - \frac{192(N-1)\big(-5+3N+3N^2\big)}{N^2(1+N)^2} \zeta_3 \Big\},
\end{aligned}
$$

The contributions to the polarized threeloop anomalous dimensions $\propto T_F$

$$
\begin{aligned}
\hat{\gamma}_{gq}^{(2)} = {} & C_F^2 T_F \Biggl\{ \frac{2P_{39}}{27(N-1)N^5(1+N)^5} + \Biggl[\frac{32(2+N)P_{36}}{27N^3(1+N)^3} + \frac{208(2+N)}{3N(1+N)} S_2 \Biggr] S_1 \\
& - \frac{16(2+N)\bigl(-3+16N+37N^2\bigr)}{9N^2(1+N)^2} S_1^2 + \frac{80(2+N)}{9N(1+N)} S_1^3 + \frac{256(2+N)}{9N(1+N)} S_3 \\
& - \frac{16(2+N)\bigl(9+46N+67N^2\bigr)}{9N^2(1+N)^2} S_2 + \frac{256}{(N-1)N^2(1+N)^2} S_{-2} - \frac{64(2+N)}{3N(1+N)} S_{2,1} \\
& - \frac{128(2+N)}{N(1+N)} \zeta_3 \Biggr\} + C_F C_A T_F \Biggl\{ \frac{8P_{38}}{27(N-1)N^3(1+N)^4} + \Biggl[-\frac{16P_{37}}{27N^3(1+N)^3} \\
& + \frac{80(2+N)}{3N(1+N)} S_2 \Biggr] S_1 + \frac{16\bigl(18+116N+129N^2+43N^3\bigr)}{9N^2(1+N)^2} S_1^2 - \frac{80(2+N)}{9N(1+N)} S_1^3 \\
& + \frac{16\bigl(-2+16N+9N^2+N^3\bigr)}{3N^2(1+N)^2} S_2 + \frac{512(2+N)}{9N(1+N)} S_3 + \Biggl[-\frac{64P_{35}}{3(N-1)N^2(1+N)^2} \\
& + \frac{256(2+N)}{3N(1+N)} S_1 \Biggr] S_{-2} + \frac{128(2+N)}{3N(1+N)} S_{-3} - \frac{128(2+N)}{3N(1+N)} S_{-2,1} + \frac{128(2+N)}{N(1+N)} \zeta_3 \Biggr\} \\
& + C_F T_F^2 \Biggl\{ \frac{64(2+N)\bigl(3+7N+N^2\bigr)}{9N(1+N)^3} + \frac{64(2+N)(2+5N)}{9N(1+N)^2} S_1 - \frac{32(2+N)}{3N(1+N)} [S_1^2 + S_2] \\
& + N_F \Biggl\{ \frac{128(2+N)\bigl(3+7N+N^2\bigr)}{9N(1+N)^3} + \frac{128(2+N)(2+5N)}{9N(1+N)^2} S_1 \\
& - \frac{64(2+N)}{3N(1+N)} [S_1^2 + S_2] \Biggr\} \Biggr\},
\end{aligned}
$$

The contributions to the polarized threeloop anomalous dimensions $\propto T_F$

$$
\begin{aligned}
\hat{\gamma}_{gg}^{(2)} = {} & C_A T_F^2 \Biggl\{ -\frac{16 P_{43}}{27 N^2 (1+N)^2} S_1 - \frac{4 P_{50}}{27 N^3 (1+N)^3} - N_F \Biggl[\frac{8 P_{50}}{27 N^3 (1+N)^3} \\
& + \frac{32 P_{43}}{27 N^2 (1+N)^2} S_1 \Biggr] \Biggr\} + C_F^2 T_F \Biggl\{ -\frac{4 P_{55}}{(N-1) N^5 (1+N)^5 (2+N)} + \Biggl[-\frac{16 P_{48}}{N^4 (1+N)^4} \\
& + \frac{32 (N-1)(2+N)}{N^2 (1+N)^2} S_2 \Biggr] S_1 + \frac{8 (N-1)(2+N)\bigl(2+3N+3N^2\bigr)}{N^3 (1+N)^3} S_1^2 \\
& + \frac{32\bigl(10+7N+7N^2\bigr)}{3 N^2 (1+N)^2} S_3 - \frac{8 (2+N)\bigl(2-11N-16N^2+9N^3\bigr)}{N^3 (1+N)^3} S_2 \\
& - \frac{32 (N-1)(2+N)}{3 N^2 (1+N)^2} S_1^3 + \Biggl[\frac{512}{N^2 (1+N)^2} S_1 - \frac{64\bigl(10+N+N^2\bigr)}{(N-1) N (1+N)(2+N)} \Biggr] S_{-2} \\
& + \frac{256}{N^2 (1+N)^2} S_{-3} - \frac{64 (N-1)(2+N)}{N^2 (1+N)^2} S_{2,1} - \frac{512}{N^2 (1+N)^2} S_{-2,1} \\
& - \frac{192\bigl(2+N+N^2\bigr)}{N^2 (1+N)^2} \zeta_3 \Biggr\} + C_A^2 T_F \Biggl\{ \frac{32 P_{41}}{9 N^2 (1+N)^2} S_2 + \frac{32 P_{45}}{9 N^2 (1+N)^2} [S_{-3} - 2 S_{-2,1}] \\
& + \frac{16 P_{46}}{9 N^2 (1+N)^2} S_3 + \frac{2 P_{56}}{27 (N-1) N^5 (1+N)^5 (2+N)} + \Biggl[\frac{1280}{9} S_2 - \frac{64}{3} S_3 - 128 \zeta_3 \\
& - \frac{8 P_{54}}{27 (N-1) N^4 (1+N)^4 (2+N)} \Biggr] S_1 + \Biggl[-\frac{32 P_{51}}{9 (N-1) N^3 (1+N)^3 (2+N)} \\
& + \frac{64 P_{49}}{9 (N-1) N^2 (1+N)^2 (2+N)} S_1 \Biggr] S_{-2} + \frac{64}{3} S_{-2}^2 + \frac{128\bigl(-3+2N+2N^2\bigr)}{N^2 (1+N)^2} \zeta_3 \Biggr\} \\
& + C_F C_A T_F \Biggl\{ \frac{8 P_{42}}{N^3 (1+N)^3} S_2 - \frac{8 P_{44}}{3 N^3 (1+N)^3} S_1^2 + \frac{2 P_{57}}{27 (N-1) N^5 (1+N)^5 (2+N)} \\
& + \Biggl[-\frac{8 P_{53}}{9 (N-1) N^4 (1+N)^4 (2+N)} - \frac{32 (N-1)(2+N)}{N^2 (1+N)^2} S_2 + 128 \zeta_3 \Biggr] S_1 \\
& + \frac{32 (N-1)(2+N)}{3 N^2 (1+N)^2} S_1^3 - \frac{32\bigl(34+N+N^2\bigr)}{3 N^2 (1+N)^2} S_3 + \Biggl[-\frac{32 P_{47}}{(N-1) N^2 (1+N)^3 (2+N)} \\
& + \frac{128 P_{40}}{(N-1) N^2 (1+N)^2 (2+N)} S_1 \Biggr] S_{-2} + \frac{192\bigl(-4+N+N^2\bigr)}{N^2 (1+N)^2} S_{-3} \\
& + \frac{64 (N-1)(2+N)}{N^2 (1+N)^2} S_{2,1} - \frac{128\bigl(-8+N+N^2\bigr)}{N^2 (1+N)^2} S_{-2,1} - \frac{64 (N-3)(4+N)}{N^2 (1+N)^2} \zeta_3 \Biggr\} \\
& + C_F T_F^2 \Biggl\{ -\frac{8 P_{52}}{27 N^4 (1+N)^4} + N_F \Biggl\{ -\frac{16 P_{52}}{27 N^4 (1+N)^4} + \frac{64 (N-1)(2+N)}{3 N^2 (1+N)^2} S_1^2 \\
& + \frac{128 (N-1)(2+N)\bigl(-6-8N+N^2\bigr)}{9 N^3 (1+N)^3} S_1 - \frac{64 (N-1)(2+N)}{N^2 (1+N)^2} S_2 \Biggr\} \\
& + \frac{64 (N-1)(2+N)\bigl(-6-8N+N^2\bigr)}{9 N^3 (1+N)^3} S_1 + \frac{32 (N-1)(2+N)}{3 N^2 (1+N)^2} S_1^2 - \frac{32 (N-1)(2+N)}{N^2 (1+N)^2} S_2 \Biggr\}
\end{aligned}
$$

The first moments

$$\gamma_{gg}^{(k)}(N=1) = -2\beta_k, \quad k \geq 0,$$

$$\beta_0 = \frac{11}{3}C_A - \frac{4}{3}T_F N_F,$$

$$\beta_1 = \frac{34}{3}C_A^2 - \frac{20}{3}C_A T_F N_F - 4C_F T_F N_F,$$

$$\hat{\beta}_2 = -\frac{1415}{27}C_A^2 T_F - \frac{205}{9}C_A C_F T_F + 2C_F^2 T_F + \frac{158}{27}C_A T_F^2 + \frac{44}{9}C_F T_F^2 + \frac{316}{27}C_A T_F^2 N_F + \frac{88}{9}C_F T_F^2 N_F \ ,$$

$$\gamma_{qq}^{(k),\mathrm{PS}}(N=1) = -4T_F N_F \gamma_{gq}^{(k-1)}(N=1), \quad k=1,2.$$

$$\gamma_{qq}^{(k),\mathrm{NS}}(N=1) = 0,$$

$$\gamma_{qg}^{(k)}(N=1) = 0, \quad k \geq 0,$$

$$\gamma_{gq}^{(0)}(N=1) = 6C_F$$

$$\gamma_{gq}^{(0)}(N=1) = -\frac{142}{3}C_F C_A + 18C_F^2 + \frac{8}{3}C_F T_F N_F.$$

$$\hat{\gamma}_{gq}^{(2)}(N=1) = -\frac{164}{3}C_A C_F T_F + 214C_F^2 T_F + \frac{104}{3}C_F T_F^2 + \frac{208}{3}C_F T_F^2 N_F + 288C_F T_F(C_A - C_F)\zeta_3.$$

The small z and large N_F expansion

Small x terms:

predicted: J. Bartels, B.I. Ermolaev and M.G. Ryskin, Z. Phys. C **70** (1996) 273; JB and A. Vogt, Phys. Lett. B **386** (1996) 350.

- direct agreement up to 2 loops
- at 3 loops: described by physical anomalous dimensions Moch et al, 2014

No phenomenological dominance of the leading small x order in all terms, e.g.:

$$\gamma_{qq}^{(2),\mathrm{PS}} = \frac{128}{3N^5}(43 - 74N) + O\left(\frac{1}{N^3}\right)$$

Large N_F terms: predicted: J.A. Gracey, Nucl. Phys. B **480** (1996) 73; J.F. Bennett and J.A. Gracey, Phys. Lett. B **432** (1998) 209.

Agree e.g. with the combination

$$\overline{\gamma}_{gg}^{(2)} + \overline{\gamma}_{gq}^{(2)} \frac{\overline{\gamma}_{qg}^{(0)}}{\overline{\gamma}_{gg}^{(0)}} = -4C_A T_F^2 \left[\frac{8Q_1}{27N^2(1+N)^2} S_1 + \frac{2Q_2}{27N^3(1+N)^3}\right] +4C_F T_F^2 \left[-\frac{4Q_3}{27N^4(1+N)^4} - \frac{64(N-1)(N+2)\left(3+7N+7N^2\right)}{9N^3(1+N)^3} S_1 \right.$$

The splitting functions in z space

Example: $P_{qq}^{(2),\mathrm{PS}}$

$$
\begin{aligned}
P_{qq}^{(2),\mathrm{PS}} = {} & C_F^2 T_F N_F \Bigg\{ -192(1-z) + 16(-25+114z)\mathrm{H}_0 - 8(32+25z)\mathrm{H}_0^2 + \frac{32}{3}(-5+6z)\mathrm{H}_0^3 \\
& -\frac{32}{3}(1+z)\mathrm{H}_0^4 - (1-z)\Bigg(2000 + 192\mathrm{H}_0 - 80\mathrm{H}_0^2\Bigg)\mathrm{H}_1 - (1-z)\Bigg(104 + 160\mathrm{H}_0\Bigg)\mathrm{H}_1^2 \\
& -\frac{160}{3}(1-z)\mathrm{H}_1^3 + \Bigg(-208(4+3z) + 32(-13+19z)\mathrm{H}_0 + 32(1+z)\mathrm{H}_0^2 \\
& +320(1-z)\mathrm{H}_1\Bigg)\mathrm{H}_{0,1} + 64(1+z)\mathrm{H}_{0,1}^2 - \Bigg(32(-1+23z) + 384(1+z)\mathrm{H}_0\Bigg)\mathrm{H}_{0,0,1} \\
& +\Bigg(64(-7+8z) - 128(1+z)\mathrm{H}_0\Bigg)H_{0,1,1} + 576(1+z)\mathrm{H}_{0,0,0,1} - 64(1+z)\mathrm{H}_{0,0,1,1} \\
& -128(1+z)\mathrm{H}_{0,1,1,1} + \Bigg(16(64+27z) - 320(-2+z)\mathrm{H}_0 + 192(1+z)\mathrm{H}_0^2\Bigg)\zeta_2 \\
& -\frac{1184}{5}(1+z)\zeta_2^2 + \Bigg(64(-11+21z) - 448(1+z)\mathrm{H}_0\Bigg)\zeta_3\Bigg\} \\
& +C_F T_F^2 N_F^2 \Bigg\{\frac{5504}{27}(1-z) - \frac{64}{27}(-65+43z)\mathrm{H}_0 + \frac{32}{9}(23+17z)\mathrm{H}_0^2 + \frac{64}{9}(1+z)\mathrm{H}_0^3 \\
& +\frac{128}{9}(1-z)\mathrm{H}_1 + \frac{160}{3}(1-z)\mathrm{H}_1^2 + \frac{128}{9}(-5+4z)\mathrm{H}_{0,1} - \frac{256}{3}(1+z)\mathrm{H}_{0,0,1} \\
& +\frac{128}{3}(1+z)\mathrm{H}_{0,1,1} + \Bigg(-\frac{128}{9}(-5+4z) + \frac{256}{3}(1+z)\mathrm{H}_0\Bigg)\zeta_2 + \frac{128}{3}(1+z)\zeta_3\Bigg\} \\
& +C_F C_A T_F N_F \Bigg\{-\frac{142048}{27}(1-z) + \Bigg(-\frac{16}{27}(2257+8899z) + 1184(1+z)\mathrm{H}_{-1} \\
& -160(1+z)\mathrm{H}_{-1}^2\Bigg)\mathrm{H}_0 + \Bigg(\frac{8}{9}(-427+1151z) - 272(1+z)\mathrm{H}_{-1}\Bigg)\mathrm{H}_0^2 - \frac{32}{9}(19 \\
& +37z)\mathrm{H}_0^3 + \frac{8}{3}(-3+4z)\mathrm{H}_0^4 + \Bigg(\frac{17024}{9}(1-z) + 544(1-z)\mathrm{H}_0 - 264(1-z)\mathrm{H}_0^2\Bigg)\mathrm{H}_1 \\
& +\Bigg(\frac{520}{3}(1-z) + 160(1-z)\mathrm{H}_0\Bigg)\mathrm{H}_1^2 + \frac{160}{3}(1-z)\mathrm{H}_1^3 + \Bigg(\frac{16}{9}(269+440z) \\
& -16(-45+31z)\mathrm{H}_0 - 112(1+z)\mathrm{H}_0^2 - 320(1-z)\mathrm{H}_1 - 128(1+z)\mathrm{H}_{-1}\Bigg)\mathrm{H}_{0,1} \\
& -64(1+z)\mathrm{H}_{0,1}^2 + \Bigg(-1184(1+z) - 64(-13+z)\mathrm{H}_0 - 96(1-z)\mathrm{H}_0^2 \\
& +320(1+z)\mathrm{H}_{-1}\Bigg)\mathrm{H}_{0,-1} + 64(1-z)\mathrm{H}_{0,-1}^2 + \Bigg(\frac{32}{3}(-44+67z) + 448(1+z)\mathrm{H}_0\Bigg)\mathrm{H}_{0,0,1} \\
& +\Bigg(224(-5+3z) - 64(-5+z)\mathrm{H}_0\Bigg)H_{0,0,-1} + \Bigg(-\frac{32}{3}(-49+41z) + 128(1+z)H_0\Bigg) \\
& \times\mathrm{H}_{0,1,1} + 128(1+z)\mathrm{H}_{0,1,-1} + 128(1+z)\mathrm{H}_{0,-1,1} - \Bigg(320(1+z) + 128(1-z)\mathrm{H}_0\Bigg) \\
& \times\mathrm{H}_{0,-1,-1} - 704(1+z)\mathrm{H}_{0,0,0,1} - 384(1+z)\mathrm{H}_{0,0,0,-1} + 64(1+z)\mathrm{H}_{0,0,1,1} \\
& +128(1+z)\mathrm{H}_{0,1,1,1} + \Bigg(-\frac{16}{9}(-91+134z) - \frac{16}{3}(29+47z)\mathrm{H}_0 - 16(1-z)\mathrm{H}_0^2 \\
& +160(1-z)\mathrm{H}_1 - 32(1+z)\mathrm{H}_{-1} + 64(1+z)\mathrm{H}_{0,1} - 64(1-z)\mathrm{H}_{0,-1}\Bigg)\zeta_2 \\
& +\frac{16}{5}(117+107z)\zeta_2^2 + \Bigg(-\frac{224}{3}(-25+26z) + 64(9+13z)\mathrm{H}_0\Bigg)\zeta_3\Bigg\}
\end{aligned}
$$

Conclusions

- We have calculated the contributions $\propto T_F$ to the polarized 3–loop anomalous dimension $\gamma_{ij}^{(2)}(N)$ and the associated splitting functions in a massive calculation.
- We agree with the previous results in S. Moch, J.A.M. Vermaseren and A. Vogt, Nucl. Phys. B **889** (2014) 351.
- The method of arbitrary high moments was instrumental to derive this result, since intermediary elliptic and higher terms are canceled in this way.
- Large difference equations have been solved by applying C. Schneider's package `Sigma`.
- A similar, but even larger difference equations have to be solved for the $O(\varepsilon^0)$ term.
- As by-products we also obtained the complete LO and NLO anomalous dimensions and $\hat{\beta}_2$.