3-Loop Heavy Flavor Corrections to Deep-Inelastic Scattering and the VFNS

Johannes Blümlein

(in collaboration with: J. Ablinger, A. Behring, G. Falcioni, A. De Freitas, A. Goedicke, S. Klein, A. von Manteuffel, C. Raab, C. Schneider, K. Schönwald, and F. Wißbrock)

DESY, Zeuthen, Germany

(RISC Linz, JKU Linz, RWTH Aachen, Edinburgh, KIT, MSU East Lansing, IHES)

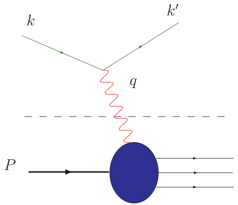
Introduction

This talk is dedicated to our friend and colleague William James Stirling.

QCD 2009, Berlin

Introduction

Unpolarized Deep–Inelastic Scattering (DIS):



$\longrightarrow \quad L_{\mu\nu}$

$\longrightarrow \quad W_{\mu\nu}$

$$Q^2 := -q^2, \quad x := \frac{Q^2}{2P.q} \quad \text{Bjorken–x}$$

$$\frac{d\sigma}{dQ^2\, dx} \sim W_{\mu\nu} L^{\mu\nu}$$

$$W_{\mu\nu}(q,P,s) = \frac{1}{4\pi}\int d^4\xi \exp(iq\xi)\langle P,s \mid [J_\mu^{em}(\xi), J_\nu^{em}(0)] \mid P,s\rangle =$$

$$\frac{1}{2x}\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) F_L(x,Q^2) + \frac{2x}{Q^2}\left(P_\mu P_\nu + \frac{q_\mu P_\nu + q_\nu P_\mu}{2x} - \frac{Q^2}{4x^2} g_{\mu\nu}\right) F_2(x,Q^2) \ .$$

Structure Functions: $F_{2,L}$ contain light and heavy quark contributions.
At 3-Loop order also graphs with two heavy quarks of different mass contribute.
$\Longrightarrow$ Single and 2-mass contributions: c and b quarks in one graph.

Introduction

Why are Heavy Flavor Contributions important ?

- They form a significant contribution to F_2 and F_L particularly at small x and high Q^2
- concise 3-loop corrections are needed to determine $\alpha_s(M_Z)$, m_c and perhaps m_b
- The accuracy of measurements at the LHC reaches a level of precision requiring 3-loop VFNS matching

NNLO: S. Alekhin, J. Blümlein, S. Moch and R. Placakyte, Phys. Rev. D **96** (2017) no.1, 014011 [arXiv:1701.05838 [hep-ph]].

$\alpha_s(M_Z^2) = 0.1147 \pm 0.0008$

$m_c(m_c) = 1.252 \pm 0.018(exp)\ {}^{+0.03}_{-0.02}\ (scale)\ {}^{+0.00}_{-0.07}\ (thy)\text{GeV},$

$m_b(m_b) = 3.84 \pm 0.12\text{GeV}$

$m_t(m_t) = 160.9 \pm 1.1\text{GeV}$ [all in $\overline{\text{MS}}$ scheme.]

Yet approximate NNLO treatment H. Kawamura et al. Nucl. Phys. B **864** (2012) 399 [arXiv:1205.5727].

NS & PS corrections are exact J. Ablinger et al. Nucl. Phys. B **886** (2014) 733 [arXiv:1406.4654 [hep-ph]]; Nucl. Phys. B **890** (2014) 48 [arXiv:1409.1135 [hep-ph]].

Factorization of the Structure Functions

At leading twist the structure functions factorize in terms of a Mellin convolution

$$F_{(2,L)}(x,Q^2) = \sum_j \underbrace{\mathbb{C}_{j,(2,L)}\left(x, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2}\right)}_{perturbative} \otimes \underbrace{f_j(x,\mu^2)}_{nonpert.}$$

into (pert.) Wilson coefficients and (nonpert.) parton distribution functions (PDFs).

$\otimes$ denotes the Mellin convolution

$$f(x) \otimes g(x) \equiv \int_0^1 dy \int_0^1 dz \, \delta(x - yz) f(y) g(z) \; .$$

The subsequent calculations are performed in Mellin space, where $\otimes$ reduces to a multiplication, due to the Mellin transformation

$$\hat{f}(N) = \int_0^1 dx \, x^{N-1} f(x) \; .$$

Wilson coefficients:

$$\mathbb{C}_{j,(2,L)}\left(N, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2}\right) = C_{j,(2,L)}\left(N, \frac{Q^2}{\mu^2}\right) + H_{j,(2,L)}\left(N, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2}\right) \; .$$

At $Q^2 \gg m^2$ the heavy flavor part

$$H_{j,(2,L)}\left(N, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2}\right) = \sum_i C_{i,(2,L)}\left(N, \frac{Q^2}{\mu^2}\right) A_{ij}\left(\frac{m^2}{\mu^2}, N\right)$$

[Buza, Matiounine, Smith, van Neerven 1996 Nucl.Phys.B]

factorizes into the light flavor Wilson coefficients C and the massive operator matrix elements (OMEs) of local operators O_i between partonic states j

$$A_{ij}\left(\frac{m^2}{\mu^2}, N\right) = \langle j \mid O_i \mid j \rangle \; .$$

$\rightarrow$ additional Feynman rules with local operator insertions for partonic matrix elements.

The unpolarized light flavor Wilson coefficients are known up to NNLO [Moch, Vermaseren, Vogt, 2005 Nucl.Phys.B].

For $F_2(x, Q^2)$: at $Q^2 \gtrsim 10m^2$ the asymptotic representation holds at the 1% level.

Status of OME calculations

Leading Order: [Witten 1976, Babcock, Sivers, Wolfram 1978, Shifman, Vainshtein, Zakharov 1978, Leveille, Weiler 1979, Glück, Reya 1979, Glück, Hoffmann, Reya 1982]

Next-to-Leading Order:

[Laenen, van Neerven, Riemersma, Smith 1993]

$Q^2 \gg m^2$: via IBP [Buza, Matiounine, Smith, Migneron, van Neerven 1996]

Compact results via ${}_pF_q$'s [Bierenbaum, Blümlein, Klein, 2007]

$O(\alpha_s^2 \varepsilon)$ (for general N) [Bierenbaum, Blümlein, Klein 2008, 2009]

Next-to-Next-to-Leading Order: $Q^2 \gg m^2$

- Moments for F_2: $N = 2...10(14)$ [Bierenbaum, Blümlein, Klein 2009] mapping large expressions to [MATAD, Steinhauser 2000]
- Contributions to transversity: $N = 1...13$ [Blümlein, Klein, Tödtli 2009]
- Two masses $m_1 \neq m_2 \rightarrow$ Moments N $= 2,4,6$ [JB, Wißbrock 2011]

At 3-loop order for general values of N: Topic of this talk.

The Wilson Coefficients at large Q^2

$$
\begin{aligned}
\text{2014}\quad L^{\mathsf{NS}}_{q,(2,L)}(N_F+1) &= a_s^2\Big[A^{(2),\mathsf{NS}}_{qq,Q}(N_F+1)\ \delta_2 + \hat{C}^{(2),\mathsf{NS}}_{q,(2,L)}(N_F)\Big] \\
&+ a_s^3\Big[A^{(3),\mathsf{NS}}_{qq,Q}(N_F+1)\ \delta_2 + A^{(2),\mathsf{NS}}_{qq,Q}(N_F+1)C^{(1),\mathsf{NS}}_{q,(2,L)}(N_F+1) + \hat{C}^{(3),\mathsf{NS}}_{q,(2,L)}(N_F)\Big] \\
\text{2010}\quad L^{\mathsf{PS}}_{q,(2,L)}(N_F+1) &= a_s^3\Big[\ A^{(3),\mathsf{PS}}_{qq,Q}(N_F+1)\ \delta_2 + A^{(2)}_{gq,Q}(N_F)\ \ N_F\tilde{C}^{(1)}_{g,(2,L)}(N_F+1) + N_F\hat{\tilde{C}}^{(3),\mathsf{PS}}_{q,(2,L)}(N_F \\
\text{2010}\quad L^{\mathsf{S}}_{g,(2,L)}(N_F+1) &= a_s^2 A^{(1)}_{gg,Q}(N_F+1)N_F\tilde{C}^{(1)}_{g,(2,L)}(N_F+1) + a_s^3\Big[\ A^{(3)}_{qg,Q}(N_F+1)\ \delta_2 \\
&+A^{(1)}_{gg,Q}(N_F+1)\ \ N_F\tilde{C}^{(2)}_{g,(2,L)}(N_F+1) + A^{(2)}_{gg,Q}(N_F+1)\ \ N_F\tilde{C}^{(1)}_{g,(2,L)}(N_F+ \\
&+\ A^{(1)}_{Qg}(N_F+1)\ \ N_F\tilde{C}^{(2),\mathsf{PS}}_{q,(2,L)}(N_F+1) + N_F\hat{\tilde{C}}^{(3)}_{g,(2,L)}(N_F)\Big]\ , \\
\text{2014}\quad H^{\mathsf{PS}}_{q,(2,L)}(N_F+1) &= a_s^2\Big[\ A^{(2),\mathsf{PS}}_{Qq}(N_F+1)\ \delta_2 +\ \tilde{C}^{(2),\mathsf{PS}}_{q,(2,L)}(N_F+1)\Big] + a_s^3\Big[\ A^{(3),\mathsf{PS}}_{Qq}(N_F+1)\ \delta_2 \\
&+\ \tilde{C}^{(3),\mathsf{PS}}_{q,(2,L)}(N_F+1) + A^{(2)}_{gq,Q}(N_F+1)\ \tilde{C}^{(1)}_{g,(2,L)}(N_F+1) \\
&+A^{(2),\mathsf{PS}}_{Qq}(N_F+1)\ C^{(1),\mathsf{NS}}_{q,(2,L)}(N_F+1)\Big]\ , \\
H^{\mathsf{S}}_{g,(2,L)}(N_F+1) &= a_s\Big[\ A^{(1)}_{Qg}(N_F+1)\ \delta_2 +\ \tilde{C}^{(1)}_{g,(2,L)}(N_F+1)\Big] + a_s^2\Big[\ A^{(2)}_{Qg}(N_F+1)\ \delta_2 \\
&+\ A^{(1)}_{Qg}(N_F+1)\ C^{(1),\mathsf{NS}}_{q,(2,L)}(N_F+1) +\ A^{(1)}_{gg,Q}(N_F+1)\ \tilde{C}^{(1)}_{g,(2,L)}(N_F+1) \\
&+\ \tilde{C}^{(2)}_{g,(2,L)}(N_F+1)\Big] +\ a_s^3\Big[\ A^{(3)}_{Qg}(N_F+1)\ \delta_2 +\ A^{(2)}_{Qg}(N_F+1)\ C^{(1),\mathsf{NS}}_{q,(2,L)}(N_F+ \\
&+\ A^{(2)}_{gg,Q}(N_F+1)\ \tilde{C}^{(1)}_{g,(2,L)}(N_F+1) +\ A^{(1)}_{Qg}(N_F+1)\Big\{C^{(2),\mathsf{NS}}_{q,(2,L)}(N_F+1) \\
&+\ \tilde{C}^{(2),\mathsf{PS}}_{q,(2,L)}(N_F+1)\Big\} +\ A^{(1)}_{gg,Q}(N_F+1)\ \tilde{C}^{(2)}_{g,(2,L)}(N_F+1) +\ \tilde{C}^{(3)}_{g,(2,L)}(N_F+1
\end{aligned}
$$

All first order factorizable contributions to $H_{Qg}^{(3)}$ are known since 2017.
All logarithmic corrections are known since 2010.

Variable Flavor Number Scheme

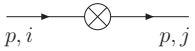
$$
\begin{aligned}
f_k(n_f+1,\mu^2) + f_{\overline{k}}(n_f+1,\mu^2) &= A_{qq,Q}^{\rm NS}\left(n_f,\frac{\mu^2}{m^2}\right) \otimes \left[f_k(n_f,\mu^2) + f_{\overline{k}}(n_f,\mu^2)\right] \\
&\quad + \tilde{A}_{qq,Q}^{\rm PS}\left(n_f,\frac{\mu^2}{m^2}\right) \otimes \Sigma(n_f,\mu^2) + \tilde{A}_{qg,Q}^{\rm S}\left(n_f,\frac{\mu^2}{m^2}\right) \otimes G(n_f,\mu^2) \\
f_{Q+\bar{Q}}(n_f+1,\mu^2) &= \tilde{A}_{Qq}^{\rm PS}\left(n_f,\frac{\mu^2}{m^2}\right) \otimes \Sigma(n_f,\mu^2) + \tilde{A}_{Qg}^{\rm S}\left(n_f,\frac{\mu^2}{m^2}\right) \otimes G(n_f,\mu^2)\,. \\
G(n_f+1,\mu^2) &= A_{gq,Q}^{\rm S}\left(n_f,\frac{\mu^2}{m^2}\right) \otimes \Sigma(n_f,\mu^2) + A_{gg,Q}^{S}\left(n_f,\frac{\mu^2}{m^2}\right) \otimes G(n_f,\mu^2)\,. \\
\Sigma(n_f+1,\mu^2) &= \sum_{k=1}^{n_f+1}\left[f_k(n_f+1,\mu^2) + f_{\overline{k}}(n_f+1,\mu^2)\right] \\
&= \left[A_{qq,Q}^{\rm NS}\left(n_f,\frac{\mu^2}{m^2}\right) + n_f\tilde{A}_{qq,Q}^{\rm PS}\left(n_f,\frac{\mu^2}{m^2}\right) + \tilde{A}_{Qq}^{\rm PS}\left(n_f,\frac{\mu^2}{m^2}\right)\right] \\
&\quad \otimes \Sigma(n_f,\mu^2) \\
&\quad + \left[n_f\tilde{A}_{qg,Q}^{\rm S}\left(n_f,\frac{\mu^2}{m^2}\right) + \tilde{A}_{Qg}^{\rm S}\left(n_f,\frac{\mu^2}{m^2}\right)\right] \otimes G(n_f,\mu^2)
\end{aligned}
$$

There are generalizations necessary in the 2-mass case.

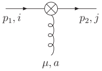
J. Ablinger et al., Nucl.Phys. B921 (2017) 585 [arXiv:1705.07030]

Calculation of the 3-loop operator matrix elements

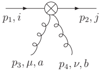
The OMEs are calculated using the QCD Feynman rules together with the following operator insertion Feynman rules:



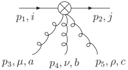
$$\delta^{ij} \not{\Delta} \gamma_\pm (\Delta \cdot p)^{N-1} , \quad N \geq 1$$



$$g t^a_{ji} \Delta^\mu \not{\Delta} \gamma_\pm \sum_{j=0}^{N-2} (\Delta \cdot p_1)^j (\Delta \cdot p_2)^{N-j-2} , \quad N \geq 2$$



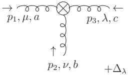
$$g^2 \Delta^\mu \Delta^\nu \not{\Delta} \gamma_\pm \sum_{j=0}^{N-3} \sum_{l=j+1}^{N-2} (\Delta p_2)^j (\Delta p_1)^{N-l-2}$$
$$\left[(t^a t^b)_{ji} (\Delta p_1 + \Delta p_4)^{l-j-1} + (t^b t^a)_{ji} (\Delta p_1 + \Delta p_3)^{l-j-1} \right] ,$$
$$N \geq 3$$



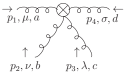
$$g^3 \Delta_\mu \Delta_\nu \Delta_\rho \not{\Delta} \gamma_\pm \sum_{j=0}^{N-4} \sum_{l=j+1}^{N-3} \sum_{m=l+1}^{N-2} (\Delta.p_2)^j (\Delta.p_1)^{N-m-2}$$
$$\Big[\ (t^a t^b t^c)_{ji} (\Delta.p_4 + \Delta.p_5 + \Delta.p_1)^{l-j-1} (\Delta.p_5 + \Delta.p_1)^{m-l-1}$$
$$+ (t^a t^c t^b)_{ji} (\Delta.p_4 + \Delta.p_5 + \Delta.p_1)^{l-j-1} (\Delta.p_4 + \Delta.p_1)^{m-l-1}$$
$$+ (t^b t^a t^c)_{ji} (\Delta.p_3 + \Delta.p_5 + \Delta.p_1)^{l-j-1} (\Delta.p_5 + \Delta.p_1)^{m-l-1}$$
$$+ (t^b t^c t^a)_{ji} (\Delta.p_3 + \Delta.p_5 + \Delta.p_1)^{l-j-1} (\Delta.p_3 + \Delta.p_1)^{m-l-1}$$
$$+ (t^c t^a t^b)_{ji} (\Delta.p_3 + \Delta.p_4 + \Delta.p_1)^{l-j-1} (\Delta.p_4 + \Delta.p_1)^{m-l-1}$$
$$+ (t^c t^b t^a)_{ji} (\Delta.p_3 + \Delta.p_4 + \Delta.p_1)^{l-j-1} (\Delta.p_3 + \Delta.p_1)^{m-l-1} \Big] ,$$
$$N \geq 4$$

$$\gamma_+ = 1 , \quad \gamma_- = \gamma_5 .$$

$$\frac{1+(-1)^N}{2} \delta^{ab} (\Delta \cdot p)^{N-2}$$
$$\left[g_{\mu\nu} (\Delta \cdot p)^2 - (\Delta_\mu p_\nu + \Delta_\nu p_\mu) \Delta \cdot p + p^2 \Delta_\mu \Delta_\nu \right] , \quad N \geq 2$$



$$-ig \frac{1+(-1)^N}{2} f^{abc} \Bigg($$
$$\left[(\Delta_\nu g_{\lambda\mu} - \Delta_\lambda g_{\mu\nu}) \Delta \cdot p_1 + \Delta_\mu (p_{1,\nu} \Delta_\lambda - p_{1,\lambda} \Delta_\nu) \right] (\Delta \cdot p_1)^{N-2}$$
$$+ \Delta_\lambda \left[\Delta \cdot p_1 p_{2,\mu} \Delta_\nu + \Delta \cdot p_2 p_{1,\nu} \Delta_\mu - \Delta \cdot p_1 \Delta \cdot p_2 g_{\mu\nu} - p_1 \cdot p_2 \Delta_\mu \Delta_\nu \right]$$
$$\times \sum_{j=0}^{N-3} (-\Delta \cdot p_1)^j (\Delta \cdot p_2)^{N-3-j}$$
$$+ \left\{ \begin{smallmatrix} p_1 \to p_2 \to p_3 \to p_1 \\ \mu \to \nu \to \lambda \to \mu \end{smallmatrix} \right\} + \left\{ \begin{smallmatrix} p_1 \to p_3 \to p_2 \to p_1 \\ \mu \to \lambda \to \nu \to \mu \end{smallmatrix} \right\} \Bigg) , \quad N \geq 2$$



$$g^2 \frac{1+(-1)^N}{2} \Bigg(f^{abe} f^{cde} O_{\mu\nu\lambda\sigma}(p_1, p_2, p_3, p_4)$$
$$+ f^{ace} f^{bde} O_{\mu\lambda\nu\sigma}(p_1, p_3, p_2, p_4) + f^{ade} f^{bce} O_{\mu\sigma\nu\lambda}(p_1, p_4, p_2, p_3) \Bigg) ,$$
$$O_{\mu\nu\lambda\sigma}(p_1, p_2, p_3, p_4) = \Delta_\nu \Delta_\lambda \Bigg\{ -g_{\mu\sigma} (\Delta \cdot p_3 + \Delta \cdot p_4)^{N-2}$$
$$+ [p_{4,\mu} \Delta_\sigma - \Delta \cdot p_4 g_{\mu\sigma}] \sum_{i=0}^{N-3} (\Delta \cdot p_3 + \Delta \cdot p_4)^i (\Delta \cdot p_4)^{N-3-i}$$
$$- [p_{1,\sigma} \Delta_\mu - \Delta \cdot p_1 g_{\mu\sigma}] \sum_{i=0}^{N-3} (-\Delta \cdot p_1)^i (\Delta \cdot p_3 + \Delta \cdot p_4)^{N-3-i}$$
$$+ [\Delta \cdot p_1 \Delta \cdot p_4 g_{\mu\sigma} + p_1 \cdot p_4 \Delta_\mu \Delta_\sigma - \Delta \cdot p_4 p_{1,\sigma} \Delta_\mu - \Delta \cdot p_1 p_{4,\mu} \Delta_\sigma]$$
$$\times \sum_{i=0}^{N-4} \sum_{j=0}^{i} (-\Delta \cdot p_1)^{N-4-i} (\Delta \cdot p_3 + \Delta \cdot p_4)^{i-j} (\Delta \cdot p_4)^j \Bigg\}$$
$$- \left\{ \begin{smallmatrix} p_1 \leftrightarrow p_2 \\ \mu \leftrightarrow \nu \end{smallmatrix} \right\} - \left\{ \begin{smallmatrix} p_3 \leftrightarrow p_4 \\ \lambda \leftrightarrow \sigma \end{smallmatrix} \right\} + \left\{ \begin{smallmatrix} p_1 \leftrightarrow p_2 , \ p_3 \leftrightarrow p_4 \\ \mu \leftrightarrow \nu , \ \lambda \leftrightarrow \sigma \end{smallmatrix} \right\} , \quad N \geq 2$$

The diagrams are generated using QGRAF [Nogueira 1993 J. Comput. Phys].

	$A_{qq,Q}^{(3),\mathrm{NS}}$	$A_{gq,Q}^{(3)}$	$A_{Qq}^{(3),\mathrm{PS}}$	$A_{gg,Q}^{(3)}$	$A_{Qg}^{(3)}$
No. diagrams	110	86	125	642	1358

A FORM [Vermaseren 2000] program was written in order to perform the γ-matrix algebra in the numerator of all diagrams, which are then expressed as a linear combination of scalar integrals.

$$\begin{aligned} A_{qq,Q}^{(3),\mathrm{NS}} &\to 7426 \ \text{scalar integrals.} \\ A_{gq,Q}^{(3)} &\to 12529 \ \text{scalar integrals.} \\ A_{Qq}^{(3),\mathrm{PS}} &\to 5470 \ \text{scalar integrals.} \end{aligned}$$

$\Longrightarrow$ Need to use integration by parts identities.

$\Longrightarrow$ The reduction for all OMEs has been completed.

$\Longrightarrow$ Use special computers: 14 units with overall 8.5 TB RAM, $>$ 250 TB fast disc, hundreds of mathematica lic. ; IBP: several TB of final relations.

The Method of Arbitrary High Moments

JB, C. Schneider, Phys. Lett. B 771 (2017) 31.

- Exploit the vast amount of IBP relations to obtain and to solve the master integrals for fixed values of N recursively to larger and larger values of N.
- Project onto each color/ζ factor to obtain a large set of rational moments.
- Use guessing methods JB, M. Kauers, S. Klein, C. Schneider, Comput. Phys. Commun. 180 (2009) 2143 to obtain a difference equation for each of these terms (usually large in both degree and order).
- Solve these difference equations using `Sigma`. It will find the solution in case of 1st order factorization or split all first order terms of the solution and returns the remaining recurrence which is not 1st order factorizing.
- One may solve all 1st order factorizable problems this way, over whatsoever alphabet (not requiring a particular choice of basis) : HPLs, Kummer iterated integrals, cyclotomic HPLs, root-iterated integrals, ...

Examples:

- 3-loop anomalous dimensions [possible also at higher loop]
- 3 loop massive OMEs, as for their 1st order factorizable parts.

The 3-loop anomalous dimensions

$$
\begin{aligned}
\gamma_{qg}^{(2)} = {} & C_A N_F^2 T_F^2 \Bigg\{ -\frac{5N^2+8N+10}{N(N+1)(N+2)} \frac{128}{9} S_{-2} - \frac{64P_8}{9N(N+1)^2(N+2)^2} S_1^2 \\
& -\frac{64P_9}{9N(N+1)^2(N+2)^2} S_2 + \frac{64P_{25}}{27N(N+1)^3(N+2)^3} S_1 + \frac{16P_{34}}{27(N-1)N^4(N+1)^4(N+2)^4} \\
& +p_{qg}^{(0)}(N) \left(\frac{32}{9} S_1^3 - \frac{32}{3} S_1 S_2 + \frac{64}{9} S_3 + \frac{128}{3} S_{-3} + \frac{128}{3} S_{2,1} \right) \Bigg\} \\
& +C_F N_F^2 T_F^2 \Bigg\{ \frac{5N^2+3N+2}{N^2(N+1)(N+2)} \frac{32}{3} S_2 + \frac{10N^3+13N^2+29N+6}{N^2(N+1)(N+2)} \frac{32}{9} S_1^2 \\
& -\frac{32P_{12}}{27N^2(N+1)^2(N+2)} S_1 + \frac{4P_{38}}{27(N-1)N^5(N+1)^5(N+2)^4} \\
& +p_{qg}^{(0)}(N) \left(-\frac{32}{9} S_1^3 - \frac{32}{3} S_1 S_2 + \frac{320}{9} S_3 \right) \Bigg\} \\
& +C_A C_F N_F T_F \Bigg\{ -128 \frac{N^3-7N^2-6N+4}{N^2(N+1)^2(N+2)} S_{-2,1} + \frac{32P_5}{N^2(N+1)^2(N+2)} S_{-3} \\
& +\frac{16P_{18}}{9(N-1)N^2(N+1)^2(N+2)^2} S_1^3 - \frac{16P_{24}}{9(N-1)N^2(N+1)^2(N+2)^2} S_3 \\
& -\frac{8P_{27}}{9(N-1)N^3(N+1)^3(N+2)^2} S_1^2 + \frac{8P_{29}}{3(N-1)N^3(N+1)^3(N+2)^3} S_2 \\
& +\frac{P_{37}}{27(N-1)N^5(N+1)^5(N+2)^4} + p_{qg}^{(0)}(N) \Bigg[\left(\frac{640}{3} S_3 - 384 S_{2,1} \right) S_1 + \frac{32}{3} S_1^4 \\
& +160 S_1^2 S_2 - 64 S_2^2 + \left(192 S_1^2 + 64 S_2\right) S_{-2} + 96 S_{-2}^2 + 224 S_{-4} - 64 S_{2,-2} + 64 S_{3,1} \\
& +192 S_{2,1,1} - 256 S_{-2,1,1} - 192 S_1 \zeta_3 \Bigg] - \frac{192P_{17}}{(N-1)N^2(N+1)^2(N+2)^2} \zeta_3 \\
& +\left(\frac{16P_{16}}{3(N-1)N^2(N+1)^2(N+2)^2} S_2 + \frac{16P_{35}}{27(N-1)N^4(N+1)^4(N+2)^4} \right) S_1 \\
& +\left[-\frac{32P_{15}}{N^3(N+1)^3(N+2)} + \frac{128\left(N^3-13N^2-14N-2\right)}{N^2(N+1)^2(N+2)} S_1 \right] S_{-2} \\
& +\frac{96N(N+1)p_{qg}^{(0)}(N)^2}{N-1} S_{2,1} \Bigg\} \\
& +C_A^2 N_F T_F \Bigg\{ -\frac{64P_{11}}{(N-1)N^2(N+1)^2(N+2)^2} S_{-2,1} - \frac{16P_{20}}{9(N-1)N^2(N+1)^2(N+2)^2} S_3 \\
& -\frac{32P_{21}}{3(N-1)N^2(N+1)^2(N+2)^2} S_{-3} - \frac{8P_{22}}{9(N-1)N^2(N+1)^2(N+2)^2} S_1^3 \\
& +\frac{16P_{32}}{9(N-1)^2N^3(N+1)^3(N+2)^3} S_1^2 + \frac{16P_{33}}{9(N-1)^2N^3(N+1)^3(N+2)^3} S_2 \\
& -\frac{8P_{39}}{27(N-1)^2N^5(N+1)^5(N+2)^5} + p_{qg}^{(0)}(N) \Bigg[-\frac{32P_{10}}{3(N-1)N(N+1)(N+2)} S_{2,1} \\
& +\left(-\frac{704}{3} S_3 + 128 S_{2,1} + 512 S_{-2,1} \right) S_1 - 512 S_{-3} S_1 - \frac{16}{3} S_1^4 - 160 S_1^2 S_2 - 16 S_2^2 - 32 S_4 \\
& +\left(-192 S_1^2 + 320 S_2 \right) S_{-2} - 96 S_{-2}^2 + 96 S_{-4} - 448 S_{2,-2} - 128 S_{3,1} + 512 S_{-3,1} \\
& -768 S_{-2,1,1} + 192 S_1 \zeta_3 \Bigg] + \frac{96(N-2)(N+3)P_4}{(N-1)N^2(N+1)^2(N+2)^2} \zeta_3 \\
& +\left(\frac{8P_{19}}{3(N-1)N^2(N+1)^2(N+2)^2} S_2 - \frac{8P_{36}}{27(N-1)^2N^4(N+1)^4(N+2)^4} \right) S_1 \\
& +\left(-\frac{64P_{13}}{(N-1)N^2(N+1)^2(N+2)^2} S_1 + \frac{32P_{30}}{9(N-1)N^3(N+1)^3(N+2)^3} \right) S_{-2} \Bigg\} \\
& +C_F^2 N_F T_F \Bigg\{ \frac{P_{31}}{N^5(N+1)^5(N+2)} - \frac{8P_3}{3N^2(N+1)^2(N+2)} S_1^3 - \frac{16P_6}{3N^2(N+1)^2(N+2)} S_3 \\
& +\frac{64P_{14}}{N^3(N+1)^2(N+2)} S_{-2} - \frac{8P_{23}}{N^3(N+1)^3(N+2)} S_1^2 + \frac{8P_{26}}{N^3(N+1)^3(N+2)} S_2 \\
& +p_{qg}^{(0)}(N) \Bigg[\left(-\frac{704}{3} S_3 + 256 S_{2,1} \right) S_1 - 256 S_{-3} S_1 - \frac{16}{3} S_1^4 - 48 S_2^2 - 160 S_4 - 64 S_{-2}^2 \\
& -192 S_{-4} - \frac{128}{N(N+1)} S_{2,1} - 128 S_{2,-2} + 64 S_{3,1} + 256 S_{-3,1} - 192 S_{2,1,1} \Bigg] \\
& +\frac{96(N-1)\left(3N^2+3N-2\right)}{N^2(N+1)^2} \zeta_3 - 256 \frac{2-N+N^2}{N^2(N+1)(N+2)} \left[S_{-2} S_1 - S_{-2,1} \right] \\
& +\left(-\frac{8P_{28}}{N^4(N+1)^4(N+2)} - \frac{8P_7}{N^2(N+1)^2(N+2)} S_2 \right) S_1 - \frac{128(N-1)}{(N+1)^2(N+2)} S_{-3} \Bigg\},
\end{aligned}
$$

Completed the calculation of all contributing 3-loop anomalous dimensions. Here, $\gamma_{qq}^{(2),\mathrm{PS}}$ and $\gamma_{qg}^{(2)}$ are complete; the others are $\propto T_F$;

Also the corresponding polarized 3-loop anomalous dimensions have been calculated very recently.

Confirmed the results of Moch, Vermaseren, Vogt, 2004 & 2014

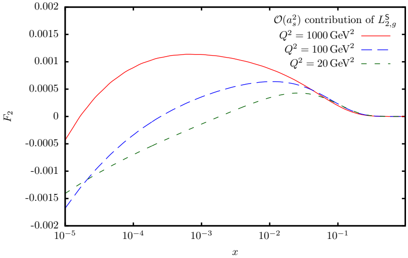
The 1st order factorizable contributions to $A_{Qg}^{(3)}$

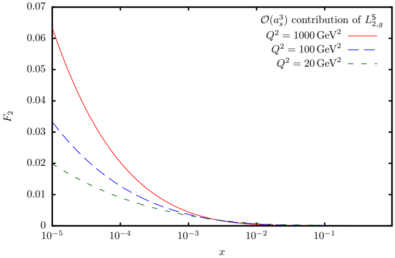
$$
\begin{aligned}
a_{Qg}^{(3)}(N) = {} & C_A^2 T_F \Big\{ t_{C_A^2 T_F}(N) + t_{C_A^2 T_F \zeta_3}(N)\zeta_3 \\
& + \frac{72(-2+N)(3+N)P_2}{(N-1)N^2(1+N)^2(2+N)^2}\zeta_4 - \frac{4P_{17}}{(N-1)N^2(1+N)^2(2+N)^2}\mathsf{B}_4 \\
& + p_{qg}^{(0)}\Big[-16\mathsf{B}_4 S_1 + 144\zeta_4 S_1 + \Big(-16S_1^3 - \frac{4P_7}{3(N-1)N(1+N)(2+N)}S_2 - 32S_1S_2 \\
& - 8S_3 + \Big(-\frac{8P_8}{3(N-1)N(1+N)(2+N)} - 48S_1 \Big) S_{-2} - 8S_{-3} + 16S_{-2,1} \Big)\zeta_2 \Big] \\
& + \Big[\frac{2P_{35}}{9(N-1)^2N^4(1+N)^4(2+N)^4} + \frac{4P_{31}}{9(N-1)^2N^3(1+N)^3(2+N)^3}S_1 \\
& - \frac{4P_{18}}{3(N-1)N^2(1+N)^2(2+N)^2}S_1^2 \Big]\zeta_2 \Big\} + C_A T_F^2 \Big\{ t_{C_A T_F^2}(N) + t_{C_A T_F^2 \zeta_3}(N)\zeta_3 \\
& + N_F \Big\{ -\frac{8P_{36}}{243(N-1)N^5(1+N)^5(2+N)^5} + p_{qg}^{(0)}\Big[\Big(\frac{1888}{27}S_3 + \frac{224}{9}S_{2,1} \Big) S_1 \\
& + \frac{32}{27}S_1^4 + \frac{176}{9}S_1^2S_2 + \frac{80}{9}S_2^2 + \frac{640}{9}S_4 + \Big(-\frac{64(2N-1)}{(N-1)N}S_1 + \frac{128}{3}S_2 \Big) S_{-2} + \frac{64}{9}S_{-4} \\
& - \frac{32}{3}S_{1,-3} - \frac{64}{3}S_{2,-2} - \frac{32}{9}S_{3,1} + \frac{64(2N-1)}{(N-1)N}S_{-2,1} + 64S_{1,1,-2} - \frac{416}{9}S_{2,1,1} \\
& + \Big(\frac{16}{3}S_1^2 + \frac{16}{3}S_2 + \frac{32}{3}S_{-2} \Big)\zeta_2 + \Big(\frac{448\big(1+N+N^2\big)}{9(N-1)N(1+N)(2+N)} - \frac{224}{9}S_1 \Big)\zeta_3 \Big] \\
& + \Big(\frac{16P_{32}}{243(N-1)N^2(1+N)^4(2+N)^4} - \frac{16P_{10}}{27N(1+N)^2(2+N)^2}S_2 \Big) S_1 \\
& + \frac{8P_{20}}{81N(1+N)^3(2+N)^3}S_1^2 - \frac{16P_9}{81N(1+N)^2(2+N)^2}S_1^3 \\
& + \frac{8P_{26}}{81(N-1)N^3(1+N)^3(2+N)^3}S_2 - \frac{32P_{22}}{81(N-1)N^2(1+N)^2(2+N)^2}S_3 \\
& + \frac{32P_{21}}{81N(1+N)^3(2+N)^3}S_{-2} + \frac{32P_{15}}{27(N-1)N^2(1+N)^2(2+N)^2}S_{-3} \\
& - \frac{64P_{16}}{9(N-1)N^2(1+N)^2(2+N)^2}S_{1,-2} - \frac{64P_4}{27N(1+N)^2(2+N)^2}S_{2,1} \\
& + \Big(-\frac{4P_{27}}{9(N-1)N^3(1+N)^3(2+N)^3} - \frac{16P_5}{9N(1+N)^2(2+N)^2}S_1 \Big)\zeta_2 \Big\} \\
& + \Big(-\frac{4P_{28}}{9(N-1)N^3(1+N)^3(2+N)^3} + \frac{160\big(4-N+N^2+4N^3+N^4\big)}{9N(1+N)^2(2+N)^2}S_1 \Big)\zeta_2 \\
& + p_{qg}^{(0)}\Big(\frac{40}{3}S_1^2 + \frac{40}{3}S_2 + \frac{80}{3}S_{-2} \Big)\zeta_2 \Big\} + C_F^2 T_F \Big\{ t_{C_F^2 T_F}(N) + t_{C_F^2 T_F \zeta_3}(N)\zeta_3 \\
& - \frac{16(N-1)\big(-2+3N+3N^2\big)}{N^2(1+N)^2}\mathsf{B}_4 + \frac{72(N-1)\big(-2+3N+3N^2\big)}{N^2(1+N)^2}\zeta_4 \\
& + \Big[\frac{P_{23}}{2N^4(1+N)^4(2+N)} + \frac{8P_{19}}{N^3(1+N)^3(2+N)}S_1 + \frac{4P_3}{N^2(1+N)^2(2+N)}S_1^2 \Big]\zeta_2 \\
& + p_{qg}^{(0)}\Big[-16S_1^3 - \frac{8\big(2+3N+3N^2\big)}{N(1+N)}S_2 + 32S_1S_2 + 16S_3 \\
& + \Big(-\frac{16}{N(1+N)} + 32S_1 \Big) S_{-2} + 16S_{-3} - 32S_{-2,1} \Big]\zeta_2 \Big\} + C_F \Big\{ C_A T_F \Big\{ t_{C_F C_A T_F}(N) + \\
& t_{C_F C_A T_F \zeta_3}(N)\zeta_3 + \frac{32P_{13}}{(N-1)N^2(1+N)^2(2+N)^2}\Big(\mathsf{B}_4 - \frac{9}{2}\zeta_4 \Big) \\
& + p_{qg}^{(0)}\Big[32\mathsf{B}_4 S_1 - 144\zeta_4 S_1 + \Big(32S_1^3 - \frac{12P_1}{(N-1)N(1+N)(2+N)}S_2 - 8S_3 \\
& + \Big(-\frac{8\big(1+3N+3N^2\big)}{N(1+N)} + 16S_1 \Big) S_{-2} - 8S_{-3} + 16S_{-2,1} \Big)\zeta_2 \Big] \\
& + \Big(\frac{P_{25}}{18(N-1)N^3(1+N)^3(2+N)^3} - \frac{4P_{29}}{9(N-1)N^3(1+N)^3(2+N)^3}S_1 \\
& + \frac{8P_{12}}{3(N-1)N^2(1+N)^2(2+N)^2}S_1^2 \Big)\zeta_2 \Big\} + T_F^2 \Big\{ t_{C_F T_F^2}(N) + t_{C_F T_F^2 \zeta_3}(N)\zeta_3 \\
& + N_F \Big\{ \frac{P_{37}}{243(-1+N)N^6(1+N)^6(2+N)^5} + p_{qg}^{(0)}\Big[\Big(-\frac{256}{27}S_3 - \frac{128}{3}S_{2,1} \Big) S_1 \\
& - \frac{32}{27}S_1^4 - \frac{64}{9}S_1^2S_2 - \frac{128}{9}S_2^2 + \frac{256}{9}S_4 - \frac{128}{3}S_{3,1} + \frac{256}{3}S_{2,1,1} - \frac{16}{3}S_1^2\zeta_2 \\
& + \Big(-\frac{56P_{14}}{9(-1+N)N^2(1+N)^2(2+N)} + \frac{224}{9}S_1 \Big)\zeta_3 \Big] + \Big(-\frac{16P_{11}}{243N^2(1+N)^3(2+N)} \\
& + \frac{32\big(24+83N+49N^2+10N^3\big)}{27N^2(1+N)(2+N)}S_2 \Big) S_1 - \frac{32P_6}{81N^2(1+N)^2(2+N)}S_1^2 \\
& + \frac{32\big(24+83N+49N^2+10N^3\big)}{81N^2(1+N)(2+N)}S_1^3 + \frac{8P_{33}}{27(-1+N)N^4(1+N)^4(2+N)^3}S_2 \\
& - \frac{16P_{24}}{81(-1+N)N^3(1+N)^3(2+N)^2}S_3 - \frac{128\big(-2-3N+N^2\big)}{3N^2(1+N)(2+N)}S_{2,1} \\
& + \Big(\frac{2(-2+N)P_{30}}{9(-1+N)N^4(1+N)^4(2+N)^3} + \frac{16\big(12+28N+11N^2+5N^3\big)}{9N^2(1+N)(2+N)}S_1 \Big)\zeta_2 \Big\} \\
& + \Big(\frac{2P_{34}}{9(-1+N)N^4(1+N)^4(2+N)^3} + \frac{80\big(6+11N+4N^2+N^3\big)}{9N^2(1+N)(2+N)}S_1 \Big)\zeta_2 \\
& + p_{qg}^{(0)}\Big(-\frac{40}{3}S_1^2 + 8S_2 \Big)\zeta_2 \Big\} \Big\} - \frac{64}{9}p_{qg}^{(0)} T_F^3 \zeta_3,
\end{aligned}
$$

18 out of 28 color/ζ factors were completed. (2000 moments)

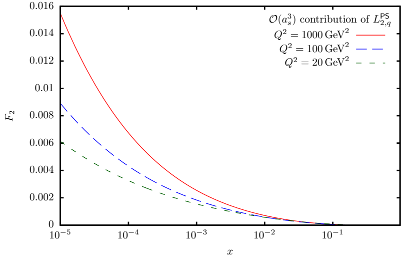
From 8000 moments we got the T_F^2 difference equations also; 4 of them are no longer first order factorizable. For them and other remaining color/ζ factors we have the 1st order terms too, i.e. we obtained the analytic results for 1122 of 1358 diagrams.

Numerical Results : $L_{g,2}^{\mathrm{S}}$ and $L_{q,2}^{\mathrm{PS}}$



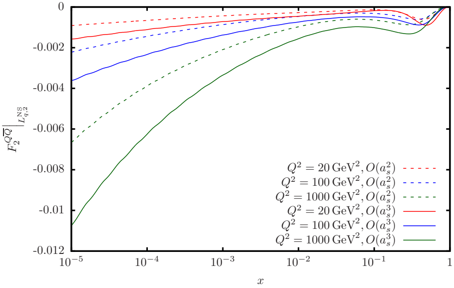


$O(a_s^2)$ $L_{2,g}^{\mathrm{S}}$ $O(a_s^3)$ $L_{2,g}^{\mathrm{S}}$

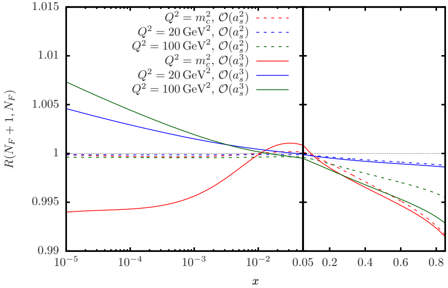


$L_{q,2}^{\mathrm{PS}}$

$L_{q,2}^{\rm NS}$

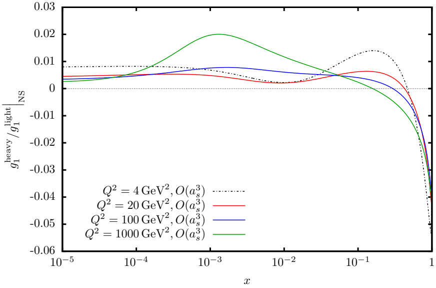


Contribution to $F_2(x, Q^2)$

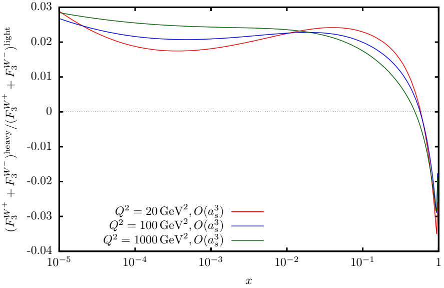


VFNS matching

NS corrections to $g_{1(2)}(x, Q^2)$ and $xF_3^{W^+ + W^-}$



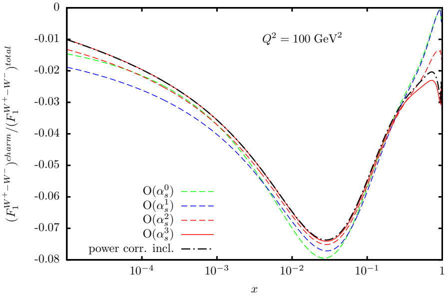
$g_1(x, Q^2)$



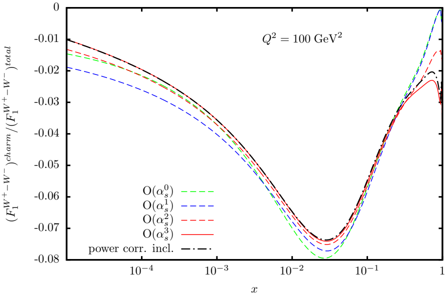
$xF_3^{W^+ + W^-}(x, Q^2)$

The corrections to $g_2(x, Q^2)$ are obtained using the Wandzura-Wilczek relation.

NS corrections to $F_1^{W^+-W^-}$ and $F_2^{W^+-W^-}$



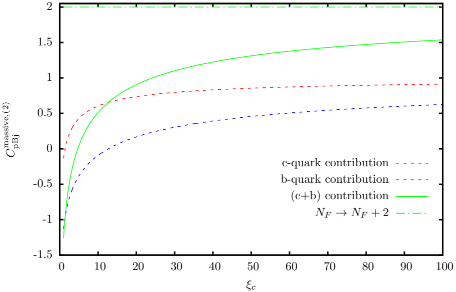
$F_1^{W^+-W^-}(x, Q^2)$



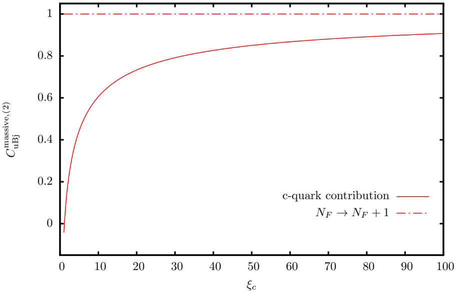
$F_2^{W^+-W^-}(x, Q^2)$

The massless corrections are due to [A. Vogt et al. arXiv:1606.08907 [hep-ph].]
from [A. Behring et al. Phys. Rev. D **94** (2016) no.11, 114006 [arXiv:1609.06255 [hep-ph]]].

$O(\alpha_s^2)$ Complete NS corrections



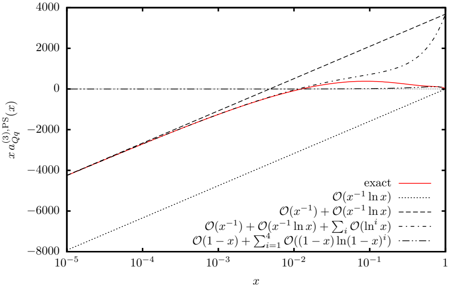
pol BJ sum rule



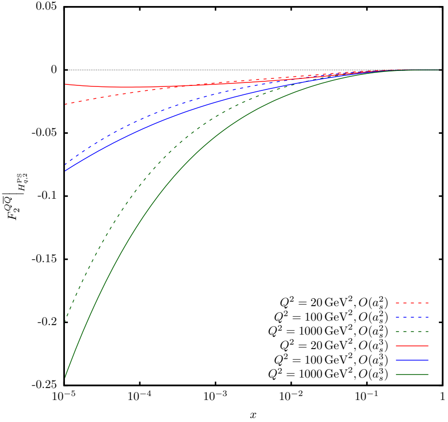
unp. BJ sum rule

Note the negative corrections at low Q^2!
Already for charm it takes quite a while to become massless.

JB, G. Falcioni, A. De Freitas, Nucl. Phys. B910 (2016) 568.



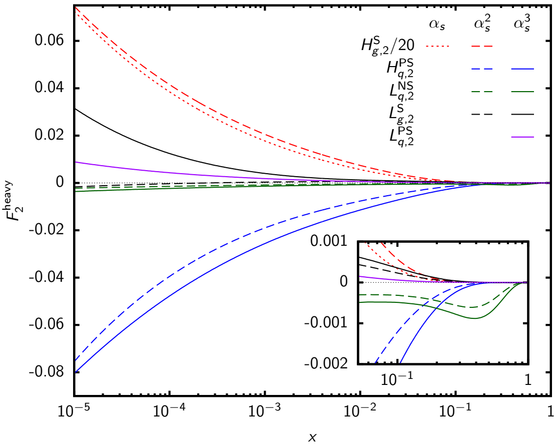
$a_{Qq}^{(3),\rm PS}$



Contribution to $F_2(x, Q^2)$

The leading small x approximation corresponding to High-energy factorization and small x heavy flavor production S. Catani, M. Ciafaloni, F. Hautmann, Nucl.Phys. B366 (1991) 135 departs from the physical result everywhere except for $x = 1$ (dotted line).

The present NC corrections to $F_2(x, Q^2)$



$Q^2 = 100\text{GeV}^2$ [$H^S_{g,2}$ scaled down by a factor 20.]

We have calculated 18 of 28 color and ζ-factors of $A^{(3)}_{Qg}$, as well as 2000 moments analytically. (MATAD, 2009: $N \leq 10$).

Here the method of arbitrary high moments proved to be crucial.

3-Loop OME: $A_{gg,Q}$

$$
\begin{aligned}
a_{gg,Q}^{(3)} = \frac{1+(-1)^N}{2}\Bigg\{ & C_F^2 T_F \Bigg[\frac{16(N^2+N+2)}{N^2(N+1)^2}\sum_{i=1}^{N}\frac{\binom{2i}{i}\left(\sum_{j=1}^{i}\frac{4^j S_1(j-1)}{\binom{2j}{j}j^2}-7\zeta_3\right)}{4^i(i+1)^2} - \frac{4P_{69}S_1^2}{3(N-1)N^4(N+1)^4(N+2)} \\
& +\tilde{\gamma}_{gq}^{(0)}\left(\frac{128\left(S_{-4}-S_{-3}S_1+S_{-3,1}+2S_{-2,2}\right)}{3N(N+1)(N+2)}+\frac{4(5N^2+5N-22)S_1^2S_2}{3N(N+1)(N+2)}+\cdots\right)+\cdots\Bigg] \\
& +C_A C_F T_F\Bigg[\frac{16P_{42}}{3(N-1)N^2(N+1)^2(N+2)}\sum_{i=1}^{N}\frac{\binom{2i}{i}\left(\sum_{j=1}^{i}\frac{4^j S_1(j-1)}{\binom{2j}{j}j^2}-7\zeta_3\right)}{4^i(i+1)^2}+\frac{32P_2S_{-2,2}}{(N-1)N^2(N+1)^2(N+2)} \\
& -\frac{64P_{14}S_{-2,1,1}}{3(N-1)N^2(N+1)^2(N+2)}-\frac{16P_{23}S_{-4}}{3(N-1)N^2(N+1)^2(N+2)}+\frac{4P_{63}S_4}{3(N-2)(N-1)N^2(N+1)^2(N+2)}+\cdots\Bigg] \\
& +C_A^2 T_F\Bigg[-\frac{4P_{46}}{3(N-1)N^2(N+1)^2(N+2)}\sum_{i=1}^{N}\frac{\binom{2i}{i}\left(\sum_{j=1}^{i}\frac{4^j S_1(j-1)}{\binom{2j}{j}j^2}-7\zeta_3\right)}{4^i(i+1)^2}+\frac{256P_5S_{-2,2}}{9(N-1)N^2(N+1)^2(N+2)} \\
& +\frac{32P_{30}S_{-2,1,1}+16P_{35}S_{-3,1}+16P_{44}S_{-4}}{9(N-1)N^2(N+1)^2(N+2)}+\frac{16P_{52}S_{-2}^2}{27(N-1)N^2(N+1)^2(N+2)}+\frac{8P_{36}S_2^2}{9(N-1)N^2(N+1)^2}+\cdots\Bigg] \\
& +C_F T_F^2\Bigg[-\frac{16P_{48}\binom{2N}{N}4^{-N}\left(\sum_{i=1}^{N}\frac{4^i S_1(i-1)}{\binom{2i}{i}i^2}-7\zeta_3\right)}{3(N-1)N(N+1)^2(N+2)(2N-3)(2N-1)}-\frac{32P_{86}S_1}{81(N-1)N^4(N+1)^4(N+2)(2N-3)(2N-1)} \\
& +\frac{16P_{45}S_1^2}{27(N-1)N^3(N+1)^3(N+2)}-\frac{16P_{45}S_2}{9(N-1)N^3(N+1)^3(N+2)}+\cdots\Bigg]+\cdots\Bigg\} \qquad (1)
\end{aligned}
$$

Also, with this calculation we were able to re-derive the three loop anomalous dimension $\gamma_{gg}^{(3)}$ for the terms $\propto T_F$, and obtained agreement with the literature. The x-space representation is underway.

The NC PS contributions to $F_2(x, Q^2)$ and $F_L(x, Q^2)$

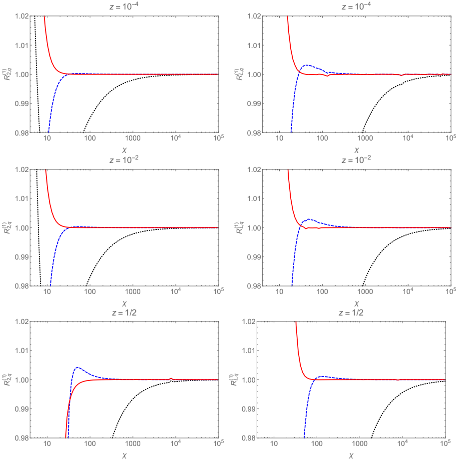
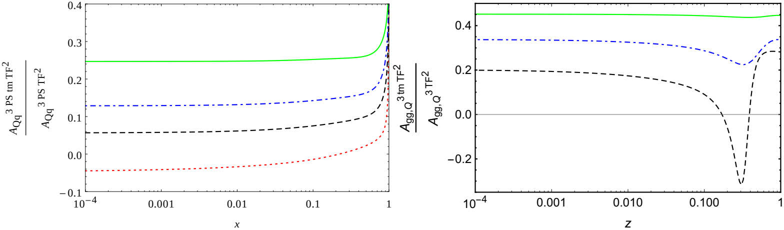


Figure 1: The ratios $R^{(1)}_{2,q}$ (left) and $R^{(1)}_{L,q}$ (right) as a function of $\chi = Q^2/m^2$ for different values of z gradually improved with κ suppressed terms. Dotted lines: asymptotic result; dashed lines: $O(m^2/Q^2)$ improved; solid lines : $O((m^2/Q^2)^2)$ improved.

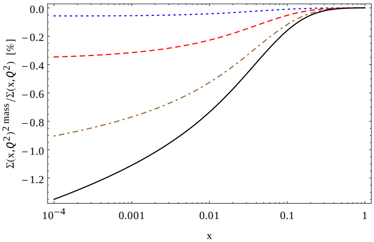
Different convergence range for F_2 and F_L w.r.t Q^2 at $O(\alpha_s^2)$.

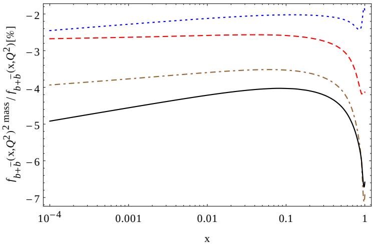
3-Loop 2 Mass contributions: PS and gg



The 2-mass contributions are a significant part of the the T_F^2 terms (also in the other channels).

2-Loop 2 Mass VFNS: Singlet and b-quark distributions





Visible effects at high luminosity @ NLO already.

Charm and bottom decouple basically at the same scale.

Conclusions

- 2009: 10-14 Mellin Moments for all massive 3-loop OMEs, coefficient functions.
  2010: Coefficient functions $L_q^{(3),\mathrm{PS}}(N)$, $L_g^{(3),\mathrm{S}}(N)$.
- 2013: Ladder, V-Graph and Benz-topologies for graphs, with no singularities in ε can be systematically calculated for general N.
- Here new functions occur (including a larger number of root-letters in iterated integrals).
- 2014 $L_q^{\mathrm{NS},(3)}$, $A_{gq,Q}^{\mathrm{S},(3)}$, $A_{qq,Q}^{\mathrm{NS,TR}(3)}$, $H_{2,q}^{\mathrm{PS}(3)}$ and $A_{Qq}^{\mathrm{PS}(3)}$ were completed.
- The $O(\alpha_s^2)$ charged current Wilson coefficients have been completed.
- All corresponding 3-loop anomalous dimensions were computed, those for transversity for the first time ab initio; those for the PS- and the qg-case independently for the first time.
- In all NS-cases [NC and CC] we also computed all power corrections at $O(a_s^2)$ and the associated sum rules in the inclusive case improving an earlier result by JB & W. van Neerven.

Conclusions

- All master integrals based on iterative integrals over whatsoever alphabet for $A_{gg,Q}^{(3)}$ and $A_{Qg}^{(3)}$ have been computed and $A_{gg,Q}^{(3)}$ is known for any even integer moment $N \geq 2$. Here all the topologies, including the ladder- and V-topologies have been solved.
- The method of high moments allowed to efficiently derive a series of important general N expressions. We are able to generate analytically moments up to $\mathrm{N} = 2000$ to 8000 at 3-loop order.
- We have all the principal means to reconstruct $A_{Qg}^{(3)}$ systematically at very high accuracy. The full analytic solution will request more mathematical efforts.
- Different new computer-algebra and mathematical technologies were developed. These efforts will continue. The technologies are certainly useful for various present and upcoming calculations for the LHC and ILC.
- Quite a series of 2-mass three loop corrections have been also obtained.

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