Recent Results on Finite Harmonic Sums in Quantum Field Theory

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- Structural Relations of Harmonic Sums up to $w = 6$
- From Moments to Functions in HO QCD

Based on :

arXiv:0901.3106, arXiv:0901.0837;

arxiv:0902.4091 [hep-ph], CPC in print with M. Kauers (RISC), S. Klein (DESY), and C. Schneider (RISC)

1. Introduction

Why are zero- and single scale quantities in Quantum Field Theory related to ζ-values, nested harmonic sums and related objects ?

- The former quantities can be obtained from the latter putting the Mellin variable N either to fixed values or $N \to \infty$.

Perturbation Theory [fixed order]

Scalar Propagators:

$$\frac{i}{p^2 + i\epsilon}$$

Combine Momenta using the Feynman Trick

$$\frac{1}{A \cdot B} = \int_0^1 \frac{dx}{[xA + (1-x)B]^2}$$

Feynman parameter integrals substitute all non-trivial angular integrals in $D = 4 + \varepsilon$–dimensions.

Introduction

Momentum Integrals yield rational functions of Γ–functions

$$\left.\frac{\Gamma(n_1+\alpha_1\varepsilon)...\Gamma(n_k+\alpha_k\varepsilon)}{\Gamma(m_1+\beta_1\varepsilon)...\Gamma(m_l+\beta_k\varepsilon)}\right|_{m_i,n_i\epsilon\mathbf{Z},\alpha_i,\beta_i\epsilon\mathbf{Q}}$$

- The scale-ratio in the diagrams factors form the Feynman parameter integrals for single scale processes

The Feynman parameter integrals can be transformed into Mellin-Barnes Integrals

$$\frac{1}{(A+B)^q}=\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}d\sigma A^\sigma B^{-q-\sigma}\frac{\Gamma(-\sigma)\Gamma(q+\sigma)}{\Gamma(q)}$$

Then all Feynman parameter integrals can again be integrated $\Longrightarrow$ rational function of Γ–functions

One or more Feynman parameters contain N as power $x_i^N, ...$ in the numerators. $\Longrightarrow$ the Γ–functions contain N as argument.

The Mellin-Barnes Integrals can be carried out using the Residue Theorem.
$\Longrightarrow$ several infinite sums over the rational functions of Γ–functions.

Introduction

- Seek compact representations for these in terms of (Generalizations of) generalized hypergeometric functions.
- The Mellin variable N is a discrete quantity in the first place for physical reasons

$\Longrightarrow$ Light-cone expansion; cut vertex method + dispersion relations

- Perform the ε–expansion.
- The respective coefficients obey Difference Equations of finite order.
- The ε–expansion of Pochhammer-Symbols & Γ–functions leads to products of single finite harmonic sums and MZV's.
- The infinite sums over the Mellin–Barnes parameters lead to the respective Nesting.
- Observation: Most of the sums occurring are Nested Finite Harmonic Sums.
- However, other related sums are possible too for individual Feynman diagrams. [Vermaseren et al. (2005)]
- General solution formalisms like, `Sigma`, will reveal this uniquely. cf. C. Schneider.

Introduction

- Single scale processes in massless Quantum Field Theories or being considered in the limit $m^2/Q^2 \to 0$ exhibit significant simplifications when calculated in Mellin space.
- This is, to some extent, due to structure of Feynman parameter intergrals which posess a Mellin symmetry.
- Harmonic sums form the appropriate language to derive compact expressions in the respective calculations.
- We will line out the relations of the harmonic sums, resp. their continuations to $N \in \mathbf{Q}, \mathbf{R}, \mathbf{C}$.

x-space results :

Nielsen-type integrals, resp. harmonic polylogarithms (E. Remiddi and J. Vermaseren (1999))

$$S_{n,p,q}(x) = \frac{(-1)^{n+p+q-1}}{\Gamma(n)p!q!} \int_0^1 \frac{dz}{z} \ln^{(n-1)}(z) \ln^p(1-zx) \ln^q(1+zx)$$

2 Loop Wilson Coefficients

W.L. van Neerven et al.: (1992) 79 functions 80 objects would be maximal.

- The high complexity is partly caused applying the the IBP–Method.
- x–space usually is not the best space to work in.

3 Loop Anomalous Dimensions & Wilson Coefficients

- $\Longrightarrow$ Harmonic Sums in linear representation.
- Still high complexity of terms.
- Compactification possible applying algebraic and structural relations.
- Observation : In all single scale calculations the same Basic Functions occur in the resp. weight.

- $\Longrightarrow$ Derive these Universal Functions and their complex analysis.

2. Algebraic Relations

cf. J.Blümlein, Comput. Phys. Commun. **159** (2004) 19

Number of harmonic sums up to weight w : 3^{w-1}.

Harmonic sums form a quasi-shuffle algebra through . (M.E. Hoffman, J. Algebraic Combin. **11** (2000) 49)

$$\begin{aligned} S_{a_1,a_2}S_{a_3,a_4} &= S_{a_1,a_2,a_3,a_4} + S_{a_1,a_3,a_2,a_4} + S_{a_1,a_2,a_4,a_2} \\ &+ S_{a_3,a_4,a_1,a_2} + S_{a_3,a_1,a_4,a_2} + S_{a_3,a_1,a_2,a_4} \quad etc. \end{aligned}$$

Solve all the linear equations possible for the harmonic sums $\Longrightarrow$ algebraic basis.

Let $\{a, a, a, ..., b, b, ..., ..., z, z\}$ a set of n_1 a's, n_2 b's etc. The number of basis elements corresponding to all words formed by ALL the above letters is:

$$l_n(n_1, ..., n_q) = \frac{1}{n}\sum_{d|n_i}\mu(d)\frac{(n/d)!}{(n_1/d)!...(n_d/d)!}, \quad \sum_i n_i = n$$

(E. Witt, 1937) $\Longrightarrow$ # Lyndon words

w	1	2	3	4	5	6
$\#_c$	2	8	26	80	242	728
$\#_r$	2	5	13	31	79	195

Algebraic Relations

Observation in Quantum Field Theory :

At least up to $O(\alpha_s^3)$ the contributing harmonic sums exhibit never any index $a_k = -1$ applying a compact representation.

The number of sums of this type is

$$N_{\neg\{-1\}}(w) = \frac{1}{2}\left[\left(1-\sqrt{2}\right)^w + \left(1+\sqrt{2}\right)^w\right]$$

$$N^{\text{basic}}_{\neg\{-1\}}(\mathsf{w}) = \frac{2}{\mathsf{w}} \sum_{d|\mathsf{w}} \mu\left(\frac{\mathsf{w}}{d}\right) N^{\text{basic}}_{\neg\{-1\}}(d) \ .$$

w	1	2	3	4	5	6
$\#_c$	1	4	11	28	69	168
$\#_r$	1	3	7	14	30	60

- Here $\#_c$ is smaller than $\#_r$ in the general case.

Algebraic Relations

Remark:

Harmonic, Generalized Harmonic Polylogarithms and Multiple Polylogarithms also form shuffle algebras. As shuffle algebras are sub-sets of the quasi-shuffle algebra studied above, the respective algebraic relations can be derived directly.

- Form the index alphabet.
- Solve the shuffle-relations $\Longrightarrow$ Basis

As the relations in J.B., Comput. Phys. Commun. **159** (2004) 19 are of arbitrary weight (general alphabet) and depth $d \leq 6$ the corresponding relations can be read off there.

Algorithms to extend this scenario are available.

3. Structural Relations

w = 1:

$$\frac{1}{1-x} \qquad \& \qquad \frac{1}{1+x}$$

$$\frac{1}{1-x^2} = \frac{1}{2}\left[\frac{1}{1-x} + \frac{1}{1+x}\right]$$

$$\mathbf{M}\left[\left(\frac{1}{1-x}\right)_+\right]\left(\frac{N}{2}\right) = \mathbf{M}\left[\left(\frac{1}{1-x}\right)_+\right](N) + \mathbf{M}\left[\frac{1}{1+x}\right](N) + \ln(2)$$

$$-\psi\left(\frac{N}{2}\right) - \gamma_E \quad = \quad -\psi(N) - \gamma_E + \beta(N) + \ln(2); \qquad \beta(N) = \frac{1}{2}\left[\psi\left(\frac{N+1}{2}\right) - \psi\left(\frac{N}{2}\right)\right]$$

- $S_{-1}(N)$ depends on $S_1(N)$ for $N \,\epsilon\, \mathbf{Q}$

Structural Relations

$N \,\epsilon\, \mathbf{R}$:

$$S_2(N) = -\frac{d}{dN} S_1(N) + \zeta_2 \qquad \text{(etc.)}$$

For $N \,\epsilon\, \mathbf{R}$: only one independent single sum occurs.

$$S_1(N) = \sum_{k=1}^{N} \frac{1}{k} = \psi(N+1) + \gamma_E$$

Harmonic sums $\cup$ $\zeta_{k_1,\ldots k_n}$ are closed under differentiation.

w = 2:

$$\mathbf{M}\left[\frac{\ln(1-x)}{1+x}\right](N) = -\mathbf{M}\left[\frac{\ln(1+x)}{1+x}\right](N) - [\psi(N) + \gamma_E + \ln(2)]\beta(N) + \beta'(N)$$

$$F_1(N) := \mathbf{M}\left[\frac{\ln(1+x)}{1+x}\right](N) \to S_{1,-1}(N)$$

Structural Relations

The relations for w = 2 were explored by N. Nielsen (1906).

$$\xi(N) = \mathbf{M}\left[\left(\frac{\ln(1-x)}{1+x}\right)_+\right](N); \qquad \eta(N) = \mathbf{M}\left[\frac{\ln(1+x)-\ln(2)}{1-x}\right](N)$$

$$\xi_1(N) = \mathbf{M}\left[\frac{\ln(1+x)}{1+x}\right](N); \qquad -\xi_2(N) = \mathbf{M}\left[\frac{\ln(1-x)}{1+x}\right](N)$$

$$\begin{aligned}
[\psi(z)+\gamma_E][\psi(1-z)+\gamma_E] &= 2\zeta_2 - \xi(z) - \xi(1-z) \\
\beta(z)[\psi(z)+\gamma_E] &= \beta'(z) + \beta(z)\ln(2) - \xi_1(z) + \xi_2(z) \\
\beta(z)\beta(1-z) &= \eta(z) + \eta(1-z) \\
\beta^2(z) &= \psi'(z) - 2\eta(z)
\end{aligned}$$

Structural Relations

If half-integer arguments in N are allowed
$\mathbf{M}[\mathrm{Li}_k(-x)/(x \pm 1)](N)$ are not independent functions :

$$\frac{1}{2^{k-2}} \frac{\mathrm{Li}_k(x^2)}{1-x^2} = \frac{\mathrm{Li}_k(x)}{1-x} + \frac{\mathrm{Li}_k(x)}{1+x} + \frac{\mathrm{Li}_k(-x)}{1-x} + \frac{\mathrm{Li}_k(-x)}{1+x} \rightarrow \frac{\mathrm{Li}_k(-x)}{1-x}$$

- There always exists another IBP relation to express also $\mathrm{Li}_k(-x)/(1+x)$

$$\begin{aligned}
(-1)^N \mathbf{M}\left[\frac{\mathrm{Li}_2(-x)}{1+x}\right](N) &= -S_{2,-1}(N) - \ln(2)[S_2(N) - S_{-2}(N)] \\
&\quad -\frac{1}{2}\zeta_2 S_{-1}(N) + \frac{1}{4}\zeta_3 - \frac{1}{2}\zeta_2 \ln(2) \\
(-1)^N \mathbf{M}\left[\frac{-\mathrm{Li}_2(x) - \ln(x)\ln(1-x) + \zeta_2}{1+x}\right](N) &= -S_{-1,2}(N) + \zeta_2 S_{-1}(N) - \zeta_3 + \frac{3}{2}\zeta_2 \ln(2) \\
S_{-1,2}(N) + S_{2,-1}(N) &= S_{-1}(N) S_2(N) + S_{-3}(N)
\end{aligned}$$

Structural Relations

$$(-1)^{(N+1)}\mathbf{M}\left[\frac{\mathrm{Li}_3(-x)}{1+x}\right](N) = -S_{3,-1}(N) - \ln(2)[S_3(N) - S_{-3}(N)] - \frac{1}{2}\zeta_2 S_{-2}(N) + \frac{3}{4}\zeta_3 S_{-1}(N) - \frac{1}{8}\zeta_2^2 + \frac{3}{4}\ln(2)\zeta_3$$

$$(-1)^{N}\mathbf{M}\left[\frac{S_{1,2}(1-x)}{1+x}\right](N) = -S_{-1,3}(N) + \zeta_3 S_{-1}(N) - \frac{19}{40}\zeta_2^2 + \frac{7}{4}\zeta_3\ln(2)$$

$$S_{1,2}(1-x) = -\mathrm{Li}_3(x) + \log(x)\mathrm{Li}_2(x) + \frac{1}{2}\log(1-x)\log^2(x) + \zeta_3$$

$$S_{-1,3}(N) + S_{3,-1}(N) = S_{-1}(N)S_3(N) + S_{-4}(N)$$

- At even w there exists an algebraic relation

$$S_{\mathsf{w}/2,\mathsf{w}/2}(N) = \frac{1}{2}\left[S^2_{\mathsf{w}/2}(N) + S_{\mathsf{w}}(N)\right]$$

which yields an additional relation for $\mathrm{Li}_k(x)/(1+x)$.

w = 3:

$$\rightarrow \frac{\mathrm{Li}_2(x)}{x\pm 1}, \qquad \frac{\ln^2(1+x)}{x\pm 1}$$

Double Sums in General

- Applying differential operators one may show :

For $N \epsilon \mathbf{R}$ double harmonic sums can always be represented by one basic function for even weight and two basic functions for odd weight.

$$\Longrightarrow \quad \frac{\mathrm{Li}_k(x)}{1+x}, \qquad \frac{\mathrm{Li}_k(x)}{1 \pm x}$$

Examples, which reduce :

$$S_{2,3}(N) = \mathbf{M}\left[\left(\frac{\ln(x)\left[S_{1,2}(1-x)-\zeta_3\right]+3\left[S_{1,3}(1-x)-\zeta_4\right]}{x-1}\right)_+\right](N)+3\zeta_4 S_1(N)$$

$$S_{-4,-2}(N) = -\mathbf{M}\left[\left(\frac{4\mathrm{Li}_5(-x)-\ln(x)\mathrm{Li}_4(-x)}{x-1}\right)_+\right](N)$$
$$+\frac{1}{2}\zeta_2\left[S_4(N)-S_{-4}(N)\right]-\frac{3}{2}\zeta_3 S_3(N)+\frac{21}{8}\zeta_4 S_2(N)-\frac{15}{4}\zeta_5 S_1(N)$$

$$S_{1,3}(1-x) = -\mathrm{Li}_4(x)+\log(x)\mathrm{Li}_3(x)-\frac{1}{2}\log^2(x)\mathrm{Li}_2(x)-\frac{1}{6}\log^3(x)\log(1-x)+\zeta_4$$

Structural Relations

w = 4; $i \neq -1$:

$$\frac{\mathrm{Li}_3(x)}{x+1}, \qquad \frac{S_{1,2}(x)}{x \pm 1}$$

The Mellin transform of

$$\left(\frac{\mathrm{Li}_3(x)}{x-1}\right)_+$$

reads

$$\mathbf{M}\left[\left(\frac{\mathrm{Li}_3(x)}{x-1}\right)_+\right](N) = \frac{1}{2}\left\{\frac{d}{dN}\mathbf{M}\left[\left(\frac{\mathrm{Li}_2(x)+\zeta_2}{x-1}\right)_+\right](N) - S_{2,2}(N-1) + \zeta_2 S_2(N-1) + 2\zeta_3 S_1(N-1)\right\}$$

and can be traced back to that of $(\mathrm{Li}_2(x)/(x-1))_+$

Structural Relations

$\mathrm{w} = 5;\ i \neq -1\ :$

$$\frac{\mathrm{Li}_4(x)}{x \pm 1} \qquad \frac{S_{1,3}(x)}{x+1} \qquad \frac{S_{2,2}(x)}{x \pm 1} \qquad \frac{\mathrm{Li}_2^2(x)}{x+1} \qquad \frac{\ln(x) S_{1,2}(-x) - \mathrm{Li}_2^2(-x)/2}{x \pm 1}$$

$$\frac{\mathrm{Li}_2^2(x)}{x-1} \qquad \frac{S_{1,3}(x)}{x-1} \qquad [\text{Occur in the } 3-\text{loop Wils. Coeff. only}]$$

$\mathrm{w} = 6;\ i \neq -1\ :$

$$\frac{\mathrm{Li}_5(x)}{x+1} \qquad \frac{S_{3,2}(x)}{x \pm 1} \qquad \frac{S_{2,3}(x)}{x \pm 1} \qquad \frac{S_{1,4}(x)}{x \pm 1} \qquad \frac{\mathrm{Li}_2(x)\mathrm{Li}_3(x)}{x \pm 1}$$

$$\frac{S_{1,2}(x)\mathrm{Li}_2(x)}{x+1} \qquad \frac{A_1(x)}{x+1} \qquad \frac{A_2(x)}{x \pm 1} \qquad \frac{A_3(x)}{x+1} \qquad \frac{H_{0,-1,0,1,1}(x)}{x \pm 1}$$

$$\frac{H_{0,0,-1,0,1}(x)}{x \pm 1} \qquad \frac{A_1(-x) + 2S_{3,2}(-x) - 2S_{2,2}(-x)\ln(x)}{x \pm 1}$$

$$\frac{A_1(-x) + 2S_{3,2}(-x) - 2S_{2,2}(-x)\ln(x) + \mathrm{Li}_2^2(-x)\ln(x)/4 - \mathrm{Li}_3(-x)\mathrm{Li}_2(-x)}{x-1}$$

New numerator functions :

$$A_1(x) = \int_0^x \frac{dy}{y} \mathrm{Li}_2^2(y), \qquad A_2(x) = \int_0^x \frac{dy}{y} \ln(1-y) S_{1,2}(y), \qquad A_3(x) = \int_0^x \frac{dy}{y} [\mathrm{Li}_4(1-y) - \zeta_4]$$

Harmonic Polylogarithms

- iterated integrals over the alphabet

$$f_a(x) = \frac{1}{x}, \qquad \frac{1}{1-x}, \qquad \frac{1}{1+x}$$

$$H_0(x) = \int_0^x \frac{dx}{x}, \qquad H_1(x) = \int_0^x \frac{dx}{1-x}, \qquad H_{-1}(x) = \int_0^x \frac{dx}{1+x}$$

$$H_{a,\vec{b}}(x) = \int_0^x dz f_a(z) H_{\vec{b}}(z)$$

Structural Relations

w = 5; $i \neq -1$:

$$\frac{\mathrm{Li}_4(x)}{x\pm1} \quad \frac{S_{1,3}(x)}{x+1} \quad \frac{S_{2,2}(x)}{x\pm1} \quad \frac{\mathrm{Li}_2^2(x)}{x+1} \quad \frac{\ln(x)S_{1,2}(-x)-\mathrm{Li}_2^2(-x)/2}{x\pm1}$$

$$\frac{\mathrm{Li}_2^2(x)}{x-1} \quad \frac{S_{1,3}(x)}{x-1}$$

[Occur in the 3 – loop Wils. Coeff. only]

w = 6; $i \neq -1$:

$$\frac{\mathrm{Li}_5(x)}{x+1} \quad \frac{S_{3,2}(x)}{x\pm1} \quad \frac{S_{2,3}(x)}{x\pm1} \quad \frac{S_{1,4}(x)}{x\pm1} \quad \frac{\mathrm{Li}_2(x)\mathrm{Li}_3(x)}{x\pm1}$$

$$\frac{S_{1,2}(x)\mathrm{Li}_2(x)}{x+1} \quad \frac{A_1(x)}{x+1} \quad \frac{A_2(x)}{x\pm1} \quad \frac{A_3(x)}{x+1} \quad \frac{H_{0,-1,0,1,1}(x)}{x\pm1}$$

$$\frac{H_{0,0,-1,0,1}(x)}{x\pm1} \quad \frac{A_1(-x)+2S_{3,2}(-x)-2S_{2,2}(-x)\ln(x)}{x\pm1}$$

$$\frac{A_1(-x)+2S_{3,2}(-x)-2S_{2,2}(-x)\ln(x)+\mathrm{Li}_2^2(-x)\ln(x)/4-\mathrm{Li}_3(-x)\mathrm{Li}_2(-x)}{x-1}$$

New numerator functions :

$$A_1(x)=\int_0^x\frac{dy}{y}\mathrm{Li}_2^2(y), \qquad A_2(x)=\int_0^x\frac{dy}{y}\ln(1-y)S_{1,2}(y), \qquad A_3(x)=\int_0^x\frac{dy}{y}[\mathrm{Li}_4(1-y)-\zeta_4]$$

Representation of Observables

- Unpolarized and Polarized Drell-Yan an Higgs-Boson Production Cross Section $O(\alpha_s^2)$,
  $\mathrm{w} = 4$ JB and V. Ravindran, Nucl. Phys. **B716** (2005) 128.
- Unpolarized and Polarized Time-like Anomalous Dimensions and Wilson Coefficients $O(\alpha_s^2)$, $\mathrm{w} = 4$ JB and V. Ravindran, Nucl. Phys. **B749** (2006) 1.
- Anomalous Dimensions and Wilson Coefficients $O(\alpha_s^3)$, $\mathrm{w} = 5, 6,$

  from: S. Moch, J. Vermaseren, A. Vogt, Nucl. Phys. **B688** (2004) 101; **691** (2004) 129; **B724** (2005) 3 → J.B., DESY 07-042
- Polarized and Unpolarized Wilson Coefficients $O(\alpha_s^2)$, $\mathrm{w} = 4$ J.B. and S. Moch
- Polarized and Unpolarized asymptotic Heavy Flavor Wilson Coefficients $O(\alpha_s^{2(3)})$, $\mathrm{w} = 4,5,$

  J.B., A. de Freitas, W. van Neerven, S. Klein, Nucl. Phys. **B755** (2006) 272; I. Bierenbaum, J.B., S. Klein, DESY 07-026,

  DESY 07-027; DESY-08-029;
- Virtual and soft corrections to Bhabha Scattering $O(\alpha^2)$, $\mathrm{w} = 4,$

  J.B. and S. Klein, arXiv:0706.2426 [hep-ph]

Example: Bhabha s+v

$$
\begin{aligned}
T_0 \quad = \quad & \frac{248+15\,N^2+N^4}{2(N-2)(N-1)N(N+1)(N+2)}S_{1,1,1,1}(N)+\frac{-2}{(N-1)(N+1)}\mathbf{S_{2,1,1}}(N)\\
&+\frac{-340+120\,N+17\,N^2+18\,N^3-31\,N^4}{2(N-2)(N-1)N(N+1)(N+2)}S_{3,1}(N)+\frac{1344-502\,N-69\,N^2-2\,N^3+57\,N^4}{8(N-2)(N-1)N(N+1)(N+2)}S_4(N)\\
&+\frac{304-328\,N-500\,N^2+330\,N^3-6\,N^4+6\,N^5-2\,N^6+4\,N^7}{(N-2)^2(N-1)^2N^2(N+1)(N+2)}\mathbf{S_{2,1}}(N)\\
&+\frac{-112-4\,N^2-4\,N^4}{(N-2)(N-1)N(N+1)(N+2)}S_{2,1}(N)\mathbf{S_1}(N)+\frac{-48+8\,N+6\,N^2+7\,N^3}{(N-1)N(N+1)(N+2)}S_3(N)S_1(N)\\
&+\frac{-1840+292\,N+5532\,N^2+827\,N^3-1978\,N^4-274\,N^5+36\,N^6+19\,N^7-22\,N^8}{4(N-2)^2(N-1)^2N^2(N+1)^2(N+2)}S_{1,1,1}(N)\\
&+\frac{128-56\,N-252\,N^2+54\,N^3+177\,N^4-91\,N^5+19\,N^6+9\,N^7}{2(N-2)(N-1)^2N^2(N+1)^2(N+2)}S_3(N)\\
&+\frac{4032-2048\,N-14200\,N^2+5036\,N^3+23610\,N^4+2521\,N^5-12342\,N^6}{4(N-2)^3(N-1)^3N^3(N+1)^3(N+2)}S_{1,1}(N)\\
&+\frac{-3365\,N^7+2148\,N^8+903\,N^9+14\,N^{10}-167\,N^{11}+50\,N^{12}}{4(N-2)^3(N-1)^3N^3(N+1)^3(N+2)}S_{1,1}(N)\\
&+\frac{-124+16\,N+24\,N^2-4\,N^3-14\,N^4}{(N-2)(N-1)N(N+1)(N+2)}S_{1,1}(N)\zeta(2)+\frac{424-118\,N+9\,N^2-2\,N^3+23\,N^4}{4(N-2)(N-1)N(N+1)(N+2)}S_2(N)S_{1,1}(N)\\
&+\frac{224+144\,N-1216\,N^2-56\,N^3+1786\,N^4+641\,N^5-406\,N^6}{4(N-2)^2(N-1)^3N^3(N+1)^3(N+2)}S_2(N)\\
&+\frac{+17\,N^7-308\,N^8+141\,N^9-56\,N^{10}+N^{11}}{4(N-2)^2(N-1)^3N^3(N+1)^3(N+2)}S_2(N)+\frac{58+21\,N+N^2+15\,N^3+10\,N^4}{(N-2)(N-1)N(N+1)(N+2)}S_2(N)\zeta(2)
\end{aligned}
$$

Example: Bhabha s+v

$$
+\frac{232-384\,N^2-17\,N^3+286\,N^4-128\,N^5-14\,N^6+N^7}{4(N-2)(N-1)^2N^2(N+1)^2(N+2)}S_2(N)S_1(N)
$$
$$
+\frac{-560-26\,N-31\,N^2-10\,N^3-33\,N^4}{8(N-2)(N-1)N(N+1)(N+2)}S_2(N)^2
$$
$$
+\frac{576+1088\,N-3280\,N^2-5136\,N^3+11764\,N^4+20392\,N^5-17385\,N^6-30114\,N^7}{4(N-2)^3(N-1)^4N^4(N+1)^4(N+2)}S_1(N)
$$
$$
+\frac{+5984\,N^8+17228\,N^9-1228\,N^{10}-2754\,N^{11}-112\,N^{12}-8\,N^{13}+33\,N^{14}-24\,N^{15}}{4(N-2)^3(N-1)^4N^4(N+1)^4(N+2)}S_1(N)
$$
$$
+\frac{-56+336\,N+522\,N^2+424\,N^3-53\,N^4-500\,N^5+60\,N^6+28\,N^7-5\,N^8}{2(N-2)^2(N-1)^2N^2(N+1)^2(N+2)}S_1(N)\zeta(2)
$$
$$
+\frac{64+6\,N^2+N^3}{(N-2)(N-1)N(N+1)}S_1(N)\zeta(3)+\frac{2112+608\,N+76\,N^2-140\,N^3+107\,N^4}{10(N-2)(N-1)N(N+1)(N+2)}\zeta(2)^2
$$
$$
+\frac{-224-136\,N+1688\,N^2+1290\,N^3-1998\,N^4-1997\,N^5+198\,N^6}{2(N-1)^3N^3(N-2)^2(N+2)(N+1)^3}\zeta(2)
$$
$$
+\frac{+405\,N^7+376\,N^8-119\,N^9+56\,N^{10}+5\,N^{11}}{2(N-1)^3N^3(N-2)^2(N+2)(N+1)^3}\zeta(2)
$$
$$
+\frac{-552+144\,N+1654\,N^2-370\,N^3-361\,N^4+19\,N^5+35\,N^6-25\,N^7}{2(N-2)^2(N-1)^2N^2(N+1)^2}\zeta(3)
$$
$$
+\frac{320-64\,N-1920\,N^2+1600\,N^3+6524\,N^4-14872\,N^5-19036\,N^6+31543\,N^7-43960\,N^8-13935\,N^9}{16(N-1)^5(N+1)^5(N-2)^3N^5(N+2)}
$$
$$
+\frac{+65372\,N^{10}+26822\,N^{11}-44576\,N^{12}-9558\,N^{13}+9840\,N^{14}+339\,N^{15}+428\,N^{16}-371\,N^{17}+128\,N^{18}}{16(N-1)^5(N+1)^5(N-2)^3N^5(N+2)}
$$
$$
+4\frac{N^4-N^2+12}{(N-2)(N-1)N(N+1)(N+2)}f_{0,2}+(-2)\frac{N^4-N^2+12}{(N-2)(N-1)N(N+1)(N+2)}f_{0,1}^2
$$

z-space: A. Penin, (2005)

$\Longrightarrow$ 3 basic sums only; no alternating sums.

5. Factorial Series

Consider

$$\Omega(z) = \int_0^1 dt\; t^{z-1}\, \varphi(t); \qquad \varphi(1-t) = \sum_{k=0}^{\infty} a_k t^k$$

$$Re(z) > 0, \qquad \Omega(z) = \sum_{k=0}^{\infty} \frac{a_{k+1} k!}{z(z+1)\ldots(z+k)}$$

- $\Omega(z)$ is meromorphic in $z \,\epsilon\, \mathbf{C}$, obeys a recursion $z \rightarrow z+1$ and has an analytic asymptotic representation.
- The poles are situated at the non-positive integers.

<u>Examples:</u>

$$\begin{aligned}
F_5(z) &= \mathbf{M}\left[\frac{\mathrm{Li}_2(z)}{1+z}\right](z) \\
F_5(z+1) &= -F_5(z) + \frac{1}{z}\left[\zeta_2 - \frac{\psi(z+1)+\gamma_E}{z}\right] \\
\text{Asymp. ser. : } \mathrm{Li}_2(z) &\rightarrow -\mathrm{Li}_2(1-z) - \ln(z)\ln(1-z) + \zeta_2 \\
\mathbf{M}\left[\frac{\mathrm{Li}_2(1-z)}{1+z}\right](N) &\sim \frac{1}{2N^2} + \frac{1}{4N^3} - \frac{7}{24}\frac{1}{N^4} - \frac{1}{3}\frac{1}{N^5} + \frac{73}{120}\frac{1}{N^6} \ldots
\end{aligned}$$

Factorial Series

$$F_{13}(z) = \mathbf{M}\left[\left(\frac{\mathrm{Li}_2^2(z)}{z-1}\right)_+\right](z)$$

$$F_{13}(z+1) = -F_{13}(z) + \frac{\zeta_2^2}{z} + \frac{4\zeta_3}{z^2} + \frac{2\zeta_2}{z^2}S_1(z) + \frac{2S_{2,1}(z)}{z^2} + \frac{2}{z^3}\left[S_1^2(z) + S_2(z)\right]$$

$$\text{Asymp. ser. : } \mathrm{Li}_2^2(z) \rightarrow \mathrm{Li}_2^2(1-z) + \ln^2(z)\ln^2(1-z) + \zeta_2^2 + 2\mathrm{Li}_2(1-z)\ln(z)\ln(1-z) + \dots$$

$$\mathbf{M}\left[\left(\frac{\mathrm{Li}_2^2(1-z)}{z-1}\right)_+\right](N) \sim \frac{1}{z^2} - \frac{7}{24}\frac{1}{z^4} + \frac{1}{12}\frac{1}{z^5} + \frac{223}{1080}\frac{1}{z^6} - \frac{7}{45}\frac{1}{z^7} - \frac{3767}{15120}\frac{1}{z^8} + \frac{38}{105}\frac{1}{z^9}$$
$$+\frac{14327}{31500}\frac{1}{z^{10}} - \frac{198}{175}\frac{1}{z^{11}} - \frac{138673}{118800}\frac{1}{z^{12}} + \frac{3263}{693}\frac{1}{z^{13}} + \frac{5265804043}{1324323000}\frac{1}{z^{14}}$$
$$-\frac{13399637}{525525}\frac{1}{z^{15}} - \frac{143341487}{8408400}\frac{1}{z^{16}} + \frac{25092}{143}\frac{1}{z^{17}} + \frac{34809672614}{402026625}\frac{1}{z^{18}}$$
$$-\frac{5749693892}{3828825}\frac{1}{z^{19}} + O\left(\frac{1}{z^{20}}\right)$$

6. The Basis

$w=1$	$1/(x-1)_+$		
$w=2$	$\ln(1+x)/(x+1)$		
$w=3$	$\mathrm{Li}_2(x)/(x\pm 1)$		
$w=4$	$\mathrm{Li}_3(x)/(x+1)$	$S_{1,2}(x)/(x\pm 1)$	
$w=5$	$\mathrm{Li}_4(x)/(x\pm 1)$	$S_{1,3}(x)/(x\pm 1)$	$S_{2,2}(x)/(x\pm 1)$
	$\mathrm{Li}_2^2(x)/(x\pm 1)$	$[\ln(x)S_{1,2}(-x)-\mathrm{Li}_2^2(-x)/2]/(x\pm 1)$	
$w=6$	$\mathrm{Li}_5(x)/(x+1)$	$S_{1,4}(x)/(x\pm 1)$	$S_{2,3}(x)/(x\pm 1)$
	$S_{3,2}(x)/(x\pm 1)$	$\mathrm{Li}_2(x)\mathrm{Li}_3(x)/(x\pm 1)$	
	$A_1(x)/(x+1)$	$A_2(x)/(x\pm 1)$	$A_3(x)/(x+1)$
	$H_{0,-1,0,1,1}(x)/(x\pm 1)$	$H_{0,0,-1,0,1}(x)/(x\pm 1)$	

$$\frac{[A_1(-x)+2S_{3,2}(-x)-2S_{2,2}(-x)\ln(x)]}{x\pm 1}$$

$$\frac{A_1(-x)+2S_{3,2}(-x)-2S_{2,2}(-x)\ln(x)+\mathrm{Li}_2^2(-x)\ln(x)/4-\mathrm{Li}_3(-x)\mathrm{Li}_2(-x)}{x-1}$$

• $O(\alpha)$	Wilson Coefficients/anom. dim.	#1
• $O(\alpha^2)$	Anomalous Dimensions	#2
• $O(\alpha^2)$	Wilson Coefficients	$\# \leq 5$
• $O(\alpha^3)$	Anomalous Dimensions	#15
• $O(\alpha^3)$	Wilson Coefficients	#35

7. Conclusions - A

- We considered mathematical structures which determine no scale and single scale quantities in Quantum Field Theories.
- The former correspond to integrated cross sections, expansion coefficients of the β–function, or anomalous dimensions at fixed moments, etc.
- The latter correspond to differential scattering cross sections of one variable, N–dependent anomalous dimensions, coefficient functions, etc.
- The single-scale quantities in Quantum Field Theories to 3 Loop Order $\Leftrightarrow$ $\mathrm{w} = 6$ can be represented in a polynomial ring spanned by a few Mellin transforms of the above basic functions, which are the same for all known processes. This points to their general nature.
- The basic Mellin transforms are meromorphic functions with single poles at the non-positive integers.
- The total amount of harmonic sums reduces due to algebraic relations [index structure], and structural relations $\mathrm{N} \in \mathbf{Q}$, $\mathrm{N} \in \mathbf{R}$.

- They can be represented in terms of factorial series up to simple "soft components". This allows an exact analytic continuation.
- Up to w = 6 physical (pseudo-) observables are free of harmonic sums with index = {-1}. Up to w = 5 all numerator functions are Nielsen integrals.

8. From Moments to Functions in QCD

- Higher order calculations in Quantum Field Theories easily become tedious due to the larger number of terms and the sophistication of the Feynman parameter integrals.
- This even applies to Zero Scale and Single Scale Quantities.
- Even more this is the case for higher scale problems.
- While in the latter case the mathematical structure of the solution for the Feynman Integrals is widely unknown, it is explored to a certain extent for Zero Scale and Single Scale quantities.
- They can be expressed by rational numbers and certain special numbers as multiple ζ-values and related quantities.

- Single Scale quantities depend on a scale $z \in [0,1]$, with z a ratio of Lorentz invariants. One may perform a Mellin Transform over z

$$\int_0^1 dz z^{N-1} f(z) = M[f](N)$$

- Here one assumes $N \in \mathbf{N}, N > 0$. Due to this the problem on hand becomes discrete.
- One may seek a description in terms of difference equations.
- Zero Scale problems are obtained from Single Scale problems treating N as a fixed integer or considering the limit $N \to \infty$.
- Can one reconstruct the general formula for Single Scale quantities out of a finite number of fixed moments ? This is possible for recurrent quantities.
- At least up to 3-loop order, presumably to higher orders, single scale quantities belong to this class.
- Goal : design a general formalism to solve the problem.

9. Single Scale Feynman Integrals as Recurrent Quantities

- Can one reconstruct the general formula for Single Scale quantities out of a finite number of fixed moments ?
- Polynomials and Nested Harmonic Sums obey recurrence relations, so do their polynomials.
- Example: Harmonic Sums or linear combinations thereof:

$$F(N+1) - F(N) = \frac{\mathrm{sign}(a)^{N+1}}{(N+1)^{|a|}}$$

is solved by $S_a(N)$; and similarly for deeper nested sums

$$S_{a,\vec{b}}(N) = \sum_{k=1}^{N} \frac{(\mathrm{sign}(a))^k}{k^{|a|}} S_{\vec{b}}(k)$$

.

Single Scale Feynman Integrals as Recurrent Quantities

- Feynman integrals have often a form like

$$\int_0^1 dz \frac{z^{N-1}-1}{1-z} H_{\vec{a}}(z), \qquad \int_0^1 dz \frac{(-z)^{N-1}-1}{1+z} H_{\vec{a}}(z)$$

- This structure leads to recurrences.
- It is very likely that single scale Feynman diagrams do always obey difference equations.

10. Establishing and Solving Recurrences

- One seeks the relation

$$\sum_{k=0}^{l}\left[\sum_{i=0}^{d} c_{i,k} N^i\right] F(N+k) = 0 \ .$$

- The corresponding linear system is dense.
- Rational number arithmetics is not applicable for the large systems to be determined; $C_{2,q,C_F^3}^{(3)}$ would require 11 Tb of memory.
- Use arithmetic in finite fields together with Chinese remaindering $\Longrightarrow$ few Gb of memory
- The linear system approximately minimizes for $l \approx d$.
- Join different recurrences found to reduce l to a minimal value.

Establishing and Solving Recurrences

- For the solution of the recurrence low degrees are clearly preferred.
- The linear difference equation of order l with polynomial coefficients is equivalent to a linear system in l variables.
- It is solved in $\Pi - \Sigma$ fields.
- Apply advanced symbolic summation methods: telescoping, creative telescoping and its refinement. Code: `sigma`.
- The solutions are found as linear combinations of rational terms in N combined with functions, which cannot be further reduced in the $\Pi - \Sigma$ fields. In the present application they turn out all to be harmonic sums $S_{\vec{b}}(N)$.
- Other or higher order applications may consist of other sums too, which are uniquely found by the algorithm.

11. Application to 3-Loop Anomalous Dimensions and Wilson coefficients

- We apply the method for the unfolding of the unpolarized anomalous dimensions and Wilson coefficients up to 3-loop order.
- $\Longrightarrow$ analyze for individual color factors; 141 contributions from $1-3$ loops
- Input: Moch, Vermaseren, Vogt, 2004/05. The expressions are given in terms of harmonic sums.
- Calculate the moments (rational numbers) recursively through recursions for the harmonic sums; MAPLE code.
- Establish the corresponding difference equation by a recurrency finder; build a difference equation of minimal order possible; test the recurrency.
- Solve the difference equation order by order with the summation package `sigma` C. Schneider.; most complicated cases: 4 weeks @ $\leq$ 10Gb, 2 GHz Proc.

Input

```
C2qq3CF^3
N=3:
#11 digits / #10 digits

-98268084191 / 1166400000

N=500:
#1262 digits / #1256 digits

1641840770424196780953020619176376506284303544481262083057197600746507008493793994
4224110323441591630311482222058287688942209570859151121677307585313995100978363179
2518952817622034037186132846974627021672678012913675099511203807811938593043910803
5044345920218696052588332036355325089998361354226882367322149037631053761764348772
5403810874264968729520075619227285471802419403727207822473765999900236383740315299
2050533601633484348249454757555344664210814111140065475391136798689167410065076749
3578709478683390573977410013520894494463909291327425815766566386397276158317387748
5945471392646089700875157445075073192328542890965462004805711998748144414379386093
9937361798029044425789953726133675199790523770427298500510063464061985840066296071
3372543015648919155964069606994597363886301185067827291937065300754786947063672848
9382081926871078600328628131936766057475970450896556667622163365895808773428119721
5352792131089063577045069693962213061198894057033606068695607123271969726981060056
0115846094360239986233917872260722277322690450132376836253549152130116645670565045
9666945920164586023958060271746606798898861360772333088030741775605546518788793327
2264368297071217405654474375844238250889238538974548421298170425909521742559494728
72017877003947396562261659860366839154407853462338171648227013134266795320251847
/
```

3057444614247225372882570514367358697278130741348282122206492932820352440850471902
7491046962105336645563654873675690796713906565688820365601907263710863954826386081
3227580037879361869941003802807590860358894142891046776447162895908787986423254678
5776778283337231702130612499429819559798501074020676282769289102955679421885795867
1982932998601320344971927374905889934059987271939760212836368619501189238215442366
3805773701929509268157747992859384837403751183019423692868569168206789710047557452
5131217382272060267681480496298975522467614707848639773185909858278799786637303834
1017166676276847525704755493166263297079720470719813623901545811953853986456533543
9994182050551827959988760121168490745476969259468454613431624179198860751513076481
0304734205926703138519418575731315944374873897873646706993620825697218523316375559
4068222004765962715924208526106008109740402380126260947524640509361283802755722132
4856690051525724685919792641506082307567956962328560073471086799287131287564668441
6256698083504233897436484702002471314330803421467773925541151273924985946178771189
2312437162213438137703896064734987157020801413153555435311326719739117599044341913
5922693587373856609594245948237469293148702516714038297077639382332251255360181047
49658623247509112659762997679737527882711111677459300352000000000000000000

N=5114:

#13388 digits / #13381 digits

Table 1: Run parameters for the unfolding of the non-singlet anomalous dimensions

	number of terms needed	order of recurrence	degree of recurrence	total time [sec]	length of recurrence [kbyte]	number of harm. sums a [b]	solution time [sec]
$P_{NS,0}$	14	2	3	0.05	0.087	1 [1]	0.55
$P^-_{NS,1,C_F^2}$	142	5	31	3.32	4.666	6 [10]	7.45
$P^-_{NS,1,C_AC_F}$	109	4	24	1.91	2.834	6 [7]	6.28
$P^-_{NS,1,C_FN_F}$	24	2	7	0.13	0.271	2 [2]	0.92
$P^+_{NS,1,C_F^2}$	142	5	31	3.35	4.707	6 [10]	7.45
$P^+_{NS,1,C_AC_F}$	109	4	23	1.88	2.703	6 [7]	6.23
$P^+_{NS,1,C_FN_F}$	24	2	7	0.09	0.271	2 [2]	0.89
$P^-_{NS,2,C_F^3}$	1079	16	192	3152.19	529.802	25 [68]	1194.41
$P^-_{NS,2,C_F^3\zeta_3}$	48	3	11	0.49	0.643	1 [1]	1.56
$P^-_{NS,2,C_AC_F^2}$	974	15	181	1736.08	450.919	25 [62]	1194.41
$P^-_{NS,2,C_AC_F^2\zeta_3}$	48	3	11	0.53	0.643	1 [1]	1.53
$P^-_{NS,2,C_A^2C_F}$	749	12	147	1004.12	242.892	25 [62]	1100.88
$P^-_{NS,2,C_A^2C_F\zeta_3}$	48	3	11	0.56	0.643	1 [1]	1.56
$P^-_{NS,2,C_FN_F^2}$	39	2	11	0.31	0.369	3 [3]	1.20
$P^-_{NS,2,C_F^2N_F}$	377	8	68	76.34	33.946	12 [24]	72.22
$P^-_{NS,2,C_F^2N_F\zeta_3}$	14	2	3	0.12	0.101	1 [1]	0.53
$P^-_{NS,2,C_AC_FN_F}$	356	7	62	65.25	23.830	12 [20]	52.67
$P^-_{NS,2,C_AC_FN_F\zeta_3}$	14	2	3	0.12	0.101	1 [1]	0.55
$P^+_{NS,2,C_F^3}$	1079	16	192	4713.27	527.094	25[68]	1165.22
$P^+_{NS,2,C_F^3\zeta_3}$	48	3	11	0.55	0.643	1[1]	1.562
$P^+_{NS,2,C_AC_F^2}$	974	15	178	1715.03	442.031	25[62]	889.047
$P^+_{NS,2,C_AC_F^2\zeta_3}$	48	3	11	0.61	0.643	1[1]	1.531
$P^+_{NS,2,C_A^2C_F}$	749	12	146	991.22	240.325	25[50]	516.812
$P^+_{NS,2,C_A^2C_F\zeta_3}$	48	3	11	0.61	0.643	1[1]	1.593
$P^+_{NS,2,C_F^2N_F}$	377	8	69	111.38	33.872	12[24]	71.235
$P^+_{NS,2,C_F^2N_F\zeta_3}$	14	2	3	0.15	0.101	1[1]	0.531
$P^+_{NS,2,C_AC_FN_F}$	307	7	61	48.62	23.808	12[24]	71.235
$P^+_{NS,2,C_AC_FN_F\zeta_3}$	14	2	3	0.15	0.101	1[1]	0.547
$P^+_{NS,2,C_FN_F^2}$	39	2	11	0.40	0.369	3[3]	1.172
$P^-_{NS,2,N_Fd_{abc}}$	39	2	11	0.55	0.369	3 [3]	1.19

Table 2: Run parameters for the unfolding of the unpolarized quarkonic Wilson Coefficients for the structure function $F_2(x, Q^2)$.

	number of terms needed	order of recurrence	degree of recurrence	total time [sec]	length of recurrence [kbyte]	number of harm. sums a [b]	solution time [sec]
$C^{(1)}_{2,q,C_F}$	35	3	7	0.26	0.429	2[3]	1.13
$C^{(2)}_{2,q,C_F^2}$	689	11	137	1134.10	177.806	13[39]	258.24
$C^{(2)}_{2,q,C_AC_F}$	545	10	121	413.33	127.893	12[35]	178.73
$C^{(2)}_{2,q,C_F^2\zeta_3}$	15	2	3	0.27	0.100	1[1]	0.54
$C_{2,q,C_AC_F\zeta_3}$	15	2	3	0.27	0.112	1[1]	0.55
C_{2,q,N_FC_F}	71	4	16	2.68	1.655	4[10]	3.95
$C^{(3)}_{2,q,C_F^3}$	5114	35	938	1.78886×10^6	30394.173	58[289]	0.50924×10^6
$C^{(3)}_{2,q,C_F^3\zeta_3}$	284	8	64	31.02	32.363	7 [18]	27.60
$C^{(3)}_{2,q,C_F^3\zeta_4}$	19	2	5	0.08	0.163	1 [1]	0.47
$C^{(3)}_{2,q,C_F^3\zeta_5}$	19	2	5	0.08	0.163	1 [1]	0.47
$C^{(3)}_{2,q,C_F^2C_A}$	5059	35	930	1.69267×10^6	30122.380	60 [290]	0.47780×10^6
$C^{(3)}_{2,q,C_F^2C_A\zeta_3}$	284	8	64	34.00	33.400	7 [18]	28.53
$C^{(3)}_{2,q,C_F^2C_A\zeta_4}$	48	3	11	0.32	0.643	1[1]	1.01
$C^{(3)}_{2,q,C_F^2C_A\zeta_5}$	19	2	5	0.08	0.167	1 [1]	0.42
$C^{(3)}_{2,q,C_FC_A^2}$	4564	33	863	1.38918×10^6	24567.518	60 [258]	0.34941×10^6
$C^{(3)}_{2,q,C_FC_A^2\zeta_3}$	284	8	63	26.83	29.918	7 [17]	30.46
$C^{(3)}_{2,q,C_FC_A^2\zeta_4}$	48	3	11	0.32	0.643	1 [1]	1.01
$C^{(3)}_{2,q,C_FC_A^2\zeta_5}$	19	2	5	0.08	0.175	1 [1]	0.42
$C^{(3)}_{2,q,C_F^2N_F}$	1762	20	348	40237.45	2339.516	29 [107]	7548.56
$C^{(3)}_{2,q,C_F^2N_F\zeta_3}$	87	4	21	1.94	2.354	3 [5]	2.83
$C^{(3)}_{2,q,C_F^2N_F\zeta_4}$	15	2	3	0.07	0.101	1 [1]	0.34
$C^{(3)}_{2,q,C_FC_AN_F}$	1847	20	360	47661.64	2507.362	29 [111]	7525.89
$C^{(3)}_{2,q,C_FC_AN_F\zeta_3}$	89	4	24	2.47	2.935	3 [8]	3.19
$C^{(3)}_{2,q,C_FC_AN_F\zeta_4}$	15	2	3	0.06	0.101	1 [1]	0.34
$C^{(3)}_{2,q,C_FN_F^2}$	131	5	30	58.00	5.347	7 [22]	8.97
$C^{(3)}_{2,q,C_FN_F^2\zeta_3}$	15	2	3	0.06	0.101	1 [1]	0.38
$C^{(3)}_{2,q,dabc}$	1199	14	242	6583.27	738.498	14 [62]	841.24
$C^{(3)}_{2,q,dabc\zeta_3}$	109	4	25	2.33	3.164	2[7]	2.40
$C^{(3)}_{2,q,dabc\zeta_5}$	8	1	2	0.03	0.041	0[0]	0.10

A complicated example

$C_{2,q} \propto C_F^3$:

- 5114 moments needed. Use a clever way to calculate the input.
- Largest moment: fraction: numerator 13388 digits; denominator 13381 digits.
- CPU time to determine the recurrence: 20.7 days.

- modular prediction of the dimension: 4 h; modular LEQ's: 5.8 days; modular operator GCDs: 11 days; Chinese Remainder + Rat. Reconstruction: 3.8 days. 140 large primes needed.

31 MB recurrence is established; largest integer: 1227 digits; order: 35; degree: 938

- Solved by `sigma` within about one week.
- 3 loop anomalous dimensions: much smaller recurrences & shorter computation times.

$\Longrightarrow$ In practice no method does yet exists to calculate such a high number of moments.

$\Longrightarrow$ Existence proof of a quite general and powerful automatic difference-equation solver, standing rather demanding tests.

Structure of the Results

- We carry out all algebraic reductions, J.B. 2003.
- Different color factor contributions lead to the same or nearly the same amount of sums at a given quantity.
- This points to the fact that the amount of harmonic sums is governed by topology rather than the fields and color.
- The linear harmonic sum representations by Vermaseren et al. 2004/05 require many more sums than our representation.
- There are reductions in the number of sums as $264 \longrightarrow 29$.
- Further use of structural relations will lead to maximally 35 sums for the 3-loop Wilson coefficients; J.B. arxiv: 0901.0837, arXiv:0901.3106.

$$P_{qq}^{0}(n) = C_F\left[4S_1 - \frac{3n^2+3n+2}{n(n+1)}\right]$$

$$P_{qq}^{1,-}(n) = C_F^2\bigg[-\frac{3n^6+9n^5+9n^4-5n^3-24n^2-32n-24}{2n^3(n+1)^3} - 16S_{-3}$$

$$+S_{-2}\left(\frac{16}{n(n+1)} - 32S_1\right) + S_1\left(\frac{8(2n+1)}{n^2(n+1)^2} - 16S_2\right) + \frac{4\,(3n^2+3n+2)}{n(n+1)}S_2$$

$$-16S_3 + 32S_{-2,1} + \frac{16(-1)^n}{(n+1)^3}\bigg]$$

$$+\ C_AC_F\bigg[-\frac{51n^5+102n^4+655n^3+484n^2+12n+144}{18n^3(n+1)^2} + 8S_{-3} + \frac{268}{9}S_1$$

$$+S_{-2}\left(16S_1 - \frac{8}{n(n+1)}\right) - \frac{44}{3}S_2 + 8S_3 - 16S_{-2,1} - \frac{8(-1)^n}{(n+1)^3}\bigg]$$

$$+\ C_FN_F\left[\frac{3n^4+6n^3+47n^2+20n-12}{9n^2(n+1)^2} - \frac{40}{9}S_1 + \frac{8}{3}S_2\right]$$

$$P_{qq}^{1,+}(n) = C_F^2\bigg[-\frac{3n^6+9n^5+9n^4+59n^3+40n^2+32n+8}{2n^3(n+1)^3} - 16S_{-3}$$

$$+S_{-2}\left(\frac{16}{n(n+1)} - 32S_1\right) + S_1\left(\frac{8(2n+1)}{n^2(n+1)^2} - 16S_2\right)$$

$$+\frac{4\,(3n^2+3n+2)}{n(n+1)}S_2 - 16S_3 + 32S_{-2,1} + \frac{16(-1)^n}{(n+1)^3}\bigg]$$

$$+\ C_AC_F\bigg[-\frac{51n^5+153n^4+757n^3+851n^2+208n-132}{18n^2(n+1)^3} + 8S_{-3} + \frac{268}{9}S_1$$

$$+S_{-2}\left(16S_1 - \frac{8}{n(n+1)}\right) - \frac{44}{3}S_2 + 8S_3 - 16S_{-2,1} - \frac{8(-1)^n}{(n+1)^3}\bigg]$$

$$+\ C_FN_F\left[\frac{3n^4+6n^3+47n^2+20n-12}{9n^2(n+1)^2} - \frac{40}{9}S_1 + \frac{8}{3}S_2\right]$$

$$P_{qq}^{2,-}(n) = C_F^3\bigg\{\left(\frac{64}{n(n+1)} - 128S_1\right)S_{-2}^2 + \bigg(\frac{16\,(3n^6+9n^5+9n^4+17n^3+6n^2+8n+2)}{n^3(n+1)^3}$$

$$+\ S_1\left(\frac{64\,(3n^2-n+1)}{n^2(n+1)^2} - 1408S_2\right) - \frac{64\,(3n^2+3n-11)\,S_2}{n(n+1)} + 1536S_3 + 128S_{-2,1}$$

$$-\ 2304S_{2,1}\bigg)S_{-2} - \frac{16\,(3n^2+3n+2)\,S_2^2}{n(n+1)} - \frac{P_1(n)}{2n^5(n+1)^5} - 576S_{-5}$$

$$+\ S_{-4}\left(-\frac{16\,(9n^2+9n-26)}{n(n+1)} - 832S_1\right)$$

$$+\ S_{-3}\left(640S_1^2 - \frac{32\,(3n^2+3n+20)\,S_1}{n(n+1)} + \frac{16\,(21n^2+17n+20)}{n^2(n+1)^2} - 320S_{-2} - 2240S_2\right)$$

$$+\ (-1)^n\left(-\frac{48\,(2n^2-n+1)}{(n+1)^5} + \frac{128S_{-2}}{(n+1)^3} + \frac{96(5n+3)S_1}{(n+1)^4} - \frac{64S_2}{(n+1)^3}\right)$$

$$+\ \frac{4\,(13n^4+26n^3+13n^2-16n-20)\,S_3}{n^2(n+1)^2} - \frac{16\,(15n^2+15n+2)\,S_4}{n(n+1)} - 192S_5 - 832S_{-4,1}$$

$$+\ \frac{896S_{-3,1}}{n(n+1)} + 1152S_{-3,2} + S_1^2\left(-\frac{32\,(3n^2+3n+1)}{n^3(n+1)^3} - 768S_{-2,1}\right) - \frac{32\,(15n^2+11n+16)\,S_{-2,1}}{n^2(n+1)^2}$$

$$+\ S_2\left(\frac{2\,(3n^6+9n^5+9n^4+19n^3+12n^2-4n-16)}{n^3(n+1)^3} + 64S_3 + 2176S_{-2,1}\right)$$

$$+\ \frac{32\,(3n^2+3n-26)\,S_{2,-2}}{n(n+1)} - 1472S_{3,-2} + \frac{64\,(3n^2+3n-2)\,S_{3,1}}{n(n+1)} + 192S_{3,2} + 192S_{4,1}$$

$$+\ 2304S_{-3,1,1} + 512S_{-2,1,-2} + \frac{384\,(n^2+n-4)\,S_{-2,1,1}}{n(n+1)} + S_1\bigg(64S_2^2 - \frac{64(2n+1)S_2}{n^2(n+1)^2}$$

$$+\ \frac{4\,(22n^6+186n^5+167n^4-40n^3-115n^2-120n-44)}{n^4(n+1)^4} - 192S_3 + 64S_4 - 1792S_{-3,1}$$

$$-\ \frac{192\,(n^2+n-4)\,S_{-2,1}}{n(n+1)} + 1664S_{2,-2} + 256S_{3,1} + 3072S_{-2,1,1}\bigg) + 2304S_{-2,2,1} + 2304S_{2,1,-2}$$

$$-\ 384S_{3,1,1} - 4608S_{-2,1,1,1}$$

$$+\ \left(C_F^3 - \frac{3}{2}C_F^2C_A\right)\zeta_3\left[-\frac{24\,(5n^4+10n^3+9n^2+4n+4)}{n^2(n+1)^2} - 192S_{-2}\right]\bigg\}$$

$$+\ C_AC_F^2\bigg\{\left(256S_1 - \frac{16\,(3n^2+3n+8)}{n(n+1)}\right)S_{-2}^2$$

$$+\ \bigg[-\frac{8\,(81n^6+243n^5-229n^4-389n^3-130n^2+228n+72)}{9n^3(n+1)^3} + \frac{32\,(31n^2+31n-81)\,S_2}{3n(n+1)}$$

$$+\ S_1\left(1728S_2 - \frac{32\,(134n^4+268n^3+215n^2+45n+54)}{9n^2(n+1)^2}\right) - 1792S_3 - 192S_{-2,1} + 2688S_{2,1}\bigg]S_{-2}$$

$$+\ \frac{176}{3}S_2^2 - \frac{P_2(n)}{36n^5(n+1)^5} + 672S_{-5} + S_{-4}\left(\frac{8\,(97n^2+97n-210)}{3n(n+1)} + 1120S_1\right)$$

$$+\ S_{-3}\bigg(-576S_1^2 + \frac{16\,(31n^2+31n+108)\,S_1}{3n(n+1)} - \frac{8\,(268n^4+536n^3+811n^2+507n+450)}{9n^2(n+1)^2}$$

$$+\ 480S_{-2} + 2656S_2\bigg) + (-1)^n\bigg(\frac{8\,(382n^2+41n-161)}{9(n+1)^5} - \frac{256S_{-2}}{(n+1)^3} - \frac{16(127n+121)S_1}{3(n+1)^4}$$

$$+\ \frac{32S_2}{(n+1)^3}\bigg) - \frac{8\,(385n^4+770n^3+427n^2+6n-126)\,S_3}{9n^2(n+1)^2} + \frac{8\,(151n^2+151n-30)\,S_4}{3n(n+1)}$$

$$+\ 384S_5 + 864S_{-4,1} - \frac{960S_{-3,1}}{n(n+1)} - 1344S_{-3,2}$$

$$
\begin{aligned}
&+\; S_2\Bigg(\frac{2\left(453n^5+906n^4+1325n^3+488n^2-120n+144\right)}{9n^3(n+1)^2}-32S_3-2624S_{-2,1}\Bigg)\\
&+\; \frac{16\left(268n^4+536n^3+625n^2+321n+414\right)S_{-2,1}}{9n^2(n+1)^2}+S_1^2(128S_3+896S_{-2,1})\\
&-\; \frac{16\left(31n^2+31n-174\right)S_{2,-2}}{3n(n+1)}+1824S_{3,-2}-\frac{32\left(29n^2+29n-24\right)S_{3,1}}{3n(n+1)}-384S_{3,2}-384S_{4,1}\\
&-\; 2688S_{-3,1,1}-768S_{-2,1,-2}+S_1\Bigg(-\frac{8\left(135n^6+731n^5+245n^4-617n^3-395n^2-309n-144\right)}{9n^4(n+1)^4}\\
&-\; \frac{2144}{9}S_2+\frac{32\left(31n^2+31n-12\right)S_3}{3n(n+1)}+160S_4+1920S_{-3,1}+\frac{32\left(31n^2+31n-84\right)S_{-2,1}}{3n(n+1)}\\
&-\; 1856S_{2,-2}-512S_{3,1}-3584S_{-2,1,1}\Bigg)-\frac{64\left(31n^2+31n-84\right)S_{-2,1,1}}{3n(n+1)}-2688S_{-2,2,1}-2688S_{2,1,-2}\\
&+\; 768S_{3,1,1}+5376S_{-2,1,1,1}\Bigg\}\\
&+\; C_A^2C_F\Bigg[\Bigg(\frac{24\left(n^2+n+2\right)}{n(n+1)}-96S_1\Bigg)S_{-2}^2+\Bigg(\frac{8\left(27n^6+81n^5-155n^4-271n^3-92n^2+78n+27\right)}{9n^3(n+1)^3}\\
&+\; S_1\Bigg(\frac{16\left(134n^4+268n^3+188n^2+54n+45\right)}{9n^2(n+1)^2}-512S_2\Bigg)-\frac{32\left(11n^2+11n-24\right)S_2}{3n(n+1)}+512S_3\\
&+\; 64S_{-2,1}-768S_{2,1}\Bigg)S_{-2}+\frac{P_3(n)}{108n^5(n+1)^5}-192S_{-5}+S_{-4}\Bigg(-\frac{8\left(35n^2+35n-66\right)}{3n(n+1)}-352S_1\Bigg)\\
&+\; (-1)^n\Bigg(-\frac{16\left(82n^2+17n-47\right)}{9(n+1)^5}+\frac{96S_{-2}}{(n+1)^3}+\frac{16(41n+47)S_1}{3(n+1)^4}\Bigg)\\
&+\; S_{-3}\Bigg(128S_1^2-\frac{16\left(11n^2+11n+24\right)S_1}{3n(n+1)}+\frac{8\left(134n^4+268n^3+311n^2+177n+135\right)}{9n^2(n+1)^2}\\
&-\; 160S_{-2}-768S_2\Bigg)+\frac{4\left(389n^4+778n^3+398n^2+9n-81\right)S_3}{9n^2(n+1)^2}-\frac{8\left(55n^2+55n-24\right)S_4}{3n(n+1)}\\
&-\; 160S_5-224S_{-4,1}+\frac{256S_{-3,1}}{n(n+1)}+384S_{-3,2}+S_1^2(-64S_3-256S_{-2,1})\\
&-\; \frac{16\left(134n^4+268n^3+245n^2+111n+135\right)S_{-2,1}}{9n^2(n+1)^2}+S_2\Bigg(768S_{-2,1}-\frac{4172}{27}\Bigg)\\
&+\; \frac{16\left(11n^2+11n-48\right)S_{2,-2}}{3n(n+1)}-544S_{3,-2}+\frac{32\left(11n^2+11n-12\right)S_{3,1}}{3n(n+1)}\\
&+\; 192S_{3,2}+192S_{4,1}+768S_{-3,1,1}+256S_{-2,1,-2}+\frac{64\left(11n^2+11n-24\right)S_{-2,1,1}}{3n(n+1)}\\
&+\; S_1\Bigg(\frac{2\left(245n^8+980n^7+1542n^6+1524n^5+851n^4+100n^3+36n^2+22n-6\right)}{3n^4(n+1)^4}\\
&-\; \frac{8\left(11n^2+11n-8\right)S_3}{n(n+1)}-128S_4-512S_{-3,1}-\frac{32\left(11n^2+11n-24\right)S_{-2,1}}{3n(n+1)}\\
&+\; 512S_{2,-2}+256S_{3,1}+1024S_{-2,1,1}\Bigg)+768S_{-2,2,1}+768S_{2,1,-2,}-384S_{3,1,1}
\end{aligned}
$$

$$
\begin{aligned}
&-\; 1536S_{-2,1,1,1}\Bigg]\\
&+\; C_A^2C_F\zeta_3\Bigg[-\frac{12\left(5n^4+10n^3+9n^2-4n-4\right)}{n^2(n+1)^2}-96S_{-2}\Bigg]\\
&+\; C_FN_F^2\Bigg[\frac{51n^6+153n^5+57n^4+35n^3+96n^2+16n-24}{27n^3(n+1)^3}-\frac{16}{27}S_1-\frac{80}{27}S_2+\frac{16}{9}S_3\Bigg]\\
&+\; C_F^2N_F\Bigg[-\frac{32}{3}S_2^2-\frac{4\left(15n^4+30n^3+79n^2+16n-24\right)S_2}{9n^2(n+1)^2}\\
&+\; \frac{207n^8+828n^7+1443n^6+1123n^5-38n^4-779n^3-632n^2+120}{9n^4(n+1)^4}-\frac{128}{3}S_{-4}\\
&+\; S_{-3}\Bigg(\frac{32\left(10n^2+10n+3\right)}{9n(n+1)}-\frac{64}{3}S_1\Bigg)+(-1)^n\Bigg(\frac{64S_1}{3(n+1)^3}-\frac{128(4n+1)}{9(n+1)^4}\Bigg)\\
&+\; S_{-2}\Bigg(-\frac{32\left(16n^2+10n-3\right)}{9n^2(n+1)^2}+\frac{640}{9}S_1-\frac{128}{3}S_2\Bigg)+\frac{16\left(29n^2+29n+12\right)S_3}{9n(n+1)}-\frac{128}{3}S_4\\
&+\; S_1\Bigg(-\frac{2\left(165n^5+330n^4+165n^3+160n^2-16n-96\right)}{9n^3(n+1)^2}+\frac{320}{9}S_2-\frac{128}{3}S_3-\frac{128}{3}S_{-2,1}\Bigg)\\
&-\; \frac{64\left(10n^2+10n-3\right)S_{-2,1}}{9n(n+1)}+\frac{64}{3}S_{2,-2}+\frac{64}{3}S_{3,1}+\frac{256}{3}S_{-2,1,1}\Bigg]\\
&+\; \left(C_F^2-C_FC_A\right)N_F\zeta_3\Bigg[32S_1-\frac{8\left(3n^2+3n+2\right)}{n(n+1)}\Bigg]\\
&+\; C_AC_FN_F\Bigg[-\frac{2\left(270\,n^7+810n^6-463n^5-1392n^4-211n^3-206n^2-156n+144\right)}{27n^4(n+1)^3}\\
&+\; \frac{64}{3}S_{-4}+S_{-3}\Bigg(\frac{32}{3}S_1-\frac{16\left(10n^2+10n+3\right)}{9n(n+1)}\Bigg)+(-1)^n\Bigg(\frac{64(4n+1)}{9(n+1)^4}-\frac{32S_1}{3(n+1)^3}\Bigg)\\
&+\; \frac{1336}{27}S_2+S_{-2}\Bigg(\frac{16\left(16n^2+10n-3\right)}{9n^2(n+1)^2}-\frac{320}{9}S_1+\frac{64}{3}S_2\Bigg)-\frac{8\left(14n^2+14n+3\right)S_3}{3n(n+1)}+\frac{80}{3}S_4\\
&+\; \frac{32\left(10n^2+10n-3\right)S_{-2,1}}{9n(n+1)}+S_1\Bigg(-\frac{4\left(209n^6+627n^5+627n^4+281n^3+36n^2+36n+18\right)}{27n^3(n+1)^3}\\
&+\; 16S_3+\frac{64}{3}S_{-2,1}\Bigg)-\frac{32}{3}S_{2,-2}-\frac{64}{3}S_{3,1}-\frac{128}{3}S_{-2,1,1}\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
P_{qq}^{2,+} \;=&\; C_F^3\Bigg[\Bigg(\frac{64}{n(n+1)}-128S_1\Bigg)S_{-2}^2+\Bigg(\frac{16\left(3n^6+9n^5+9n^4+n^3+2n^2+4n+2\right)}{n^3(n+1)^3}\\
&+\; S_1\Bigg(-\frac{64\left(3n^2+7n+5\right)}{n^2(n+1)^2}-1408S_2\Bigg)-\frac{64\left(3n^2+3n-11\right)S_2}{n(n+1)}+1536S_3+128S_{-2,1}
\end{aligned}
$$

$$- \quad 2304S_{2,1}\Big)S_{-2} - \frac{16\left(3n^2+3n+2\right)S_2^2}{n(n+1)} - \frac{P_4(n)}{2n^5(n+1)^5} - 576S_{-5}$$

$$+ \quad S_{-4}\Big(-\frac{16\left(9n^2+9n-26\right)}{n(n+1)} - 832S_1\Big) + S_{-3}\Big(640S_1^2 - \frac{32\left(3n^2+3n+20\right)S_1}{n(n+1)}$$

$$+ \quad \frac{16\left(9n^2+5n+8\right)}{n^2(n+1)^2} - 320S_{-2} - 2240S_2\Big) + (-1)^n\Big(\frac{16\left(2n^2+11n+1\right)}{(n+1)^5} + \frac{128S_{-2}}{(n+1)^3}$$

$$+ \quad \frac{96(5n+3)S_1}{(n+1)^4} - \frac{64S_2}{(n+1)^3}\Big) + \frac{4\left(13n^4+26n^3+13n^2-16n-20\right)S_3}{n^2(n+1)^2}$$

$$- \quad \frac{16\left(15n^2+15n+2\right)S_4}{n(n+1)} - 192S_5 - 832S_{-4,1} + \frac{896S_{-3,1}}{n(n+1)} + 1152S_{-3,2}$$

$$+ \quad S_1^2\Big(-\frac{32\left(3n^2+3n+1\right)}{n^3(n+1)^3} - 768S_{-2,1}\Big) - \frac{32\left(3n^2-n+4\right)S_{-2,1}}{n^2(n+1)^2}$$

$$+ \quad S_2\Big(\frac{2\left(3n^6+9n^5+9n^4+83n^3+76n^2+60n+16\right)}{n^3(n+1)^3} + 64S_3 + 2176S_{-2,1}\Big)$$

$$+ \quad \frac{32\left(3n^2+3n-26\right)S_{2,-2}}{n(n+1)} - 1472S_{3,-2} + \frac{64\left(3n^2+3n-2\right)S_{3,1}}{n(n+1)} + 192S_{3,2} + 192S_{4,1}$$

$$+ \quad 2304S_{-3,1,1} + 512S_{-2,1,-2} + \frac{384\left(n^2+n-4\right)S_{-2,1,1}}{n(n+1)} + S_1\Big(64S_2^2 - \frac{64(2n+1)S_2}{n^2(n+1)^2}$$

$$+ \quad \frac{4\left(22n^6-54n^5+23n^4+88n^3+197n^2+160n+52\right)}{n^4(n+1)^4} - 192S_3 + 64S_4 - 1792S_{-3,1}$$

$$- \quad \frac{192\left(n^2+n-4\right)S_{-2,1}}{n(n+1)} + 1664S_{2,-2} + 256S_{3,1} + 3072S_{-2,1,1}\Big) + 2304S_{-2,2,1}$$

$$+ \quad 2304S_{2,1,-2} - 384S_{3,1,1} - 4608S_{-2,1,1,1}\Big]$$

$$+ \quad C_F^3\zeta_3\Big[-\frac{24\left(5n^4+10n^3+n^2-4n-4\right)}{n^2(n+1)^2} - 192S_{-2}\Big]$$

$$+ \quad C_AC_F^2\Big\{\Big(256S_1 - \frac{16\left(3n^2+3n+8\right)}{n(n+1)}\Big)S_{-2}^2$$

$$+ \quad \Big(-\frac{8\left(81n^5+243n^4-337n^3-1181n^2-526n-60\right)}{9n^2(n+1)^3} + \frac{32\left(31n^2+31n-81\right)S_2}{3n(n+1)}$$

$$+ \quad S_1\Big(1728S_2 - \frac{32\left(134n^4+268n^3+89n^2-81n-72\right)}{9n^2(n+1)^2}\Big) - 1792S_3 - 192S_{-2,1} + 2688S_{2,1}\Big)S_{-2}$$

$$+ \quad \frac{176}{3}S_2^2 - \frac{P_5(n)}{36n^4(n+1)^4} + 672S_{-5} + S_{-4}\Big(\frac{8\left(97n^2+97n-210\right)}{3n(n+1)} + 1120S_1\Big)$$

$$+ \quad S_{-3}\Big(-576S_1^2 + \frac{16\left(31n^2+31n+108\right)S_1}{3n(n+1)} - \frac{8\left(268n^4+536n^3+487n^2+183n+126\right)}{9n^2(n+1)^2}$$

$$+ \quad 480S_{-2} + 2656S_2\Big) + (-1)^n\Big(\frac{8(346n-125)}{9(n+1)^4} - \frac{256S_{-2}}{(n+1)^3} - \frac{16(103n+73)S_1}{3(n+1)^4} + \frac{32S_2}{(n+1)^3}\Big)$$

$$- \quad \frac{8\left(385n^4+770n^3+427n^2+6n-126\right)S_3}{9n^2(n+1)^2} + \frac{8\left(151n^2+151n-30\right)S_4}{3n(n+1)} + 384S_5$$

$$+ \quad 864S_{-4,1} - \frac{960S_{-3,1}}{n(n+1)} - 1344S_{-3,2} + S_2\Big(\frac{2\left(453n^5+1359n^4+2231n^3+1525n^2+80n-264\right)}{9n^2(n+1)^3}$$

$$- \quad 32S_3 - 2624S_{-2,1}\Big) + \frac{16\left(268n^4+536n^3+301n^2-3n+90\right)S_{-2,1}}{9n^2(n+1)^2} + S_1^2(128S_3 + 896S_{-2,1})$$

$$- \quad \frac{16\left(31n^2+31n-174\right)S_{2,-2}}{3n(n+1)} + 1824S_{3,-2} - \frac{32\left(29n^2+29n-24\right)S_{3,1}}{3n(n+1)} - 384S_{3,2} - 384S_{4,1}$$

$$- \quad 2688S_{-3,1,1} - 768S_{-2,1,-2} + S_1\Big(-\frac{8\left(135n^6-649n^5-1039n^4-569n^3+487n^2+621n+216\right)}{9n^4(n+1)^4}$$

$$- \quad \frac{2144}{9}S_2 + \frac{32\left(31n^2+31n-12\right)S_3}{3n(n+1)} + 160S_4 + 1920S_{-3,1} + \frac{32\left(31n^2+31n-84\right)S_{-2,1}}{3n(n+1)}$$

$$- \quad 1856S_{2,-2} - 512S_{3,1} - 3584S_{-2,1,1}\Big) - \frac{64\left(31n^2+31n-84\right)S_{-2,1,1,n}}{3n(n+1)} - 2688S_{-2,2,1}$$

$$- \quad 2688S_{2,1,-2} + 768S_{3,1,1} + 5376S_{-2,1,1,1}\Big]$$

$$+ \quad C_AC_F^2\zeta_3\Big[\frac{36\left(5n^4+10n^3+n^2-4n-4\right)}{n^2(n+1)^2} + 288S_{-2}\Big]$$

$$+ \quad C_A^2C_F\Big(\frac{24\left(n^2+n+2\right)}{n(n+1)} - 96S_1\Big)S_{-2}^2 + \Big(\frac{8\left(27n^6+81n^5-209n^4-595n^3-272n^2-48n-9\right)}{9n^3(n+1)^3}$$

$$+ \quad S_1\Big(\frac{16\left(134n^4+268n^3+116n^2-18n-27\right)}{9n^2(n+1)^2} - 512S_2\Big) - \frac{32\left(11n^2+11n-24\right)S_2}{3n(n+1)} + 512S_3$$

$$+ \quad 64S_{-2,1} - 768S_{2,1}\Big)S_{-2} + \frac{P_6(N)}{108n^3(n+1)^5} - 192S_{-5} + S_{-4}\Big(-\frac{8\left(35n^2+35n-66\right)}{3n(n+1)} - 352S_1\Big)$$

$$+ \quad (-1)^n\Big(-\frac{16\left(91n^2+80n-29\right)}{9(n+1)^5} + \frac{96S_{-2}}{(n+1)^3} + \frac{16(29n+23)S_1}{3(n+1)^4}\Big)$$

$$+ \quad S_{-3}\Big(128S_1^2 - \frac{16\left(11n^2+11n+24\right)S_1}{3n(n+1)} + \frac{8\left(134n^4+268n^3+203n^2+69n+27\right)}{9n^2(n+1)^2}$$

$$- \quad 160S_{-2} - 768S_2\Big) + \frac{4\left(389n^4+778n^3+398n^2+9n-81\right)S_3}{9n^2(n+1)^2} - \frac{8\left(55n^2+55n-24\right)S_4}{3n(n+1)}$$

$$- \quad 160S_5 - 224S_{-4,1} + \frac{256S_{-3,1}}{n(n+1)} + 384S_{-3,2} + S_1^2(-64S_3 - 256S_{-2,1})$$

$$- \quad \frac{16\left(134n^4+268n^3+137n^2+3n+27\right)S_{-2,1}}{9n^2(n+1)^2} + S_2\Big(768S_{-2,1} - \frac{4172}{27}\Big)$$

$$+ \quad \frac{16\left(11n^2+11n-48\right)S_{2,-2}}{3n(n+1)} - 544S_{3,-2} + \frac{32\left(11n^2+11n-12\right)S_{3,1}}{3n(n+1)} + 192S_{3,2}$$

$$
\begin{aligned}
&+\ 192S_{4,1}+768S_{-3,1,1}+256S_{-2,1,-2}+\frac{64\left(11n^2+11n-24\right)S_{-2,1,1}}{3n(n+1)}\\
&+\ S_1\Bigg(\frac{2\left(245n^8+980n^7+1542n^6+964n^5+211n^4-60n^3+156n^2+222n+90\right)}{3n^4(n+1)^4}\\
&-\ \frac{8\left(11n^2+11n-8\right)S_3}{n(n+1)}-128S_4-512S_{-3,1}\\
&-\ \frac{32\left(11n^2+11n-24\right)S_{-2,1}}{3n(n+1)}+512S_{2,-2}+256S_{3,1}+1024S_{-2,1,1}\Bigg)+768S_{-2,2,1}\\
&+\ 768S_{2,1,-2}-384S_{3,1,1}-1536S_{-2,1,1,1}\Bigg\}\\
&+\ C_A^2C_F\zeta_3\left[-\frac{12\left(5n^4+10n^3+n^2-4n-4\right)}{n^2(n+1)^2}-96S_{-2}\right]\\
&+\ C_F^2N_F\Bigg\{-\frac{32}{3}S_2^2-\frac{4\left(15n^4+30n^3+79n^2+16n-24\right)S_2}{9n^2(n+1)^2}+\frac{P_7(n)}{9n^4(n+1)^4}-\frac{128}{3}S_{-4}\\
&+\ S_{-3}\left(\frac{32\left(10n^2+10n+3\right)}{9n(n+1)}-\frac{64}{3}S_1\right)+(-1)^n\left(\frac{64S_1}{3(n+1)^3}-\frac{128(4n+1)}{9(n+1)^4}\right)\\
&+\ S_{-2}\left(-\frac{32\left(16n^2+10n-3\right)}{9n^2(n+1)^2}+\frac{640}{9}S_1-\frac{128}{3}S_2\right)+\frac{16\left(29n^2+29n+12\right)S_3}{9n(n+1)}-\frac{128}{3}S_4\\
&+\ S_1\left(-\frac{2\left(165n^5+495n^4+495n^3+517n^2+336n+80\right)}{9n^2(n+1)^3}+\frac{320}{9}S_2-\frac{128}{3}S_3-\frac{128}{3}S_{-2,1}\right)\\
&-\ \frac{64\left(10n^2+10n-3\right)S_{-2,1}}{9n(n+1)}+\frac{64}{3}S_{2,-2}+\frac{64}{3}S_{3,1}+\frac{256}{3}S_{-2,1,1}\Bigg\}\\
&+\ C_F^2N_F\zeta_3\left[32S_1-\frac{8\left(3n^2+3n+2\right)}{n(n+1)}\right]\\
&+\ C_FN_F^2\left[\frac{51n^6+153n^5+57n^4+35n^3+96n^2+16n-24}{27n^3(n+1)^3}-\frac{16}{27}S_1-\frac{80}{27}S_2+\frac{16}{9}S_3\right]\\
&+\ C_AC_FN_F\Bigg[-\frac{2\left(270n^7+1080n^6+383n^5-979n^4-571n^3+507n^2+106n-132\right)}{27n^3(n+1)^4}\\
&+\ \frac{64}{3}S_{-4}+S_{-3}\left(\frac{32}{3}S_1-\frac{16\left(10n^2+10n+3\right)}{9n(n+1)}\right)+(-1)^n\left(\frac{64(4n+1)}{9(n+1)^4}-\frac{32S_1}{3(n+1)^3}\right)\\
&+\ \frac{1336}{27}S_2+S_{-2}\left(\frac{16\left(16n^2+10n-3\right)}{9n^2(n+1)^2}-\frac{320}{9}S_1+\frac{64}{3}S_2\right)-\frac{8\left(14n^2+14n+3\right)S_3}{3n(n+1)}+\frac{80}{3}S_4\\
&+\ \frac{32\left(10n^2+10n-3\right)S_{-2,1}}{9n(n+1)}+S_1\Bigg(-\frac{4\left(209n^6+627n^5+627n^4+137n^3-108n^2-108n-54\right)}{27n^3(n+1)^3}\\
&+\ 16S_3+\frac{64}{3}S_{-2,1}\Bigg)-\frac{32}{3}S_{2,-2}-\frac{64}{3}S_3,1-\frac{128}{3}S_{-2,1,1}\Bigg]\\
&+\ C_AC_FN_F\zeta_3\left[\frac{8\left(3n^2+3n+2\right)}{n(n+1)}-32S_1\right]
\end{aligned}
$$

$$
\begin{aligned}
P_{qq}^{2,-,dabc} &= \frac{d_{abc}d^{abc}}{N_c}N_F\Bigg[-\frac{P_8(n)}{3n^5(n+1)^5(n+2)^3}+\frac{4\left(n^2+n+2\right)S_{-3}}{n^2(n+1)^2}-\frac{P_9(n)S_1}{3n^4(n+1)^4(n+2)^3}\\
&+\ S_{-2}\left(-\frac{8S_1\left(n^2+n+2\right)^2}{(n-1)n^2(n+1)^2(n+2)}-\frac{4\left(n^6+3n^5-8n^4-21n^3-23n^2-12n-4\right)}{(n-1)n^3(n+1)^3(n+2)}\right)\\
&+\ (-1)^n\Bigg(\frac{16\left(5n^6+29n^5+78n^4+118n^3+114n^2+72n+16\right)S_1}{3(n-1)n^2(n+1)^3(n+2)^3}\\
&-\ \frac{4\left(13n^8+74n^7+179n^6+314n^5+644n^4+1000n^3+816n^2+352n+64\right)}{3(n-1)n^3(n+1)^4(n+2)^3}\Bigg)\\
&-\ \frac{2\left(n^2+n+2\right)S_3}{n^2(n+1)^2}-\frac{8\left(n^2+n+2\right)S_{-2,1}}{n^2(n+1)^2}\Bigg]
\end{aligned}
$$

12. Conclusions - B

- We established a general algorithm to calculate the exact expression for single scale quantities from a finite (suitably large) number of moments (zero scale quantities).
- The latter ones are much more easily calculable.
- We applied the method to the anomalous dimensions and Wilson coefficients up to 3-loop order.
- To solve 3-loop problems this way is not possible at present, since the number of required moments is too large for the methods available.
- We attempted to solve the quantities for all color projections at once. This problem is too voluminous.
- Yet we showed that giant difference equations [order 35; degree ~ 1000] can be reliably and fast established and solved unconditionally for advanced problems in Quantum Field Theory.