Higher order corrections in QCD and Gravity

QMC London, SAGEX meeting

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DESY

Outline

1. Introduction
   - Heavy Quark Corrections to Deep-Inelastic Scattering
   - Massive 3-Loop Formfactors
   - Precision QED Corrections to $e^+e^- \to \gamma^*/Z^*$
   - 3–loop anomalous dimensions

2. Higher Order Corrections in Classical Gravity
   - Up to 5 PN corrections
     - 5 PN: the potential corrections
     - 5 PN: ‘tail’ terms
     - 5 PN: phenomenological results
   - Test of PM results at 6PN

3. Conclusions

Survey on the node activities

- ESR: Marco Saragnese, defending his PhD Thesis on July 14th (UHH); internship at Linz
- 50 papers
- 40 journal publications, 10 proceedings
- 1 Book: Anti-Differentiation an the Calculation of Feynman Amplitudes, Texts & Monographs in Symbolic Computation, (Springer, Berlin, 2021)
- KMPB Workshop: 'From Classical Gravity to Quantum Amplitudes and Back', Berlin, November 17-22, 2019

Papers with the ESR:

- J. Blümlein, M. Saragnese and C. Schneider, *Hypergeometric Structures in Feynman Integrals*, [arXiv:2111.15501 [math-ph]].
- J. Blümlein and M. Saragnese, *The N^3LO scheme-invariant QCD evolution of the non-singlet structure functions $F_2^{NS}(x, Q^2)$ and $g_1^{NS}(x, Q2)$*, Phys. Lett. B **820** (2021) 136589.
- J. Blümlein, A. De Freitas, M. Saragnese, C. Schneider and K. Schönwald, *Logarithmic contributions to the polarized $O(\alpha_s^3)$ asymptotic massive Wilson coefficients and operator matrix elements in deeply inelastic scattering*, Phys. Rev. D **104** (2021) no.3, 034030.
- J. Ablinger, J. Blümlein, A. De Freitas, A. Goedicke, M. Saragnese, C. Schneider and K. Schönwald, *The two-mass contribution to the three-loop polarized gluonic operator matrix element $A_{gg,Q}^{(3)}$* Nucl. Phys. B **955** (2020) 115059.
- J. Ablinger, J. Blümlein, A. De Freitas, M. Saragnese, C. Schneider and K. Schönwald, *The three-loop polarized pure singlet operator matrix element with two different masses*, Nucl. Phys. B **952** (2020) 114916.
- `Fortran` library for DIS evolution to 3-loop order, in preparation.
- 2 more conference proceedings, together 242 pages.

Research Topics:

- Higher order QCD and QED corrections for massive and massless processes (2- and 3-loops)
- Higher order corrections to classical gravity (5 PN, 6PN)
- Mathematical and computer-algebraic methods $\Longrightarrow$ Talk by Carsten Schneider.

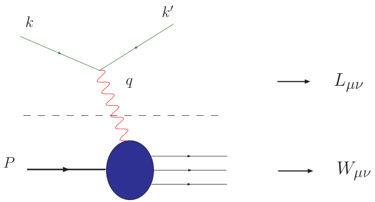
Higher order QCD and QED corrections for massive and massless processes

- Heavy Quark Corrections to Deep-Inelastic Scattering (Behring, JB, De Freitas, Saragnese, Schneider, Schönwald)
- Massive Formfactors (JB, De Freitas, Marquard, Rana, Schneider)
- Precision QED Corrections to $e^+e^- \rightarrow \gamma^*/Z^*$ (JB, De Freitas, Raab, Schönwald)
- 3–loop anomalous dimensions (JB, Marquard, Schönwald,Schneider)

Heavy Quark Corrections to Deep-Inelastic Scattering

- The scaling violations of the heavy quark corrections are quite different from those of massless quarks.
- Work in the region $Q^2 \gg m_Q^2$, also to avoid higher twist corrections.
- Under these conditions the heavy flavor corrections are given by the massive operator matrix elements (OMEs) A_{ij} and the massless process-dependent Wilson coefficients.
- Analytic calculations are possible to the 3–loop level for single and two mass corrections.
- The corrections are needed in the unpolarized and the polarized case.
- The massive OMEs also form the transition matrix element in the Variable Flavor Number Scheme describing the behaviour of massive and massless partons in the high energy range at colliders.
- Measurement goals: improve the current values at the theoretical side.
  - $\alpha_s(M_Z^2) = 0.1147 \pm 0.0008$
  - $m_c(m_c) = 1.252 \pm 0.018(exp) \begin{array}{c}+0.03\\-0.02\end{array} (scale), \begin{array}{c}+0.00\\-0.07\end{array} (thy)\mathrm{GeV}$ ($\overline{\mathrm{MS}}$-scheme)

Deep–inelastic scattering



- Kinematic invariants:

$$Q^2 = -q^2, \qquad x = \frac{Q^2}{2P.q}$$

- The cross section factorizes into leptonic and hadronic tensor:

$$\frac{d^2\sigma}{dQ^2dx} \sim L_{\mu\nu} W^{\mu\nu}$$

- The hadronic tensor can be expressed through structure functions:

$$W_{\mu\nu} = \frac{1}{4\pi}\int d^4\xi \exp(iq\xi) \langle P, | \left[J_\mu^{em}(\xi), J_\nu^{em}(\xi)\right] |P\rangle$$

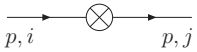
$$= \frac{1}{2x}\left(g_{\mu\nu} + \frac{q_\mu q_\nu}{Q^2}\right) F_L(x,Q^2) + \frac{2x}{Q^2}\left(P_\mu P_\nu + \frac{q_\mu P_\nu + q_\nu P_\mu}{2x} - \frac{Q^2}{4x^2} g_{\mu\nu}\right) F_2(x,Q^2)$$

$$+ i\epsilon_{\mu\nu\rho\sigma}\frac{q^\rho S^\sigma}{q\cdot P} g_1(x,Q^2) + i\epsilon_{\mu\nu\rho\sigma}\frac{q^\rho\left(q\cdot PS^\sigma - q\cdot SP^\sigma\right)}{(q\cdot P)^2} g_2(x,Q^2)$$

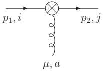
- F_L, F_2, g_1 and g_2 contain contributions from both, charm and bottom quarks.

Calculation of the 3-loop operator matrix elements

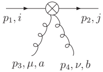
The OMEs are calculated using the QCD Feynman rules together with the following operator insertion Feynman rules:



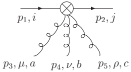
$$\delta^{ij} \not\Delta \gamma_\pm (\Delta \cdot p)^{N-1}\,, \quad N \geq 1$$



$$g t^a_{ji} \Delta^\mu \not\Delta \gamma_\pm \sum_{j=0}^{N-2} (\Delta \cdot p_1)^j (\Delta \cdot p_2)^{N-j-2}\,, \quad N \geq 2$$



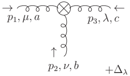
$$g^2 \Delta^\mu \Delta^\nu \not\Delta \gamma_\pm \sum_{j=0}^{N-3} \sum_{l=j+1}^{N-2} (\Delta p_2)^j (\Delta p_1)^{N-l-2}$$
$$\left[(t^a t^b)_{ji} (\Delta p_1 + \Delta p_4)^{l-j-1} + (t^b t^a)_{ji} (\Delta p_1 + \Delta p_3)^{l-j-1} \right]\,,$$
$$N \geq 3$$



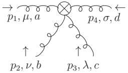
$$g^3 \Delta_\mu \Delta_\nu \Delta_\rho \not\Delta \gamma_\pm \sum_{j=0}^{N-4} \sum_{l=j+1}^{N-3} \sum_{m=l+1}^{N-2} (\Delta.p_2)^j (\Delta.p_1)^{N-m-2}$$
$$\Big[\; (t^a t^b t^c)_{ji} (\Delta.p_4 + \Delta.p_5 + \Delta.p_1)^{l-j-1} (\Delta.p_5 + \Delta.p_1)^{m-l-1}$$
$$+ (t^a t^c t^b)_{ji} (\Delta.p_4 + \Delta.p_5 + \Delta.p_1)^{l-j-1} (\Delta.p_4 + \Delta.p_1)^{m-l-1}$$
$$+ (t^b t^a t^c)_{ji} (\Delta.p_3 + \Delta.p_5 + \Delta.p_1)^{l-j-1} (\Delta.p_5 + \Delta.p_1)^{m-l-1}$$
$$+ (t^b t^c t^a)_{ji} (\Delta.p_3 + \Delta.p_5 + \Delta.p_1)^{l-j-1} (\Delta.p_3 + \Delta.p_1)^{m-l-1}$$
$$+ (t^c t^a t^b)_{ji} (\Delta.p_3 + \Delta.p_4 + \Delta.p_1)^{l-j-1} (\Delta.p_4 + \Delta.p_1)^{m-l-1}$$
$$+ (t^c t^b t^a)_{ji} (\Delta.p_3 + \Delta.p_4 + \Delta.p_1)^{l-j-1} (\Delta.p_3 + \Delta.p_1)^{m-l-1} \Big]\,,$$
$$N \geq 4$$

$$\gamma_+ = 1\,, \quad \gamma_- = \gamma_5\,.$$

$$\frac{1+(-1)^N}{2} \delta^{ab} (\Delta \cdot p)^{N-2}$$
$$\left[g_{\mu\nu} (\Delta \cdot p)^2 - (\Delta_\mu p_\nu + \Delta_\nu p_\mu) \Delta \cdot p + p^2 \Delta_\mu \Delta_\nu \right]\,, \quad N \geq 2$$



$$-ig \frac{1+(-1)^N}{2} f^{abc} \Bigg($$
$$\left[(\Delta_\nu g_{\lambda\mu} - \Delta_\lambda g_{\mu\nu}) \Delta \cdot p_1 + \Delta_\mu (p_{1,\nu} \Delta_\lambda - p_{1,\lambda} \Delta_\nu) \right] (\Delta \cdot p_1)^{N-2}$$
$$+ \Delta_\lambda \left[\Delta \cdot p_1 p_{2,\mu} \Delta_\nu + \Delta \cdot p_2 p_{1,\nu} \Delta_\mu - \Delta \cdot p_1 \Delta \cdot p_2 g_{\mu\nu} - p_1 \cdot p_2 \Delta_\mu \Delta_\nu \right]$$
$$\times \sum_{j=0}^{N-3} (-\Delta \cdot p_1)^j (\Delta \cdot p_2)^{N-3-j}$$
$$+ \left\{ \begin{smallmatrix} p_1 \to p_2 \to p_3 \to p_1 \\ \mu \to \nu \to \lambda \to \mu \end{smallmatrix} \right\} + \left\{ \begin{smallmatrix} p_1 \to p_3 \to p_2 \to p_1 \\ \mu \to \lambda \to \nu \to \mu \end{smallmatrix} \right\} \Bigg)\,, \quad N \geq 2$$



$$g^2 \frac{1+(-1)^N}{2} \Big(f^{abe} f^{cde} O_{\mu\nu\lambda\sigma}(p_1, p_2, p_3, p_4)$$
$$+ f^{ace} f^{bde} O_{\mu\lambda\nu\sigma}(p_1, p_3, p_2, p_4) + f^{ade} f^{bce} O_{\mu\sigma\nu\lambda}(p_1, p_4, p_2, p_3) \Big)\,,$$
$$O_{\mu\nu\lambda\sigma}(p_1, p_2, p_3, p_4) = \Delta_\nu \Delta_\lambda \Big\{ -g_{\mu\sigma} (\Delta \cdot p_3 + \Delta \cdot p_4)^{N-2}$$
$$+ [p_{4,\mu} \Delta_\sigma - \Delta \cdot p_4 g_{\mu\sigma}] \sum_{i=0}^{N-3} (\Delta \cdot p_3 + \Delta \cdot p_4)^i (\Delta \cdot p_4)^{N-3-i}$$
$$- [p_{1,\sigma} \Delta_\mu - \Delta \cdot p_1 g_{\mu\sigma}] \sum_{i=0}^{N-3} (-\Delta \cdot p_1)^i (\Delta \cdot p_3 + \Delta \cdot p_4)^{N-3-i}$$
$$+ [\Delta \cdot p_1 \Delta \cdot p_4 g_{\mu\sigma} + p_1 \cdot p_4 \Delta_\mu \Delta_\sigma - \Delta \cdot p_4 p_{1,\sigma} \Delta_\mu - \Delta \cdot p_1 p_{4,\mu} \Delta_\sigma]$$
$$\times \sum_{i=0}^{N-4} \sum_{j=0}^{i} (-\Delta \cdot p_1)^{N-4-i} (\Delta \cdot p_3 + \Delta \cdot p_4)^{i-j} (\Delta \cdot p_4)^j \Big\}$$
$$- \left\{ \begin{smallmatrix} p_1 \leftrightarrow p_2 \\ \mu \leftrightarrow \nu \end{smallmatrix} \right\} - \left\{ \begin{smallmatrix} p_3 \leftrightarrow p_4 \\ \lambda \leftrightarrow \sigma \end{smallmatrix} \right\} + \left\{ \begin{smallmatrix} p_1 \leftrightarrow p_2,\; p_3 \leftrightarrow p_4 \\ \mu \leftrightarrow \nu,\; \lambda \leftrightarrow \sigma \end{smallmatrix} \right\}\,, \quad N \geq 2$$

The Wilson Coefficients at large Q^2

$$L_{q,(2,L)}^{\rm NS}(N_F+1) = a_s^2\left[A_{qq,Q}^{(2),\rm NS}(N_F+1)\delta_2 + \hat{C}_{q,(2,L)}^{(2),\rm NS}(N_F)\right] + a_s^3\left[A_{qq,Q}^{(3),\rm NS}(N_F+1)\delta_2 + A_{qq,Q}^{(2),\rm NS}(N_F+1)C_{q,(2,L)}^{(1),\rm NS}(N_F+1) + \hat{C}_{q,(2,L)}^{(3),\rm NS}(N_F)\right]$$

$$L_{q,(2,L)}^{\rm PS}(N_F+1) = a_s^3\left[A_{qq,Q}^{(3),\rm PS}(N_F+1)\delta_2 + N_F A_{gq,Q}^{(2),\rm NS}(N_F)\tilde{C}_{g,(2,L)}^{(1),\rm NS}(N_F+1) + N_F\hat{\tilde{C}}_{q,(2,L)}^{(3),\rm PS}(N_F)\right]$$

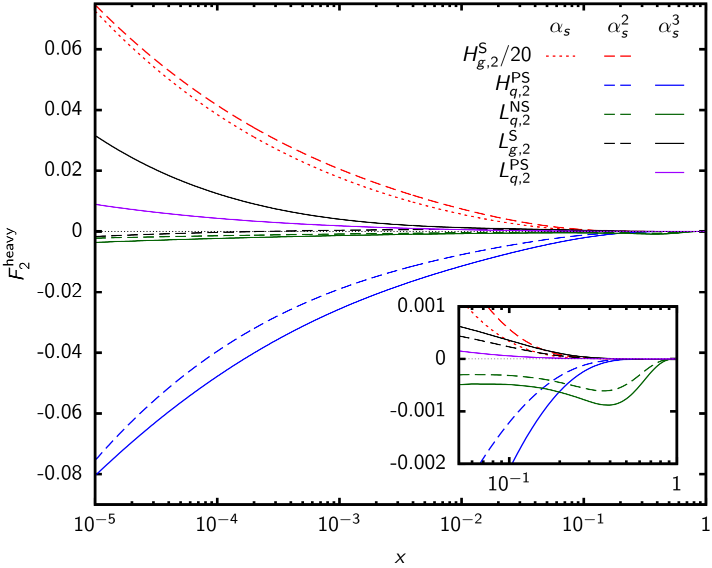
$$L_{g,(2,L)}^{\rm S}(N_F+1) = a_s^2 A_{gg,Q}^{(1)}(N_F+1)N_F\tilde{C}_{g,(2,L)}^{(2)}(N_F+1) + a_s^3\left[A_{qg,Q}^{(3)}(N_F+1)\delta_2 + A_{gg,Q}^{(1)}(N_F+1)N_F\tilde{C}_{g,(2,L)}^{(2)}(N_F+1)\right.$$
$$\left. + A_{gg,Q}^{(2)}(N_F+1)N_F\tilde{C}_{g,(2,L)}^{(1)}(N_F+1) + A_{Qg}^{(1)}(N_F+1)N_F\tilde{C}_{q,(2,L)}^{(2),\rm PS}(N_F+1) + N_F\hat{\tilde{C}}_{g,(2,L)}^{(3)}(N_F)\right]$$

$$H_{q,(2,L)}^{\rm PS}(N_F+1) = a_s^2\left[A_{Qq}^{(2),\rm PS}(N_F+1)\delta_2 + \tilde{C}_{q,(2,L)}^{(2),\rm PS}(N_F+1)\right]$$
$$+ a_s^3\left[A_{Qq}^{(3),\rm PS}(N_F+1)\delta_2 + A_{gq,Q}^{(2)}(N_F+1)\tilde{C}_{g,(1,L)}^{(2)}(N_F+1) + A_{Qq}^{(2),\rm PS}(N_F+1)\tilde{C}_{q,(2,L)}^{(1),\rm NS}(N_F+1) + \tilde{C}_{q,(2,L)}^{(3),\rm PS}(N_F+1)\right]$$

$$H_{g,(2,L)}^{\rm S}(N_F+1) = a_s\left[A_{Qg}^{(1)}(N_F+1)\delta_2 + \tilde{C}_{g,(2,L)}^{(1)}(N_F+1)\right]$$
$$+ a_s^2\left[A_{Qg}^{(2)}(N_F+1)\delta_2 + A_{Qg}^{(1)}(N_F+1)\tilde{C}_{q,(2,L)}^{(1)}(N_F+1) + A_{gg,Q}^{(1)}(N_F+1)\tilde{C}_{g,(2,L)}^{(1)}(N_F+1) + \tilde{C}_{g,(2,L)}^{(2)}(N_F+1)\right]$$
$$+ a_s^3\left[A_{Qg}^{(3)}(N_F+1)\delta_2 + A_{Qg}^{(2)}(N_F+1)\tilde{C}_{q,(2,L)}^{(1)}(N_F+1) + A_{gg,Q}^{(2)}(N_F+1)\tilde{C}_{g,(2,L)}^{(1)}(N_F+1)\right.$$
$$\left. + A_{Qg}^{(1)}(N_F+1)\tilde{C}_{q,(2,L)}^{(2),\rm S}(N_F+1) + A_{gg,Q}^{(1)}(N_F+1)\tilde{C}_{g,(2,L)}^{(1)}(N_F+1) + \tilde{C}_{g,(2,L)}^{(3)}(N_F+1)\right]$$

- All first order factorizable contributions and $O(1000)$ fixed moments of $A_{Qg}^{(3)}$ are known.
- The case for two different masses obeys an analogous representation.

Heavy Flavor contribution to F_2



The variable flavor number scheme

- Matching conditions for parton distribution functions:

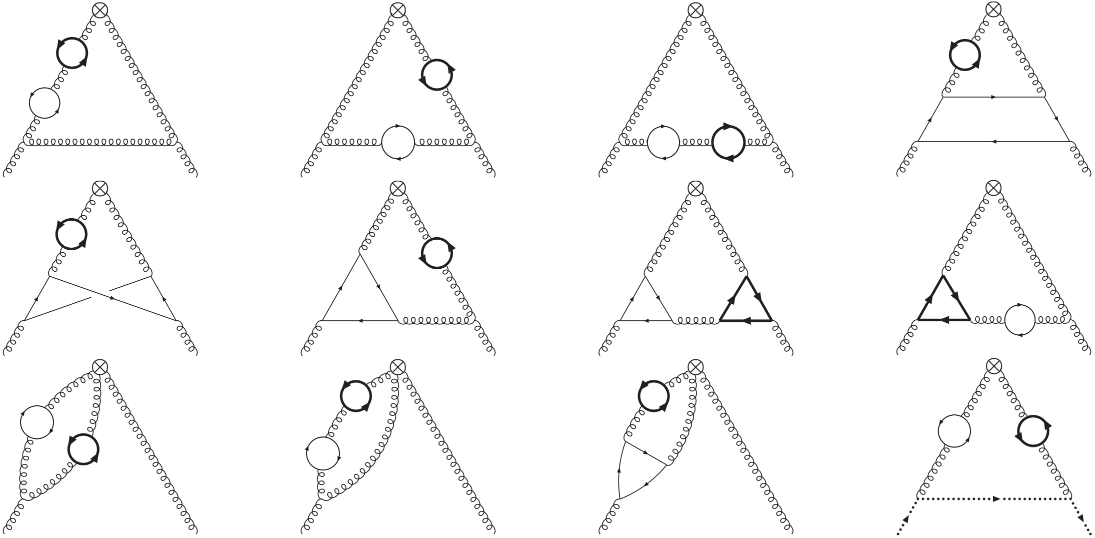
$$f_k(N_F+2)+f_{\overline{k}}(N_F+2)=A_{qq,Q}^{NS}\left(N_F+2,\frac{m_1^2}{\mu^2},\frac{m_2^2}{\mu^2}\right)\cdot\left[f_k(N_F)+f_{\overline{k}}(N_F)\right]+\frac{1}{N_F}A_{qq,Q}^{PS}\left(N_F+2,\frac{m_1^2}{\mu^2},\frac{m_2^2}{\mu^2}\right)\cdot\Sigma(N_F)$$
$$+\frac{1}{N_F}A_{qg,Q}\left(N_F+2,\frac{m_1^2}{\mu^2},\frac{m_2^2}{\mu^2}\right)\cdot G(N_F)\,,$$

$$f_Q(N_F+2)+f_{\overline{Q}}(N_F+2)=A_{Qq}^{PS}\left(N_F+2,\frac{m_1^2}{\mu^2},\frac{m_2^2}{\mu^2},\right)\cdot\Sigma(N_F)+A_{Qg}\left(N_F+2,\frac{m_1^2}{\mu^2},\frac{m_2^2}{\mu^2}\right)\cdot G(N_F)\,,$$

$$\Sigma(N_F+2)=\left[A_{qq,Q}^{NS}\left(N_F+2,\frac{m_1^2}{\mu^2},\frac{m_2^2}{\mu^2}\right)+A_{qq,Q}^{PS}\left(N_F+2,\frac{m_1^2}{\mu^2},\frac{m_2^2}{\mu^2}\right)+A_{Qq}^{PS}\left(N_F+2,\frac{m_1^2}{\mu^2},\frac{m_2^2}{\mu^2}\right)\right]\cdot\Sigma(N_F)$$
$$+\left[A_{qg,Q}\left(N_F+2,\frac{m_1^2}{\mu^2},\frac{m_2^2}{\mu^2}\right)+A_{Qg}\left(N_F+2,\frac{m_1^2}{\mu^2},\frac{m_2^2}{\mu^2}\right)\right]\cdot G(N_F)\,,$$

$$G(N_F+2)=A_{gq,Q}\left(N_F+2,\frac{m_1^2}{\mu^2},\frac{m_2^2}{\mu^2}\right)\cdot\Sigma(N_F)+A_{gg,Q}\left(N_F+2,\frac{m_1^2}{\mu^2},\frac{m_2^2}{\mu^2}\right)\cdot G(N_F)\,.$$

2-mass contributions



2-mass contributions

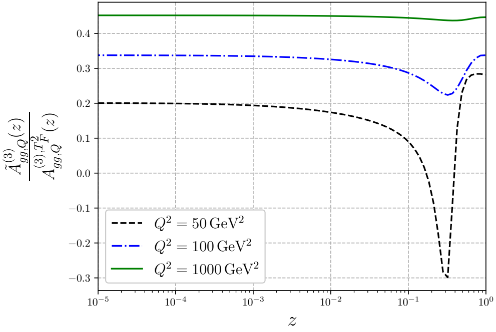
$A_{qq,Q}^{(3),\mathrm{NS}}$, $A_{gq,Q}^{(3)}$	Harmonic Sums [Vermaseren '98; Blümlein, Kurth '98] $\sum_{i=1}^{N} \frac{1}{i^3} \sum_{j=1}^{i} \frac{1}{j}$	HPLs [Remiddi, Vermaseren '99] $\int_0^x \frac{d\tau_1}{1+\tau_1} \int_0^{\tau_1} \frac{d\tau_2}{1-\tau_2}$
$A_{gg,Q}^{(3)}$	Generalized harmonic and binomial sums [Ablinger, Blülein, Schneider '13] [Ablinger, Blülein, Raab, Schneider '14] $\sum_{i=1}^{N} \frac{4^i (1-\eta)^{-i}}{i \binom{2i}{i}} \sum_{j=1}^{i} \frac{(1-\eta)^j}{j^2}$	Iterated integrals over root and η valued letters [Ablinger, Blümlein, Raab, Schneider '14] $\int_0^x d\tau_1 \frac{\sqrt{\tau_1(1-\tau_1)}}{1-\tau_1(1-\eta)} \int_0^{\tau_1} \frac{d\tau_2}{\tau_2}$
$A_{Qq}^{(3),\mathrm{PS}}$	–	Iterated integrals over root valued letters with restricted support $\theta(x-\eta_+) \int_0^{x(1-x)/\eta} d\tau \frac{\sqrt{1-4\tau}}{\tau}$

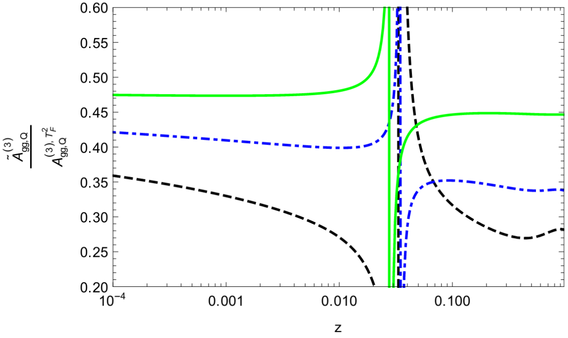
Results: $A_{gg,Q}^{(3)}$

$$
\begin{aligned}
\tilde{a}_{gg,Q}^{(3)}(N) \quad = \quad & \frac{1}{2}\left(1+(-1)^N\right)\Biggl\{T_F^3\Biggl\{\frac{32}{3}\left(L_1^3+L_2^3\right)+\frac{64}{3}L_1L_2\left(L_1+L_2\right)+32\zeta_2\left(L_1+L_2\right)+\frac{128}{9}\zeta_3\Biggr\} \\
+ \quad & C_F T_F^2\Biggl\{\cdots+32\left(\mathrm{H}_0^2(\eta)-\frac{1}{3}S_2\right)S_1+\frac{128}{3}S_{2,1}-\frac{64}{3}S_{1,1,1}\left(\frac{1}{1-\eta},1-\eta,1,N\right) \\
& -\frac{4P_{41}}{3(N-1)N^3(N+1)^2(N+2)(2N-3)(2N-1)}\left(\frac{\eta}{1-\eta}\right)^N\Biggl[\mathrm{H}_0^2(\eta) \\
& -2\mathrm{H}_0(\eta)S_1\left(\frac{\eta-1}{\eta},N\right)-2S_2\left(\frac{\eta-1}{\eta},N\right)+2S_{1,1}\left(\frac{\eta-1}{\eta},1,N\right)\Biggr]+\ldots\Biggr\} \\
+ \quad & C_A T_F^2\Biggl\{\cdots+\Biggl[\frac{8P_{65}}{3645\eta(N-1)N^3(N+1)^3(N+2)(2N-3)(2N-1)} \\
& +\frac{8P_{37}\mathrm{H}_0(\eta)}{45\eta(N-1)N^2(N+1)^2(N+2)}+\frac{2P_{23}\mathrm{H}_0^2(\eta)}{9\eta(N-1)N(N+1)^2}+\frac{32}{27}\mathrm{H}_0^3(\eta)+\frac{128}{9}\mathrm{H}_{0,0,1}(\eta) \\
& +\frac{64}{9}\mathrm{H}_0^2(\eta)\mathrm{H}_1(\eta)-\frac{128}{9}\mathrm{H}_0(\eta)\mathrm{H}_{0,1}(\eta)\Biggr]S_1 \\
& +\frac{2^{-1-2N}P_{47}}{45\eta^2(N-1)N(N+1)^2(N+2)(2N-3)(2N-1)}\binom{2N}{N}\sum_{i=1}^{N}\frac{4^i\left(\frac{\eta}{\eta-1}\right)^i}{i\binom{2i}{i}}\Biggl\{\frac{1}{2}\mathrm{H}_0^2(\eta) \\
& S_{1,1}\left(\frac{\eta-1}{\eta},1,i\right)\Biggr\}+\ldots\Biggr\}
\end{aligned}
$$

Results: $A_{gg,Q}^{(3)}$

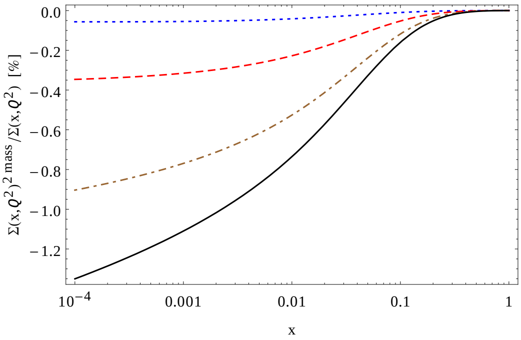
The two mass contributions over the whole T_F^2- contributions to the OME $A_{gg,Q}^{(3)}$:



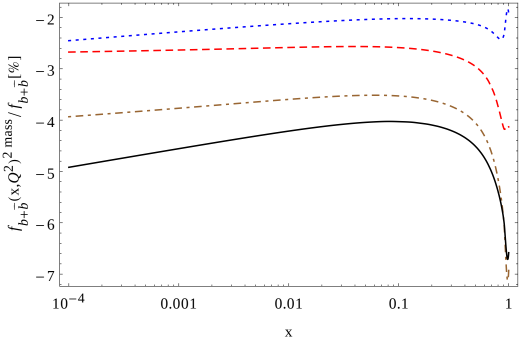


The variable flavor number scheme at NLO

$\Sigma(x, Q^2)^{\text{2-mass}} / \Sigma(x, Q^2)$



$f_{b+\bar{b}}(x, Q^2)^{\text{2-mass}} / f_{b+\bar{b}}(x, Q^2)$



- The ratio of the 2-mass contributions to the singlet parton distribution $\Sigma(x, Q^2)$ (left) and the heavy flavor parton distribution $f_{b+\bar{b}}(x, Q^2)$ (right) over their full form in percent for $m_c = 1.59$ GeV, $m_b = 4.78$ GeV in the on-shell scheme. Dash-dotted line: $Q^2 = 30$ GeV^2; Dotted line: $Q^2 = 30$ GeV^2; Dashed line: $Q^2 = 100$ GeV^2; Dash-dotted line: $Q^2 = 1000$ GeV^2; Full line: $Q^2 = 10000$ GeV^2.
- For the PDFs the `NNLO` variant of ABMP16 with $N_f = 3$ flavors was used.
  Alekhin et al., Phys. Rev. D 96 (2017) 1

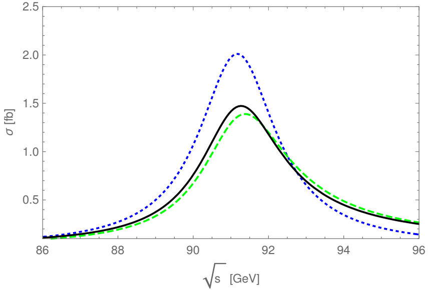
Massive 3-Loop Formfactors

- Quarkonic corrections finished
- Using difference eq. technologies; all first order factorizing contributions have been obtained.
- Non-first order factorizing contributions are analytically continued using differential equations and the matching method.
- Here the problem can be traced back to a series of new special numbers thanks to the use of computer algebra.
- Gluonic corrections are in preparation.
- The latter case can be solved in a similar way.
- The calculation of these quantities are of importance for the precision determination of the mass of the top quark, as fundamental parameter of the Standard Model, at future high energy e^+e^- colliders.

Precision QED Corrections to $e^+e^- \to \gamma^*/Z^*$

- In the future the properties of the Z-boson shall be measured down to a few MeV, where $M_Z = 91187.6$ MeV and $\Gamma_Z = 2495.2$ MeV for the energy-dependent scattering cross section.
- This requires to calculate the initial state QED radiative corrections to very high orders.
- Errors on the level of the $O(\alpha^2)$ corrections (Berends, Burgers, van Neerven, 1988) had to be corrected [several years of intense work, developing new computation methods.]
- Correction now known at $O(\alpha^2)$ completely and all the first three logarithmic terms in higher orders up to $O(\alpha^6 L^5)$, with $L = \ln(M_Z^2/m_e^2)$.
- Note that $m_e^2 \approx M_Z^2 10^{-10}$ and numerics in this small parameter has to be exactly controlled before expanding in m_e^2/M_Z^2.
- Two-valued root letter alphabets emerge, including incomplete elliptic integrals.
- The forward-backward asymmetry A_{FB} has also been calculated to high order in the leading log approximation to $O((\alpha L)^6)$.
- It allows to measure the fine structure constant α_{QED} at high energies at high precision.
- This provides to measure non-perturbative hadronic contributions.

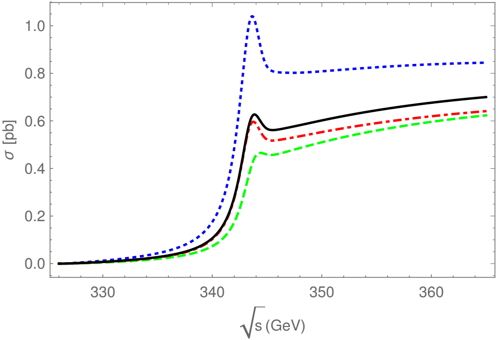
Z peak

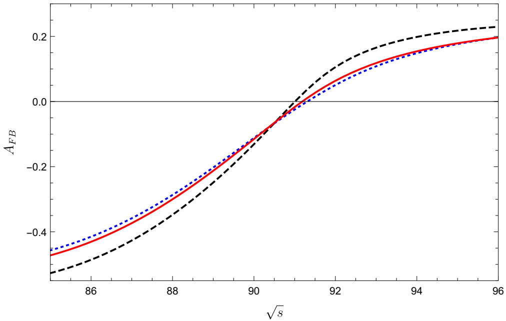


	Fixed width		s dep. width	
	Peak (MeV)	Width (MeV)	Peak (MeV)	Width (MeV)
$O(\alpha)$ correction	185.638	539.408	181.098	524.978
$O(\alpha^2 L^2)$:	– 96.894	–177.147	– 95.342	–176.235
$O(\alpha^2 L)$:	6.982	22.695	6.841	21.896
$O(\alpha^2)$:	0.176	– 2.218	0.174	– 2.001
$O(\alpha^3 L^3)$:	23.265	38.560	22.968	38.081
$O(\alpha^3 L^2)$:	– 1.507	– 1.888	– 1.491	– 1.881
$O(\alpha^3 L)$:	– 0.152	0.105	– 0.151	– 0.084
$O(\alpha^4 L^4)$:	– 1.857	0.206	– 1.858	0.146
$O(\alpha^4 L^3)$:	0.131	– 0.071	0.132	– 0.065
$O(\alpha^4 L^2)$:	0.048	– 0.001	0.048	0.001
$O(\alpha^5 L^5)$:	0.142	– 0.218	0.144	– 0.212
$O(\alpha^5 L^4)$:	– 0.000	0.020	– 0.001	0.020
$O(\alpha^5 L^3)$:	– 0.008	0.009	– 0.008	0.008
$O(\alpha^6 L^6)$:	– 0.007	0.027	– 0.007	0.027
$O(\alpha^6 L^5)$:	– 0.001	0.000	– 0.001	0.000

Table 1: Shifts in the Z-mass and the width due to the different contributions to the ISR QED radiative corrections for a fixed width of $\Gamma_Z = 2.4952$ GeV and s-dependent width using $M_Z = 91.1876$ GeV and $s_0 = 4m_\tau^2$.

Top-threshold corrections and A_{FB}





dotted: $O(\alpha^0)$, dashed: $O(\alpha)$, dash-dotted: $O(\alpha^2)$, full: $O(\alpha^2)$ + soft resummation

dotted: $O(\alpha^0)$, dashed: $O(\alpha)$, full line: including $O(\alpha^6 L^6)$

3–loop anomalous dimensions

- All non-singlet anomalous dimensions were calculated in a fully automated way.
- Also the polarized singlet anomalous dimensions were computed.
- The method used are off-shell gauge variant massless operator matrix elements, using our method of arbitrary high Mellin moments.
- In the unpolarized case the complete singlet anomalous dimensions are obtained at 2-loop order, correcting errors in the foregoing literature.
- The anomalous dimensions for transversity were calculated first.
- In earlier calculations (Moch, Vermaseren, Vogt 2004a,b 2014) various special assumptions have been made, which we all did not assume. Those were now verified for the case of general N for the first time by the present calculations.

The evolution equations

The different anomalous dimensions

- 3 non–singlet anomalous dimensions (starting at 1-, 2-, and 3-loop) $\gamma_{qq}^{\rm NS,\pm}, \gamma_{qq}^{\rm NS,s}$
- 4 singlet anomalous dimensions $\gamma_{qq}^{\rm PS}, \gamma_{qg}, \gamma_{gq}, \gamma_{gg}$ [$\gamma_{qq}^{\rm PS}$ contributes from 2-loops.]
- both in the unpolarized and polarized case: $\gamma_{ij} \rightarrow \Delta\gamma_{ij}$
- + expansion coefficients for the so called alien operators, including new anomalous dimensions.

Evolution equations

$$\Sigma(N) = \sum_{k=1}^{N_F} q_k(N) + \bar{q}_k(N)$$

$$\gamma_{ij} = \sum_{k=0}^{\infty} a_s^{k+1} \gamma_{ij}$$

$$\frac{dq^{\rm NS,\pm,(s)}(N, a_s)}{d\ln(\mu^2)} = \sum_{k=0}^{\infty} a_s^{k+1} \gamma_{qq}^{(k),\rm NS,\pm(s)}(N) \cdot q^{\rm NS,\pm(s)}(N, a_s)$$

$$\frac{d}{d\ln(\mu^2)} \begin{pmatrix} \Sigma(N) \\ G(N) \end{pmatrix} = \sum_{k=0}^{\infty} a_s^{k+1} \begin{pmatrix} \gamma_{qq}^{(k)} & \gamma_{qg}^{(k)} \\ \gamma_{gq}^{(k)} & \gamma_{gg}^{(k)} \end{pmatrix} \cdot \begin{pmatrix} \Sigma(N) \\ G(N) \end{pmatrix}$$

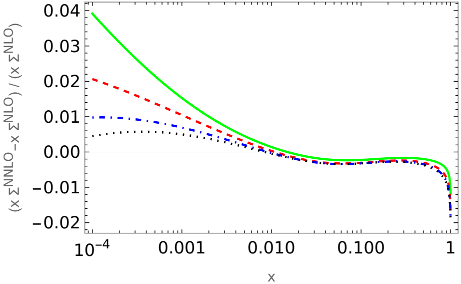
The non-singlet anomalous dimension $\gamma_{qq}^{+,\mathrm{NS}}$

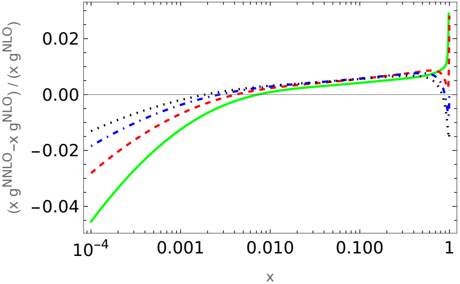
$$\gamma_{\mathrm{NS}}^{(2),+} = \frac{1}{2}\left[1+(-1)^N\right]$$
$$\times\Bigg\{C_F^2\Bigg\{C_A\Bigg[\frac{72P_3}{N^2(1+N)^2}\zeta_3 + \frac{32P_{15}}{9N^2(1+N)^2}S_{-2,1} - \frac{16P_{17}}{9N^2(1+N)^2}S_3 + \frac{P_{33}}{18N^4(1+N)^4}$$
$$+\Bigg(-\frac{16P_{29}}{9N^4(1+N)^4} - \frac{4288}{9}S_2 + \frac{64(-12+31N+31N^2)}{3N(1+N)}S_3 + 320S_4 - 1024S_{3,1}$$
$$+\frac{64(-84+31N+31N^2)}{3N(1+N)}S_{-2,1} + 3712S_{-2,2} + 3840S_{-3,1} - 7168S_{-2,1,1}\Bigg)S_1 + \Bigg(256S_3$$
$$+1792S_{-2,1}\Bigg)S_1^2 + \Bigg(\frac{4P_{19}}{9N^2(1+N)^2} - 832S_3 - 5248S_{-2,1}\Bigg)S_2 + \frac{352}{3}S_2^2$$
$$+\frac{16(-30+151N+151N^2)}{3N(1+N)}S_4 + \Bigg(-\frac{16P_{22}}{9N^2(1+N)^3} + (-\frac{64P_9}{9N^2(1+N)^2} - 256S_2\Bigg)S_1$$
$$+\frac{32(12+31N+31N^2)S_2}{3N(1+N)} + 64S_3 + 5376S_{2,1} - 384S_{-2,1} + 576\zeta_3\Bigg)S_{-2}$$
$$+\Bigg(-\frac{32(8+3N+3N^2)}{N(1+N)} + 512S_1\Bigg)S_{-2}^2 + \Bigg(\frac{32(108+31N+31N^2)}{3N(1+N)}S_1 - \frac{16P_{16}}{9N^2(1+N)^2}$$
$$-1152S_1^2 + 2624S_2 + 960S_{-2}\Bigg)S_{-3} + \Bigg(\frac{16(138+35N+35N^2)}{3N(1+N)} - 1472S_1\Bigg)S_{-4}$$
$$+2304S_{-5} + 768S_{2,3} + 2688S_{2,-3} - \frac{64(-24+29N+29N^2)}{3N(1+N)}S_{3,1} - 768S_{4,1}$$
$$+\frac{32(-174+31N+31N^2)}{3N(1+N)}S_{-2,2} - 3648S_{-2,3} - \frac{1920}{N(1+N)}S_{-3,1} + 1728S_{-4,1}$$
$$-5376S_{2,1,-2} + 1536S_{3,1,1} - \frac{128(-84+31N+31N^2)}{3N(1+N)}S_{-2,1,1} - 1536S_{-2,1,-2}$$
$$-5376S_{-2,2,1} - 5376S_{-3,1,1} + 10752S_{-2,1,1,1}\Bigg] + T_FN_F\Bigg[-\frac{16P_5}{9N^2(1+N)^2}S_2 + \frac{4P_{34}}{9N^4(1+N)^4}$$
$$+\Bigg(-\frac{8P_{13}}{9N^2(1+N)^2} + \frac{1280}{9}S_2 - \frac{512}{3}S_3 - \frac{512}{3}S_{-2,1} + 128\zeta_3\Bigg)S_1 - \frac{128}{3}S_2^2$$
$$+\frac{64(12+29N+29N^2)}{9N(1+N)}S_3 - \frac{512}{3}S_4 + \Bigg(-\frac{128(-3+10N+16N^2)}{9N^2(1+N)^2} + \frac{2560}{9}S_1 - \frac{256}{3}S_2\Bigg)$$
$$\times S_{-2} + \Bigg(\frac{128(3+10N+10N^2)}{9N(1+N)} - \frac{256}{3}S_1\Bigg)S_{-3} - \frac{256}{3}S_{-4} - \frac{256(-3+10N+10N^2)}{9N(1+N)}S_{-2,1}$$
$$+\frac{256}{3}S_{3,1} - \frac{256}{3}S_{-2,2} + \frac{1024}{3}S_{-2,1,1} - \frac{32(2+3N+3N^2)}{N(1+N)}\zeta_3\Bigg]\Bigg\}$$
$$+C_F\Bigg\{T_F^2N_F^2\Bigg[\frac{8P_{28}}{27N^3(1+N)^3} - \frac{128}{27}S_1 - \frac{640}{27}S_2 + \frac{128}{9}S_3\Bigg] + C_A^2\Bigg[-\frac{24P_3}{N^2(1+N)^2}\zeta_3$$
$$\frac{32P_{11}}{9N^2(1+N)^2}S_{-2,1} + \frac{8P_{18}}{9N^2(1+N)^2}S_3 + \frac{P_{32}}{54N^3(1+N)^3} + \Bigg(\frac{4P_{35}}{3N^4(1+N)^4}$$
$$-\frac{16(-8+11N+11N^2)}{N(1+N)}S_3 - 256S_4 + 512S_{3,1} - \frac{64(-24+11N+11N^2)}{3N(1+N)}S_{-2,1}$$
$$-1024S_{-2,2} - 1024S_{-3,1} + 2048S_{-2,1,1}\Bigg)S_1 + \Bigg(-128S_3 - 512S_{-2,1}\Bigg)S_1^2 + \Bigg(-\frac{8344}{27}$$
$$+384S_3 + 1536S_{-2,1}\Bigg)S_2 - \frac{16(-24+55N+55N^2)}{3N(1+N)}S_4 + 64S_5 + \Bigg(\frac{32P_{10}}{9N^2(1+N)^2}S_1$$
$$+\frac{16P_{27}}{9N^3(1+N)^3} - \frac{352}{3}S_2 - 64S_3 - 1536S_{2,1} + 128S_{-2,1} - 192\zeta_3\Bigg)S_{-2}$$
$$+\Bigg(\frac{48(2+N+N^2)}{N(1+N)} - 192S_1\Bigg)S_{-2}^2 + \Bigg(\frac{16P_{12}}{9N^2(1+N)^2} - \frac{32(24+11N+11N^2)}{3N(1+N)}S_1$$
$$+256S_1^2 - 768S_2 - 320S_{-2}\Bigg)S_{-3} + \Bigg(-\frac{16(30+13N+13N^2)}{3N(1+N)} + 320S_1\Bigg)S_{-4}$$
$$-704S_{-5} - 384S_{2,3} - 768S_{2,-3} + \frac{64(-12+11N+11N^2)}{3N(1+N)}S_{3,1} + 384S_{4,1}$$
$$-\frac{32(-48+11N+11N^2)}{3N(1+N)}S_{-2,2} + 1088S_{-2,3} + \frac{512}{N(1+N)}S_{-3,1} - 448S_{-4,1}$$
$$+1536S_{2,1,-2} - 768S_{3,1,1} + \frac{128(-24+11N+11N^2)}{3N(1+N)}S_{-2,1,1} + 512S_{-2,1,-2} + 1536[S_{-2,2,1}$$
$$+S_{-3,1,1}] - 3072S_{-2,1,1,1}\Bigg] + C_AT_FN_F\Bigg[-\frac{8P_{31}}{27N^3(1+N)^3} + \Bigg(-\frac{16P_{30}}{27N^3(1+N)^3} + 64S_3$$
$$+\frac{256}{3}S_{-2,1} - 128\zeta_3\Bigg)S_1 + \frac{5344}{27}S_2 - \frac{32(3+14N+14N^2)}{3N(1+N)}S_3 + \frac{320}{3}S_4 + \Bigg(-\frac{1280}{9}S_1$$
$$+\frac{64(-3+10N+16N^2)}{9N^2(1+N)^2} + \frac{128}{3}S_2\Bigg)S_{-2} + \Bigg(-\frac{64(3+10N+10N^2)}{9N(1+N)} + \frac{128}{3}S_1\Bigg)S_{-3}$$
$$+\frac{128}{3}S_{-4} - \frac{256}{3}S_{3,1} + \frac{128(-3+10N+10N^2)}{9N(1+N)}S_{-2,1} + \frac{128}{3}S_{-2,2} - \frac{512}{3}S_{-2,1,1}$$
$$+\frac{32(2+3N+3N^2)}{N(1+N)}\zeta_3\Bigg]\Bigg\} + C_F^3\Bigg\{-\frac{48P_3}{N^2(1+N)^2}\zeta_3 + \frac{8P_4}{N^2(1+N)^2}S_3 + \frac{P_{36}}{N^5(1+N)^5}$$
$$+\Bigg(\frac{8P_{26}}{N^4(1+N)^4} - \frac{128(1+2N)}{N^2(1+N)^2}S_2 + 128S_2^2 - 384S_3 + 128S_4 + 512S_{3,1} - 3328S_{-2,2}$$
$$-\frac{384(-4+N+N^2)}{N(1+N)}S_{-2,1} - 3584S_{-3,1} + 6144S_{-2,1,1}\Bigg)S_1 + \Bigg(-\frac{64(1+3N+3N^2)}{N^3(1+N)^3}$$
$$-1536S_{-2,1}\Bigg)S_1^2 + \Bigg(\frac{4P_{25}}{N^3(1+N)^3} + 512S_3 + 4352S_{-2,1}\Bigg)S_2 - \frac{32(2+3N+3N^2)}{N(1+N)}S_2^2$$
$$\frac{32(2+15N+15N^2)}{N(1+N)}S_4 + (\frac{32P_{24}}{N^3(1+N)^3} + \Bigg(-\frac{128(5+7N+3N^2)}{N^2(1+N)^2} + 512S_2\Bigg)S_1$$
$$-\frac{64(4+3N+3N^2)}{N(1+N)}S_2 + 128S_3 - 4608S_{2,1} + 256S_{-2,1} - 384\zeta_3\Bigg)S_{-2} + \Bigg(\frac{128}{N(1+N)}$$
$$-256S_1\Bigg)S_{-2}^2 + \Bigg(\frac{32(8+5N+9N^2)}{N^2(1+N)^2} - \frac{64(20+3N+3N^2)}{N(1+N)}S_1 + 1280S_1^2 - 2176S_2$$
$$-640S_{-2}\Bigg)S_{-3} + \Bigg(-\frac{32(26+3N+3N^2)}{N(1+N)} + 1664S_1\Bigg)S_{-4} - 1792S_{-5} - 384S_{2,3}$$
$$-2304S_{2,-3} + \frac{128(-2+3N+3N^2)}{N(1+N)}S_{3,1} + 384S_{4,1} - \frac{64(4-N+3N^2)}{N^2(1+N)^2}S_{-2,1}$$
$$-\frac{64(-26+3N+3N^2)}{N(1+N)}S_{-2,2} + 2944S_{-2,3} + \frac{1792}{N(1+N)}S_{-3,1} - 1664S_{-4,1}$$
$$+4608S_{2,1,-2} - 768S_{3,1,1} + \frac{768(-4+N+N^2)}{N(1+N)}S_{-2,1,1} + 1024S_{-2,1,-2}$$
$$+4608[S_{-2,2,1} + S_{-3,1,1}] - 9216S_{-2,1,1,1}\Bigg\}\Bigg\},$$

The polarized singlet anomalous dimension: $\Delta\gamma_{gg}^{(2)}$

$$
\begin{aligned}
\Delta\gamma_{gg}^{(2)} = {} & C_A T_F^2 N_F^2 \Bigg[-\frac{16P_8}{27N^2(1+N)^2}S_1 - \frac{4P_{48}}{27N^3(1+N)^3}\Bigg] + C_F \Bigg[T_F^2 N_F^2 \Bigg[-\frac{8P_{59}}{27N^4(1+N)^4} \\
& + \frac{64(N-1)(2+N)\big(-6-8N+N^2\big)}{9N^3(1+N)^3}S_1 + \frac{32(N-1)(2+N)}{3N^2(1+N)^2}S_1^2 \\
& - \frac{32(N-1)(2+N)}{N^2(1+N)^2}S_2\Bigg] + C_A T_F N_F \Bigg[\frac{8P_6}{N^3(1+N)^3}S_2 - \frac{8P_9}{3N^3(1+N)^3}S_1^2 \\
& + \frac{2P_{77}}{27(N-1)N^5(1+N)^5(2+N)} + \Bigg(-\frac{8P_{67}}{9(-1+N)N^4(1+N)^4(2+N)} \\
& - \frac{32(N-1)(2+N)}{N^2(1+N)^2}S_2 + 128\zeta_3\Bigg)S_1 + \frac{32(N-1)(2+N)}{3N^2(1+N)^2}S_1^3 - \frac{32\big(34+N+N^2\big)}{3N^2(1+N)^2} \\
& \times S_3 + \Bigg(\frac{128P_2}{(N-1)N^2(1+N)^2(2+N)}S_1 - \frac{32P_{23}}{(N-1)N^2(1+N)^3(2+N)}\Bigg)S_{-2} \\
& - \frac{192\big(4-N-N^2\big)}{N^2(1+N)^2}S_{-3} + \frac{64(N-1)(2+N)}{N^2(1+N)^2}S_{2,1} - \frac{128\big(-8+N+N^2\big)}{N^2(1+N)^2}S_{-2,1} \\
& - \frac{64(-3+N)(4+N)}{N^2(1+N)^2}\zeta_3\Bigg]\Bigg] + C_A^3\Bigg[\frac{64P_{16}}{9N^2(1+N)^2}S_{-2,1} - \frac{32P_{18}}{9N^2(1+N)^2}S_3 \\
& + \frac{P_{74}}{27(N-1)N^5(1+N)^5(2+N)} + \Bigg(\frac{4P_{69}}{9(N-1)N^4(1+N)^4(2+N)} \\
& - \frac{64P_{17}}{9N^2(1+N)^2}S_2 + 128S_2^2 + \frac{16\big(-96+11N+11N^2\big)}{3N(1+N)}S_3 + 192S_4 \\
& + \frac{1024}{N(1+N)}S_{-2,1} - 640S_{-2,2} - 768S_{-3,1} + 1024S_{-2,1,1}\Bigg)S_1 \\
& + \Bigg(-\frac{256\big(1+3N+3N^2\big)}{N^3(1+N)^3} + 128S_3 - 256S_{-2,1}\Bigg)S_1^2 + \Bigg(-\frac{16P_{41}}{9N^3(1+N)^3} \\
& + 64S_3 + 640S_{-2,1}\Bigg)S_2 - \frac{256}{N(1+N)}S_2^2 - \frac{384}{N(1+N)}S_4 + 64S_5 \\
& + \Bigg(\frac{32P_{52}}{9(N-1)N^3(1+N)^3(2+N)} + \Big(-\frac{64P_{32}}{9(-1+N)N(1+N)^2(2+N)} + 256S_2\Bigg) \\
& \times S_1 - \frac{512}{N(1+N)}S_2 + 128S_3 - 768S_{2,1}\Bigg)S_{-2} + \Bigg(-\frac{16\big(24+11N+11N^2\big)}{3N(1+N)} \\
& + 64S_1\Bigg)S_{-2}^2 + \Bigg(-\frac{32P_{15}}{9N^2(1+N)^2} - \frac{1536}{N(1+N)}S_1 + 384S_1^2 - 320S_2\Bigg)S_{-3} \\
& + \Bigg(-\frac{1024}{N(1+N)} + 512S_1\Bigg)S_{-4} - 192S_{-5} - 384S_{2,-3} + \frac{1280}{N(1+N)}S_{-2,2} \\
& + 384S_{-2,3} + \frac{1536}{N(1+N)}S_{-3,1} - 384S_{-4,1} + 768S_{2,1,-2} - \frac{2048}{N(1+N)}S_{-2,1,1} \\
& + 768[S_{-2,2,1} + S_{-3,1,1}] - 1536S_{-2,1,1,1}\Bigg] \\
& + C_F^2 T_F N_F\Bigg[-\frac{4P_{75}}{(N-1)N^5(1+N)^5(2+N)} + \Bigg(\frac{32(N-1)(2+N)S_2}{N^2(1+N)^2} \\
& - \frac{16P_{42}}{N^4(1+N)^4}\Bigg)S_1 + \frac{8(N-1)(2+N)\big(2+3N+3N^2\big)}{N^3(1+N)^3}S_1^2 - \frac{32(N-1)(2+N)}{3N^2(1+N)^2} \\
& \times S_1^3 - \frac{8(2+N)\big(2-11N-16N^2+9N^3\big)}{N^3(1+N)^3}S_2 + \frac{32\big(10+7N+7N^2\big)}{3N^2(1+N)^2}S_3 \\
& + \Bigg(-\frac{64\big(10+N+N^2\big)}{(N-1)N(1+N)(2+N)} + \frac{512}{N^2(1+N)^2}S_1\Bigg)S_{-2} + \frac{256}{N^2(1+N)^2}S_{-3} \\
& - \frac{64(N-1)(2+N)}{N^2(1+N)^2}S_{2,1} - \frac{512}{N^2(1+N)^2}S_{-2,1} + \frac{192\big(-2-N-N^2\big)}{N^2(1+N)^2}\zeta_3\Bigg] \\
& + C_A^2 T_F N_F\Bigg[\frac{32P_4}{9N^2(1+N)^2}S_2 + \frac{32P_{11}}{9N^2(1+N)^2}S_{-3} - \frac{64P_{11}}{9N^2(1+N)^2}S_{-2,1} \\
& + \frac{16P_{13}}{9N^2(1+N)^2}S_3 + \frac{2P_{76}}{27(N-1)N^5(1+N)^5(2+N)} + \Bigg(\frac{1280}{9}S_2 - \frac{64}{3}S_3 \\
& - \frac{8P_{68}}{27(-1+N)N^4(1+N)^4(2+N)} - 128\zeta_3\Bigg)S_1 + \frac{64}{3}S_{-2}^2 \\
& + \Bigg(\frac{64P_{45}}{9(N-1)N^2(1+N)^2(2+N)}S_1 - \frac{32P_{50}}{9(N-1)N^3(1+N)^3(2+N)}\Bigg)S_{-2} \\
& + \frac{128\big(-3+2N+2N^2\big)}{N^2(1+N)^2}\zeta_3\Bigg]
\end{aligned}
$$

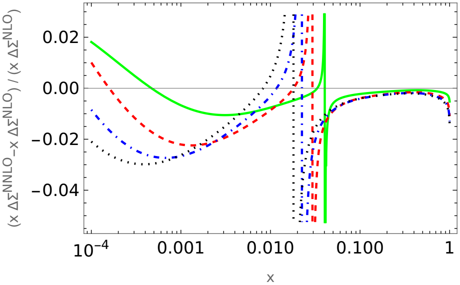
The unpolarized NNLO evolution

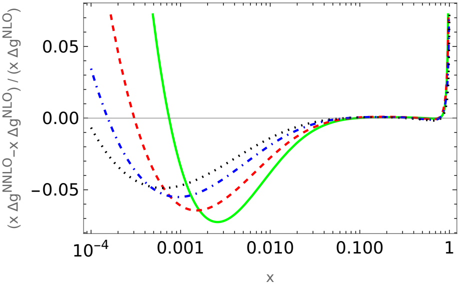




$Q^2 = 10, 10^2, 10^3, 10^4$ GeV^2 dotted, dash-dotted, dashed, full lines.

The polarized NNLO evolution





Higher Order Corrections in Classical Gravity

Topics:

- 5 PN corrections
- Test of the PM results at 6PN
- Study the inspiraling phase of 2 massive objects
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The topic has been inspired by J. Plefka's talk at QMC in 2018. This has been the time of the 3PN / 4PN static potential corrections using effective field-theory methods (i.e. 4PN incomplete). Foffa et al. [1612.00482]

However, the complete 4 PN corrections were known by using other technologies (ADM), Damour et al. [1401.4548]

- Current Status:
  - Post Minkowskian approach:
    - G^4 : Bern et al. [2112.10750], Dlapa et al. [2112.11296]
    - potential contributions are checked up to 6PN in Blümlein et al. [2101.08630]
    - Blümlein et al. [2003.07145] proofed that the G^3 terms of Bern et al. [1901.04424] are correct and a hypothesis in Damour [1912.02139] does not apply.
    - Many recent research results using the post Minkowskian approach: see the extensive list of Refs. given in Blümlein et al. [2003.07145]

Higher Order Corrections in Classical Gravity

- Current Status:
  - Post Newtonian approach:
    - 4 PN
    - complete: [A lot of groups, working in at least 3 different gauges.] Canonical transformations cf.: Blümlein et al. [2003.01692]
    - Results of different approaches can only be compared by referring to observables.
    - 5 PN
    - partial results Bini et al. [2003.11891] tutti frutti; two constants cannot be determined
    - 5 PN potential terms Blümlein et al. [2010.13672] EFT complete
    - 5 PN tail terms through multipole expansion Blümlein et al. [2110.13822] EFT (see discussion below)
    - Bini et al. [2107.08896]: disagreement of the multipole 'tail' contributions of Foffa et al. [1907.02869] with χ_4 ν constraint.
    - 6 PN
    - partial results Bini et al. [2007.11239] tutti frutti; various more constants cannot be determined
    - However, 5 PN is not yet finished, which would be a conditio sine qua non to understand 6 PN.
  - The complete results can only be obtained by a full calculation.

Up to the 5 PN static potential

Hamiltonian and Lagrange formalism:

[applicable to the bound state and to the scattering problem]

EFT approach to Einstein gravity, cf. Kol & Smolkin [0712.4116 [hep-th]].

- 5 PN static potential
  - Foffa et al. [1902.10571] by geometric trick
  - Blümlein et al. [1902.11180] calculated within EFT ab initio
  - The papers were submitted within half a day independently.

$$\mathcal{L}^S_{5\mathrm{PN}} = -\frac{G_N^6}{r^6} m_1 m_2 \left[\frac{5}{16}(m_1^5 + m_2^5) + \frac{91}{6} m_1 m_2 (m_1^3 + m_2^3) + \frac{653}{6} m_1^2 m_2^2 (m_1 + m_2)\right]$$

- 4 PN complete by EFT
  - ADM Damour et al. [1401.4548]
  - harmonic coordinates Blanchet et al. [1610.07934] Foffa & Sturani [1903.05113] Blümlein et al. [2003.01692]
  - EOB Bini et al. [2003.11891]
  - isotropic coordinates Bern et al. [2112.10750] and earlier papers

5 PN: the potential corrections

Blümlein et al. [2010.13672]:

- calculation ab initio in harmonic coordinates
- treatment of potential and singular 'tail' terms together in D dimensions: pole cancellation up to a canonical transformation
- pole-free Hamiltonian
- adding the non-local 'tail' terms [agreement with the literature]
- γ_5-like treatment of ε_{ijk} in D dimensions: leading to the correct terms $O(\nu)$; see also the later paper: Foffa et al. [2110.14146]
- obtaining all terms but the rational terms $O(\nu^2)$
- The remaining finite rational $O(\nu^2)$ terms come all from **the 'tail'.**
- The potential terms have been mulitiply verified (static potential, up to $O(G^4)$ terms by Bern et al.)
- Blümlein et al. [2010.13672] introduced the expansion by regions to classical (EFT) gravity; only potential and ultra-soft modes contribute in the classical limit.

5 PN: the potential corrections

First obtained:

$$\bar{d}_5^{\pi^2\nu^2} = \frac{306545}{512}\pi^2\nu^2,$$
$$a_6^{\pi^2\nu^2} = \frac{25911}{256}\pi^2\nu^2.$$

#loops	QGRAF	source irred.	no source loops	no tadpoles	masters
0	3	3	3	3	0
1	72	72	72	72	1
2	3286	3286	3286	2702	1
3	81526	62246	60998	41676	1
4	545812	264354	234934	116498	7
5	332020	128080	101570	27582	4

Table: Numbers of contributing diagrams at the different loop levels and master integrals.

5 PN: 'tail' terms

There is no generally agreed field theoretic approach to the non-potential terms yet, but would be utterly needed.

Blümlein et al. [2110.13822]:

- It is assumed at present that all non-potential terms can be obtained from multi-pole insertions in the sense of an EFT approach. Foffa & Sturani et al. [1903.05113], Marchand et al. [2003.13672] Larrouturou et al. [2110.02243,2110.02240].
- Partly different propagator treatment in the literature.
- A consistent description is possible by using the in-in formalism.
- Unfortunately the ν constraint hypothesis Bini et al. [2003.11891] is not met for the finite $O(\nu^2)$ terms.
- Closer analysis in the EOB representation.

5 PN: EOB representation

- Our results obtained in harmonic coordinates can be re-parameterized in EOB form for all local terms.
- The nonlocal terms do already agree between different approaches.

$$
\begin{aligned}
H_{\rm EOB}^{\rm loc,eff} &= \sqrt{A(1 + AD\eta^2(p.n)^2 + \eta^2(p^2 - (p.n)^2) + Q)}, \\
A &= 1 + \sum_{k=1}^{6} a_k(\nu)\eta^{2k}u^k, \quad a_2 = 0, \\
D &= 1 + \sum_{k=2}^{5} d_k(\nu)\eta^{2k}u^k, \\
Q &= \eta^4(p.n)^4[q_{42}(\nu)\eta^4u^2 + q_{43}(\nu)\eta^6u^3 + q_{44}(\nu)\eta^8u^4] + \eta^6(p.n)^6[q_{62}(\nu)\eta^4u^2 + q_{63}(\nu)\eta^6u^3] \\
&\quad +\eta^{12}(p.n)^8u^2q_{82}(\nu)\ .
\end{aligned}
$$

Here $u = 1/r$ and $\eta = 1/c$.

5 PN: EOB representation

$$\mathrm{5PN}, u^2 : q_{82} = \frac{6}{7}\nu + \frac{18}{7}\nu^2 + \frac{24}{7}\nu^3 - 6\nu^4,$$

$$u^3 : q_{63} = \frac{123}{10}\nu - \frac{69}{5}\nu^2 + 116\nu^3 - 14\nu^4,$$

$$u^4 : q_{44} = \left(\frac{1580641}{3150} - \frac{93031}{1536}\pi^2\right)\nu + \left(-\frac{3670222}{4725} + \frac{31633}{512}\pi^2\right)\nu^2 + \left(640 - \frac{615}{32}\pi^2\right)\nu^3,$$

$$u^5 : \bar{d}_5 = \left(\frac{331054}{175} - \frac{63707}{512}\pi^2\right)\nu + \bar{d}_5^{\nu^2}\nu^2 + \left(\frac{1069}{3} - \frac{205}{16}\pi^2\right)\nu^3,$$

$$u^6 : a_6 = \left(-\frac{1026301}{1575} + \frac{246367}{3072}\pi^2\right)\nu + a_6^{\nu^2}\nu^2 + 4\nu^3.$$

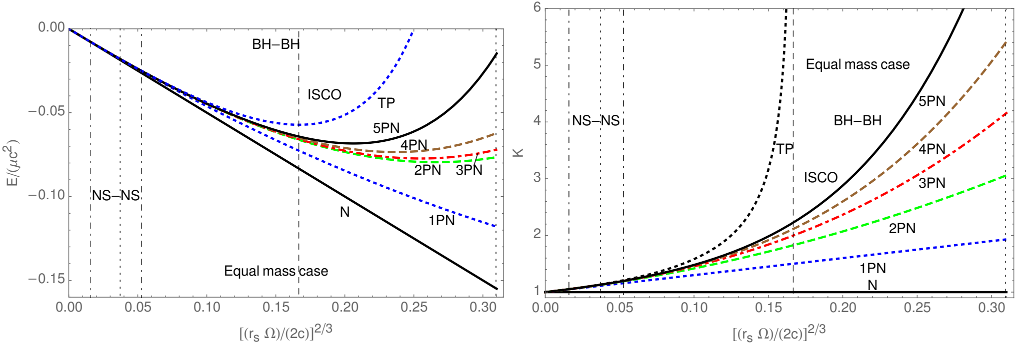
New:

$$\bar{d}_5^{\nu^2} = \left(-\frac{31295104}{4725} + \frac{306545}{512}\pi^2\right), \quad a_6^{\nu^2} = \left(-\frac{1749043}{1575} + \frac{25911}{256}\pi^2\right)$$

$$q_{44}^{\nu^2,r} = -\frac{9367}{15}$$: Bini et al. [2003.11891]

5 PN: phenomenological results

Binding energy & periastron advance in the circular limit



A numerical remark on the scattering angle: Khalil et al. [2204.05047]

The usual scattering angle takes values of ~ 120 degrees and larger. The remaining numerical difference is of the order of 10^{-3} degrees for velocities $< 1/2$.

Yet it has to be clarified.

Test of PM results at 6PN

- We have calculated the 6 PN contributions up to G^4 in Blümlein et al. [2003.07145], [2101.08630]
- This confirmed

$$C_B = \frac{2}{3}\gamma(14\gamma^2 + 25) + 4(4\gamma^4 - 12\gamma^2 - 3)\frac{as(\gamma)}{\sqrt{\gamma^2 - 1}}$$

  from Bern et al. [1901.04424]
- and ruled out

$$C_c = \gamma(35 + 26\gamma^2) - (18 + 96\gamma^2)\frac{as(\gamma)}{\sqrt{\gamma^2 - 1}}$$

  from Damour [1912.02139v1]
- The results also agree with Bini et al. [2004.05407]
  Here

$$as(\gamma) = arcsinh(\sqrt{(\gamma - 1)/2}), \quad \gamma = \sqrt{p_\infty^2 + 1})$$

  and C_i contributes to $\chi_3(\gamma, \nu)$.

Conclusions

- Significant progress has been made in the field of analytic massive 3-loop corrections in QCD and QED (up to two scales).
- Technologies are available to solve problems footing on complex alphabets (also with more than one variable).
- Calculation of QCD power corrections at 2-loop order.
- 3-loop massless results on anomalous dimensions.
- Application of effective field theory methods to classical Einstein gravity.
- reached the level of 5 and 6PN.
- Many interesting topics ahead for the coming years.
- Current and future experiments testing QCD, QED, the SM, and Gravity need
  Analytic und Numeric Precision Calculations.