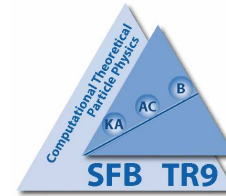


# $O(\alpha_s^3)$ Heavy Flavor Wilson Coefficients in DIS @ $Q^2 \gg m^2$

Johannes Blümlein, DESY

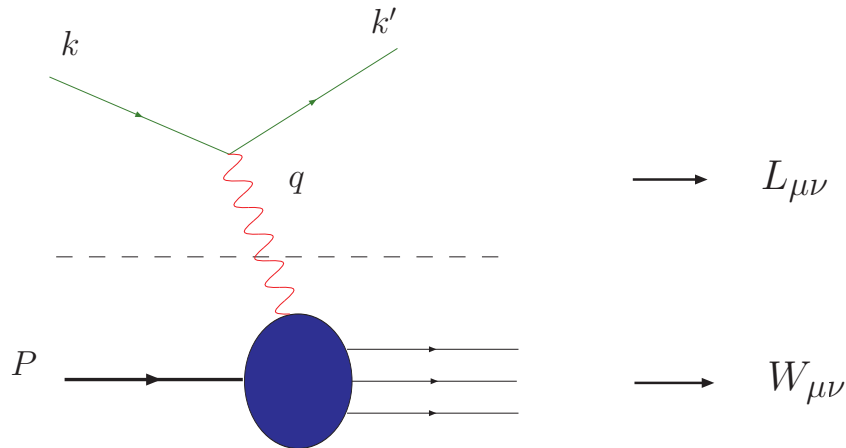
[in collaboration with I. Bierenbaum and S. Klein]



- Introduction
- Theory Status
- The Method
- Asymptotic 2 Loop Results (all N)
- Asymptotic 3 Loop Results (fixed moments)
- Conclusions

# 1. Introduction

Deep-Inelastic Scattering (DIS):



$$Q^2 := -q^2, \quad x := \frac{Q^2}{2pq} \quad \text{Bjorken-}x$$

$$\nu := \frac{Pq}{M},$$

$$\frac{d\sigma}{dQ^2 dx} \sim W_{\mu\nu} L^{\mu\nu}$$

$$W_{\mu\nu}(q, P, s) = \frac{1}{4\pi} \int d^4\xi \exp(iq\xi) \langle P, s | [J_\mu^{em}(\xi), J_\nu^{em}(0)] | P, s \rangle$$

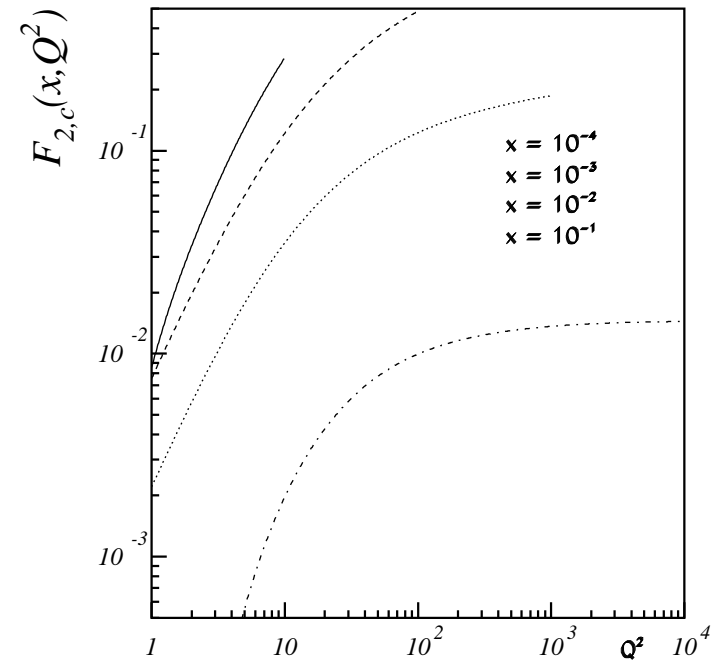
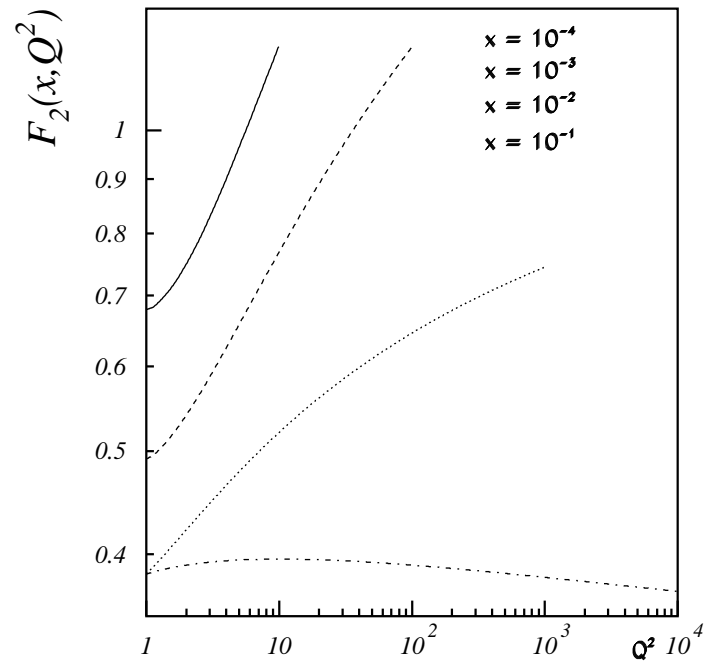
$$\text{unpol.} \left\{ = \frac{1}{2x} \left( g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right) F_L(x, Q^2) + \frac{2x}{Q^2} \left( P_\mu P_\nu + \frac{q_\mu P_\nu + q_\nu P_\mu}{2x} - \frac{Q^2}{4x^2} g_{\mu\nu} \right) F_2(x, Q^2) \right.$$

$$\left. \text{pol.} \left\{ -\frac{M}{2Pq} \varepsilon_{\mu\nu\alpha\beta} q^\alpha \left[ s^\beta g_1(x, Q^2) + \left( s^\beta - \frac{sq}{Pq} p^\beta \right) g_2(x, Q^2) \right] \right. \right.$$

Structure Functions:  $F_{2,L}, g_{1,2}$

contain light and heavy quark contributions

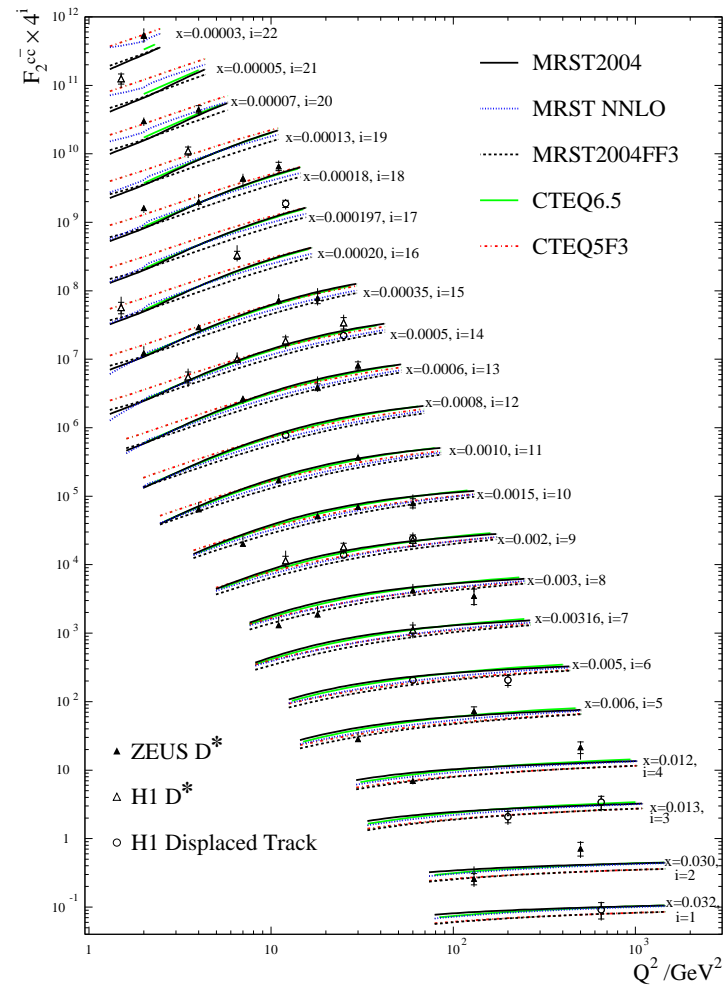
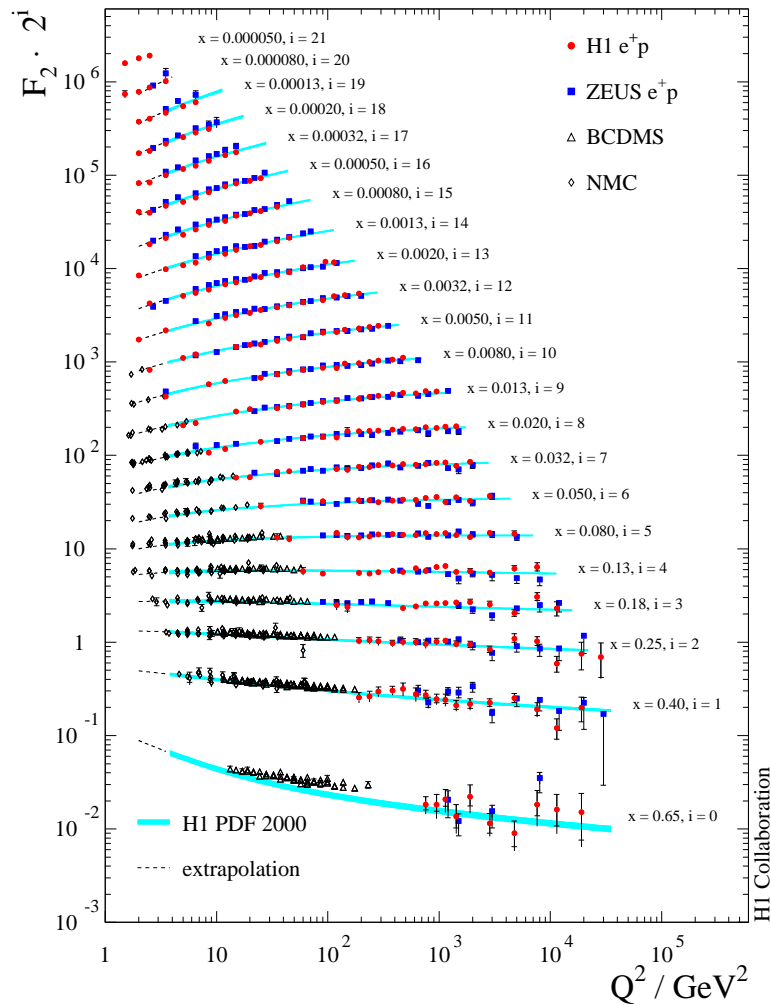
Let us compare the scaling violations in the massless and massive case.



LO charm contributions : PDFs from [Alekhin, Melnikov, Petriello, 2006.]

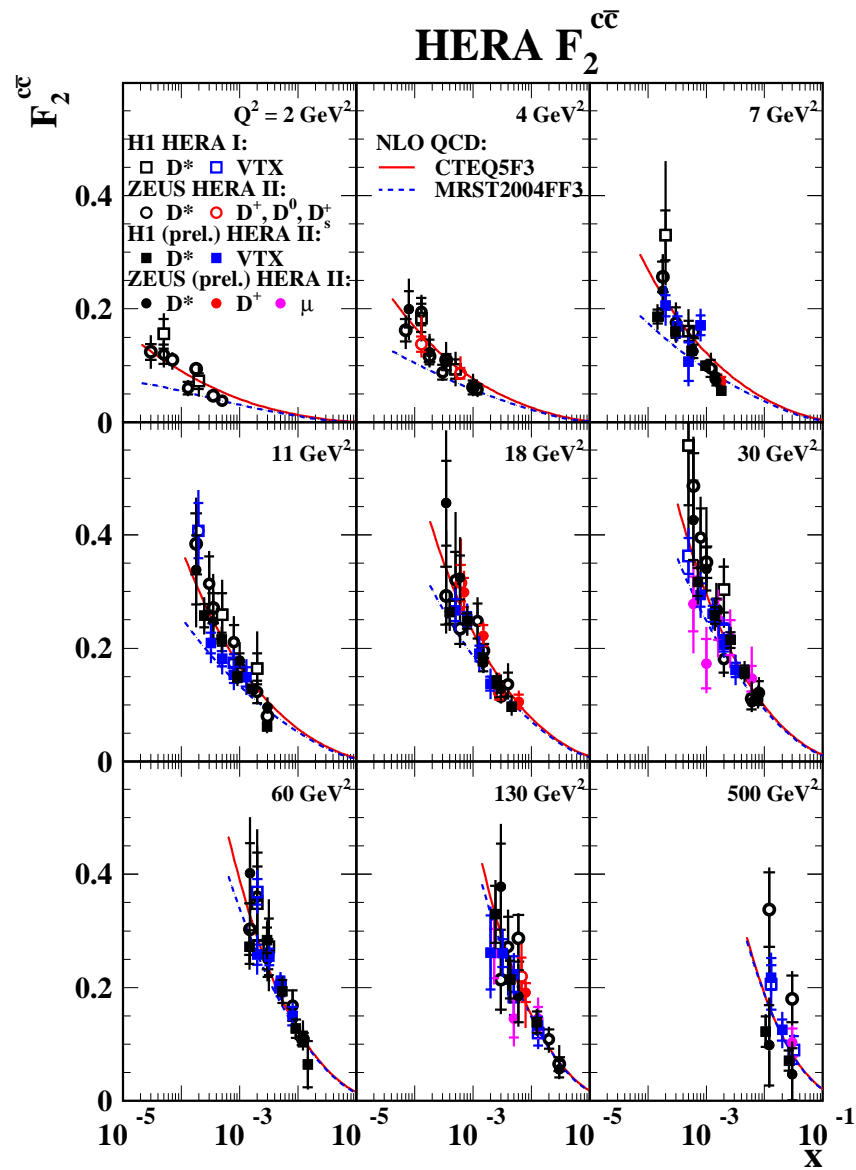
$\Rightarrow$  different scaling violations!

# Light and Heavy Quark Contributions

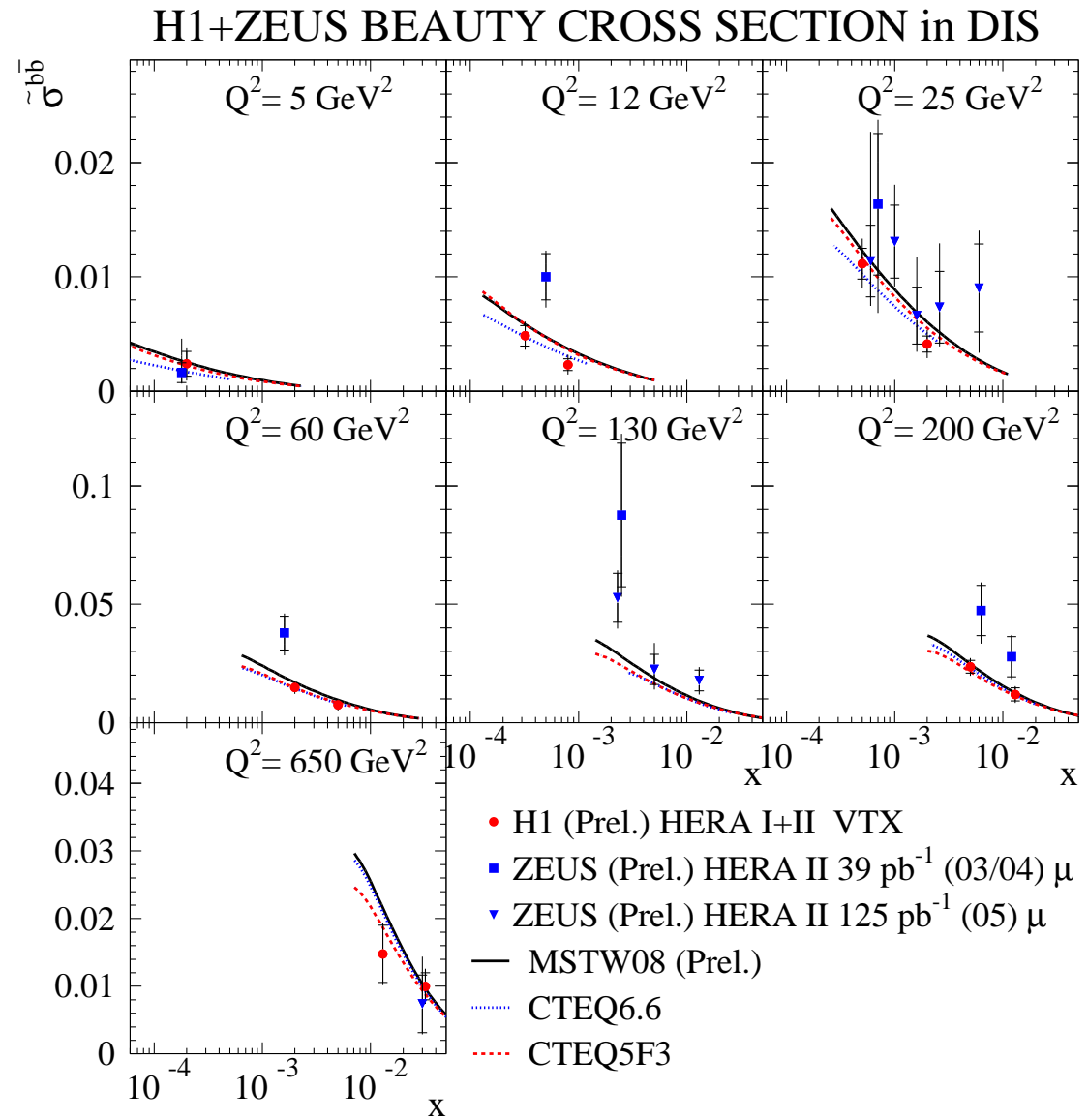


[Thompson, 2007]

High statistics for  $F_2$  and  $F_2^{c\bar{c}}$   $\implies$  Accuracy will increase in the future.



[Krüger (H1 and Z. Coll.), 2008.]



[Krüger (H1 and Z. Coll.), 2008.]

# Evolution of Light Quark Distributions

- The **scaling violations** are described by the **splitting functions**  $P_{ij}(x, a_s)$ .
- They describe the **probability** to find parton  $i$  radiated from parton  $j$  and carrying its momentum fraction  $x$ .
- They are related to the **anomalous dimensions** via a **Mellin-Transform**:

$$\mathbf{M}[f](N) := \int_0^1 dz z^{N-1} f(z) , \quad \gamma_{ij}(N, a_s) := -\mathbf{M}[P_{ij}](N, a_s) .$$

- The **splitting functions** govern the **scale-evolution** of the **parton densities**.

$$\frac{d}{d \ln Q^2} \begin{pmatrix} \Sigma(N, Q^2) \\ G(N, Q^2) \end{pmatrix} = - \begin{pmatrix} \gamma_{qq} & \gamma_{qg} \\ \gamma_{gq} & \gamma_{gg} \end{pmatrix} \otimes \begin{pmatrix} \Sigma(N, Q^2) \\ G(N, Q^2) \end{pmatrix} ,$$

$$\frac{d}{d \ln Q^2} q_{NS}(N, Q^2) = -\gamma_{qq,NS} \otimes q_{NS} .$$

- The **singlet light flavor density** is defined by

$$\Sigma(n_f, \mu^2) = \sum_{i=1}^{n_f} (f_i(n_f, \mu^2) + \bar{f}_i(n_f, \mu^2)) .$$

- The **anomalous dimensions** are presently known at **NNLO** [Moch, Vermaseren, Vogt, 2004] and **N<sup>3</sup>LO** for NS<sup>+</sup>, N=2 [Baikov & Chetyrkin, 2006]

## 2. Theory Status

Leading Order :  $F_{2,L}(x, Q^2)$  [Witten, 1976; Babcock, Sivers, 1978; Shifman, Vainshtein, Zakharov, 1978; Leveille, Weiler, 1979; Glück, Reya, 1979; Glück, Hoffmann, Reya, 1982.]

Leading Order :  $g_1(x, Q^2)$  [Watson, 1982; Glück, Reya, Vogelsang, 1991; Vogelsang, 1991]

Leading Order :  $g_2(x, Q^2)$  [J.B., Ravindran, van Neerven, 2003]  $\implies$  holds to all orders  
Wandzura-Wilczek relation.

Soft resummation:  $F_{2,L}(x, Q^2)$  [Laenen & Moch, 1998; Alekhin & Moch, 2008]

Next-to-Leading Order :  $F_{2,L}(x, Q^2)$  [Laenen, Riemersma, Smith, van Neerven, 1993, 1995]

asymptotic: [Buza, Matiounine, Smith, Migneron, van Neerven, 1996; Bierenbaum, J.B., Klein, 2007]

Next-to-Leading Order :  $g_1(x, Q^2)$  asymptotic: [Buza, Matiounine, Smith, Migneron, van Neerven, 1997; Bierenbaum, J.B., Klein, 2009]

- Fast semi-analytic representation in Mellin space: [Alekhin & J.B., 2003] This includes power corrections.

$\implies$  3-Loop corrections needed to line up with the accuracy reached for light flavor contributions.

## 3. The Method

- massless RGE and light-cone expansion in Bjorken-limit  $\{Q^2, \nu\} \rightarrow \infty$ ,  $x$  fixed:

$$\lim_{\xi^2 \rightarrow 0} \left[ J(\xi), J(0) \right] \propto \sum_{i, N, \tau} c_{i, \tau}^N(\xi^2, \mu^2) \xi_{\mu_1} \dots \xi_{\mu_N} O_{i, \tau}^{\mu_1 \dots \mu_N}(0, \mu^2) .$$

- Operators: flavor non-singlet ( $\leq 3$ ), pure-singlet and gluon; consider leading twist.
- RGE for collinear singularities: mass factorization of the structure functions into Wilson coefficients and parton densities:

$$F_i(x, Q^2) = \sum_j \underbrace{C_i^j \left( x, \frac{Q^2}{\mu^2} \right)}_{\text{perturbative}} \otimes \underbrace{f_j(x, \mu^2)}_{\text{non-perturbative}}$$

- Light-flavor Wilson coefficients: process dependent ( $O(a_s^3)$ ): [Moch, Vermaseren, Vogt, 2005.]

$$C_{(2,L);i}^{\text{fl}} \left( \frac{Q^2}{\mu^2} \right) = \delta_{i,q} + \sum_{l=1}^{\infty} a_s^l C_{(2,L),i}^{\text{fl},(l)}, \quad i = q, g$$

- Heavy quark contributions given by heavy quark Wilson coefficients,  $H_{(2,L),i}^{\text{S,NS}} \left( \frac{Q^2}{\mu^2}, \frac{m^2}{Q^2} \right)$ .



- In the limit  $Q^2 \gg m_h^2$  [ $Q^2 \approx 10 m^2$  for  $F_2, g_1$ ]:  
**massive RGE**, derivative  $m^2 \partial / \partial m^2$  acts on Wilson coefficients only: all terms but power corrections calculable through **partonic operator matrix elements**,  $\langle i | A_l | j \rangle$ , which are **process independent objects!**

$$H_{(2,L),i}^{\text{S,NS}} \left( \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2} \right) = \underbrace{A_{k,i}^{\text{S,NS}} \left( \frac{m^2}{\mu^2} \right)}_{\text{massive OMEs}} \otimes \underbrace{C_{(2,L),k}^{\text{S,NS}} \left( \frac{Q^2}{\mu^2} \right)}_{\text{light-parton-Wilson coefficients}}.$$

- holds for **polarized** and **unpolarized** case. OMEs obey expansion

$$A_{k,i}^{\text{S,NS}} \left( \frac{m^2}{\mu^2} \right) = \langle i | O_k^{\text{S,NS}} | i \rangle = \delta_{k,i} + \sum_{l=1}^{\infty} a_s^l A_{k,i}^{\text{S,NS},(l)} \left( \frac{m^2}{\mu^2} \right), \quad i = q, g$$

[Buza, Matiounine, Migneron, Smith, van Neerven, 1996; Buza, Matiounine, Smith, van Neerven, 1997.]

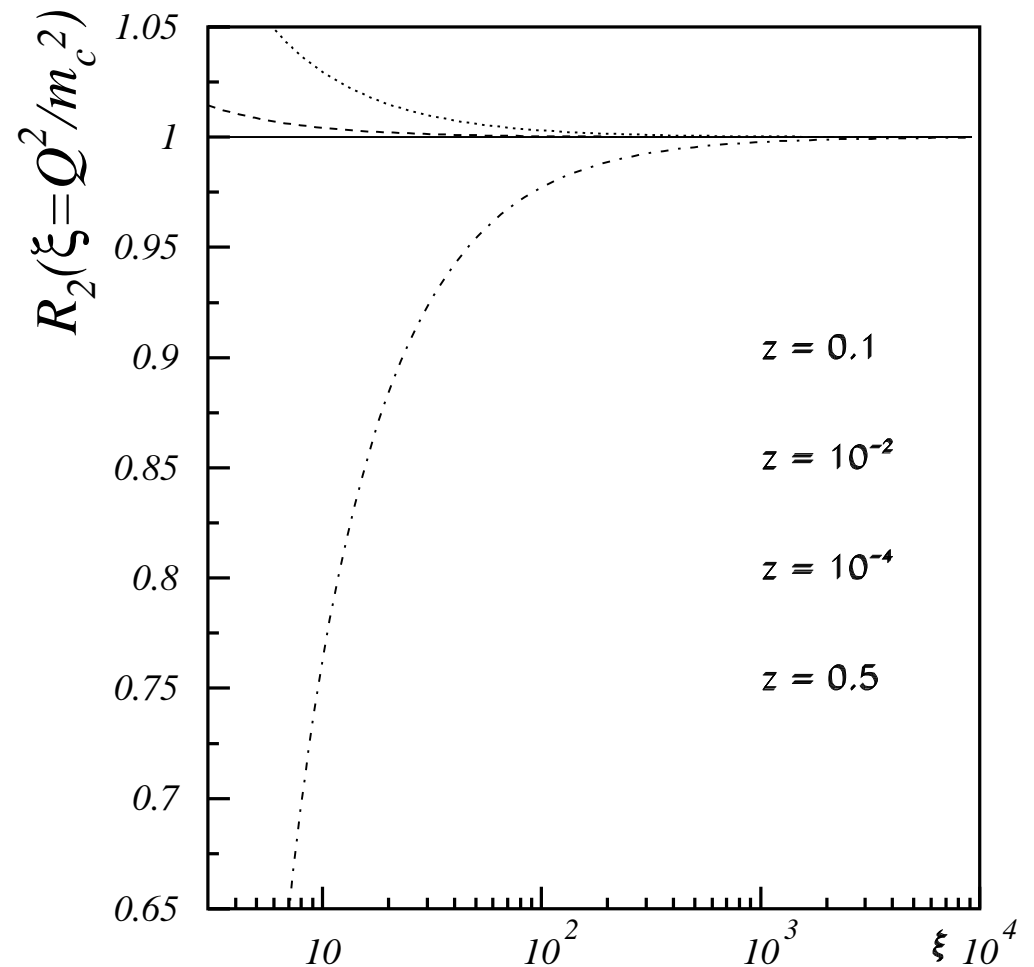
- **Heavy OMEs** occur as well as transition functions to define the **VFNS** starting from a **fixed flavor number scheme (FFNS)**.

[Buza, Matiounine, Smith, van Neerven, 1998; Chuvakin, Smith, van Neerven, 1998.]

- Comparison for **LO**:

$$R_2\left(\xi \equiv \frac{Q^2}{m^2}\right) \equiv \frac{H_{2,g}^{(1)}}{H_{2,g,(asym)}^{(1)}} .$$

- Comparison to exact order  $O(a_s^2)$  result: asymptotic formulae valid for  $Q^2 \gtrsim 20$   $(\text{GeV}/c)^2$  in case of  $F_2^{c\bar{c}}(x, Q^2)$  and  $Q^2 \gtrsim 800$   $(\text{GeV}/c)^2$  for  $F_L^{c\bar{c}}(x, Q^2)$
- **Drawbacks**:
  - **Power corrections**  $(m^2/Q^2)^k$  cannot be calculated using this method.
  - The case of **two different heavy quark masses** is still too complicated  $\implies$  **2 scale** problem to be treated semi-analytically.
  - Only inclusive quantities can be calculated  $\implies$  **structure functions**.



# Renormalization

$$\hat{A}_{ij} = \delta_{ij} + \sum_{k=0}^{\infty} \hat{a}_s^k \hat{A}_{ij}^{(k)}$$

- Mass renormalization (on-mass shell scheme)
- Charge renormalization

MOM scheme  $\rightarrow$   $\overline{\text{MS}}$  scheme ( $D = 4 + \varepsilon$ ):  $\rightarrow$  decoupling formalism  
 [Ovrut, Schnitzer, 1981; Bernreuther, Wetzel, 1982].

- Renormalization of ultraviolet singularities  
 $\implies$  are absorbed into  $Z$ -factors given in terms of anomalous dimensions  $\gamma_{ij}$ .
- Factorization of collinear singularities  
 $\implies$  are factored into  $\Gamma$ -factors  $\Gamma_{NS}$ ,  $\Gamma_{ij,S}$  and  $\Gamma_{qq,PS}$ .  
 For massless quarks it would hold:  $\Gamma = Z^{-1}$ .

Here:  $\Gamma$ -matrices apply to parts of the diagrams with massless lines only .

Generic formula for operator renormalization and mass factorization:

$$A_{ij} = Z_{il}^{-1} \hat{A}_{lm} \Gamma_{mj}^{-1}$$

$\implies$   $O(\varepsilon)$ -terms of the 2-loop OMEs are needed for renormalization at 3-loops.

## 4. $O(a_s^2)$ Contributions to $O(\varepsilon)$

Refs.: [I. Bierenbaum, J.B., S. Klein, Phys. Lett. B672 (2009) 401; B648 (2007) 195; Nucl. Phys. B780 (2007) 1; I. Bierenbaum, J.B., S. Klein, C. Schneider, Nucl. Phys. B803 (2008) 1 ]

- Use of **generalized hypergeometric functions** for general analytic results  
 $\implies$  allows **feasible computation** of **higher orders in  $\varepsilon$**  & **automated check** for fixed values of  $N$ .
- use of **Mellin-Barnes integrals** for numerical checks (**MB**, [Czakon, 2006.] )
- Summation of lots of **new infinite one-parameter sums** into **harmonic sums**. E.g.:

$$N \sum_{i,j=1}^{\infty} \frac{S_1(i)S_1(i+j+N)}{i(i+j)(j+N)} = 4S_{2,1,1} - 2S_{3,1} + S_1 \left( -3S_{2,1} + \frac{4S_3}{3} \right) - \frac{S_4}{2} - S_2^2 + S_1^2 S_2 + \frac{S_1^4}{6} + 6S_1 \zeta_3 + \zeta_2 (2S_1^2 + S_2) .$$

use of **integral techniques** and the **Mathematica package SIGMA** [Schneider, 2007.], [Bierenbaum, J.B., Klein, Schneider, 2007, 2008.]

- Partial checks for fixed values of  $N$  using **SUMMER**, [Vermaseren, 1999.]
- Computation within general  $R_\xi$  gauge.

We calculated all 2-loop  $O(\varepsilon)$ -terms in the unpolarized case  
and several 2-loop  $O(\varepsilon)$ -terms in the polarized case:

$$\bar{a}_{Qg}^{(2)}, \bar{a}_{Qq}^{(2),\mathbf{PS}}, \bar{a}_{gg,Q}^{(2)}, \bar{a}_{gq,Q}^{(2)}, \bar{a}_{qq,Q}^{(2),\mathbf{NS}}.$$

$$\Delta\bar{a}_{Qg}^{(2)}, \Delta\bar{a}_{Qq}^{(2),\mathbf{PS}}, \Delta\bar{a}_{qq,Q}^{(2),\mathbf{NS}}.$$

We verified all corresponding 2-loop  $O(\varepsilon^0)$ -results by van Neerven et. al.

- A remark on the appearing functions:

van Neerven et al. to  $O(1)$ : unpolarized: 48 basic functions; polarized: 24 basic functions.

$$O(1): \quad \{S_1, S_2, S_3, S_{-2}, S_{-3}\}, \quad S_{-2,1} \implies 2 \text{ basic objects.}$$

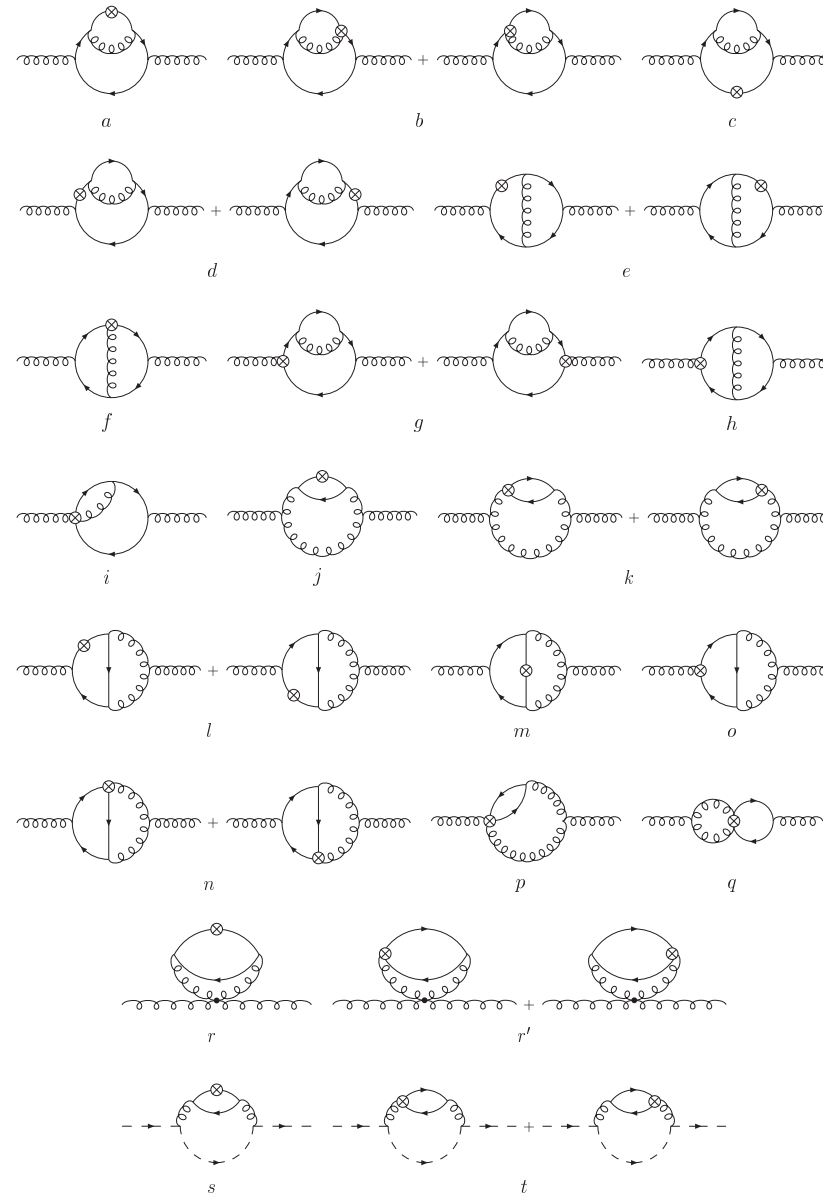
$$O(\varepsilon): \quad \{S_1, S_2, S_3, S_4, S_{-2}, S_{-3}, S_{-4}\}, \quad S_{2,1}, \quad S_{-2,1}, \quad S_{-3,1}, \quad S_{2,1,1}, \quad S_{-2,1,1}$$

$$\implies 6 \text{ basic objects}$$

- harmonic sums with index  $\{-1\}$  cancel (holds even for each diagram)  
J.B., 2004; J.B., Ravindran, 2005,2006; J.B., Klein, 2007; J.B., Moch in preparation;  
J.B. 0901.3106 [hep-ph]; 0901.0837 [math-ph]
- Expectation for 3-loops: weight 5 (6) harmonic sums

- Diagrams contain **two scales**: the mass  $m$  and the Mellin-parameter  $N$ .
- **2-point functions** with on-shell external momentum,  $p^2 = 0$ .  
 → reduce for  $N = 0$  to **massive tadpoles**.
- E.g. diagrams contributing to the **gluonic OME**

$$\hat{A}_{Qg}^{(2)} \implies$$



Example: Unpolarized case, Singlet,  $O(\varepsilon)$

$$\begin{aligned}
\bar{a}_{Qg}^{(2)} = & T_F C_F \left\{ \frac{2}{3} \frac{(N^2 + N + 2)(3N^2 + 3N + 2)}{N^2(N+1)^2(N+2)} \zeta_3 + \frac{P_1}{N^3(N+1)^3(N+2)} S_2 + \frac{N^4 - 5N^3 - 32N^2 - 18N - 4}{N^2(N+1)^2(N+2)} S_1^2 \right. \\
& + \frac{N^2 + N + 2}{N(N+1)(N+2)} \left( 16S_{2,1,1} - 8S_{3,1} - 8S_{2,1}S_1 + 3S_4 - \frac{4}{3}S_3S_1 - \frac{1}{2}S_2^2 - S_2S_1^2 - \frac{1}{6}S_1^4 + 2\zeta_2S_2 - 2\zeta_2S_1^2 - \frac{8}{3}\zeta_3S_1 \right) \\
& - 8 \frac{N^2 - 3N - 2}{N^2(N+1)(N+2)} S_{2,1} + \frac{2}{3} \frac{3N+2}{N^2(N+2)} S_1^3 + \frac{2}{3} \frac{3N^4 + 48N^3 + 43N^2 - 22N - 8}{N^2(N+1)^2(N+2)} S_3 + 2 \frac{3N+2}{N^2(N+2)} S_2S_1 + 4 \frac{S_1}{N^2} \zeta_2 \\
& + \left. \frac{N^5 + N^4 - 8N^3 - 5N^2 - 3N - 2}{N^3(N+1)^3} \zeta_2 - 2 \frac{2N^5 - 2N^4 - 11N^3 - 19N^2 - 44N - 12}{N^2(N+1)^3(N+2)} S_1 + \frac{P_2}{N^5(N+1)^5(N+2)} \right\} \\
& + T_F C_A \left\{ \frac{N^2 + N + 2}{N(N+1)(N+2)} \left( 16S_{-2,1,1} - 4S_{2,1,1} - 8S_{-3,1} - 8S_{-2,2} - 4S_{3,1} - \frac{2}{3}\beta''' + 9S_4 - 16S_{-2,1}S_1 \right. \right. \\
& + \frac{40}{3}S_1S_3 + 4\beta''S_1 - 8\beta'S_2 + \frac{1}{2}S_2^2 - 8\beta'S_1^2 + 5S_1^2S_2 + \frac{1}{6}S_1^4 - \frac{10}{3}S_1\zeta_3 - 2S_2\zeta_2 - 2S_1^2\zeta_2 - 4\beta'\zeta_2 - \frac{17}{5}\zeta_2^2 \left. \right) \\
& + \frac{4(N^2 - N - 4)}{(N+1)^2(N+2)^2} \left( -4S_{-2,1} + \beta'' - 4\beta'S_1 \right) - \frac{2}{3} \frac{N^3 + 8N^2 + 11N + 2}{N(N+1)^2(N+2)^2} S_1^3 + 8 \frac{N^4 + 2N^3 + 7N^2 + 22N + 20}{(N+1)^3(N+2)^3} \beta' \\
& + 2 \frac{3N^3 - 12N^2 - 27N - 2}{N(N+1)^2(N+2)^2} S_2S_1 - \frac{16}{3} \frac{N^5 + 10N^4 + 9N^3 + 3N^2 + 7N + 6}{(N-1)N^2(N+1)^2(N+2)^2} S_3 - 8 \frac{N^2 + N - 1}{(N+1)^2(N+2)^2} \zeta_2S_1 \\
& - \frac{2}{3} \frac{9N^5 - 10N^4 - 11N^3 + 68N^2 + 24N + 16}{(N-1)N^2(N+1)^2(N+2)^2} \zeta_3 - \frac{P_3}{(N-1)N^3(N+1)^3(N+2)^3} S_2 - \frac{2P_4}{(N-1)N^3(N+1)^3(N+2)^2} \zeta_2 \\
& - \left. \frac{P_5}{N(N+1)^3(N+2)^3} S_1^2 + \frac{2P_6}{N(N+1)^4(N+2)^4} S_1 - \frac{2P_7}{(N-1)N^5(N+1)^5(N+2)^5} \right\}.
\end{aligned}$$

## 5. Fixed moments at 3–Loop: $F_2^{Q\bar{Q}}$

$F_L^{Q\bar{Q}}(x, Q^2)$  @ 3-loop: [J.B., De Freitas, Klein, van Neerven, Nucl. Phys. B755 (2006) 272.]

Refs.: [I. Bierenbaum, J.B., S. Klein, Nucl.Phys.Proc.Suppl. 183 (2008) 162; arXiv:0812.2427; PoS]

Contributing OMEs:

$$\begin{array}{lcl}
 \text{Singlet} & A_{Qg} & A_{qg,Q} & A_{gg,Q} & A_{gq,Q} & \left. \vphantom{\begin{array}{l} \text{Singlet} \\ \text{Pure-Singlet} \\ \text{Non-Singlet} \end{array}} \right\} \text{ mixing} \\
 \text{Pure-Singlet} & & A_{Qq}^{\text{PS}} & A_{qq,Q}^{\text{PS}} & & \\
 \text{Non-Singlet} & & A_{qq,Q}^{\text{NS,+}} & & & 
 \end{array}$$

- All 2-loop  $O(\varepsilon)$ -terms in the **unpolarized** case are known.
- **Unpolarized anomalous dimensions** are known up to  $O(a_s^3)$  [Moch, Vermaseren, Vogt, 2004.]  
 $\implies$  All terms needed for the renormalization of **unpolarized 3-Loop heavy OMEs** are present.  
 $\implies$  The calculation will provide first independent checks on  $\gamma_{qg}^{(2)}$ ,  $\gamma_{qq}^{(2),\text{PS}}$  and on respective color projections of  $\gamma_{qq}^{(2),\text{NS+}}$ ,  $\gamma_{gg}^{(2)}$  and  $\gamma_{gq}^{(2)}$ .
- The calculation proceeds in the same way in the **polarized** case.
- Independent checks provided by pole terms (**anomalous dimensions, lower order OMEs**) and **sum rules** for  $N = 2$ .



## Fixed moments using MATAD

- three-loop “self-energy” type diagrams with an operator insertion
- **Extension:** additional scale compared to massive propagators: Mellin variable  $N$
- Genuine tensor integrals due to

$$\Delta^{\mu_1} \dots \Delta^{\mu_n} \langle p | O_{\mu_1 \dots \mu_n} | p \rangle = \Delta^{\mu_1} \dots \Delta^{\mu_n} \langle p | S \bar{\Psi} \gamma_{\mu_1} D_{\mu_2} \dots D_{\mu_n} \Psi | p \rangle = A(N) \cdot (\Delta p)^N$$

$$D_\mu = \partial_\mu - i g t_a A_\mu^a, \quad \Delta^2 = 0.$$

- Construction of a projector to obtain the desired moment in  $N$  [undo  $\Delta$ -contraction]
- 3-loop OMEs are generated with QGRAF [Nogueira, 1993.]
- Color factors are calculated with [van Ritbergen, Schellekens, Vermaseren, 1998.]
- Translation to suitable input for MATAD [Steinhauser, 2001.]

### Tests performed:

- Various 2-loop calculations for  $N = 2, 4, 6, \dots$  were repeated  
→ agreement with our previous calculation.
- Several non-trivial scalar 3-loop diagrams were calculated using Feynman-parameters for all  $N$   
→ agreement with MATAD.

## General structure of the result: the PS –case

$$\begin{aligned}
A_{Qq}^{(3),\text{PS}} + A_{qq,Q}^{(3),\text{PS}} &= \frac{\hat{\gamma}_{qg}^{(0)} \gamma_{gq}^{(0)}}{48} \left\{ \gamma_{gg}^{(0)} - \gamma_{qq}^{(0)} + 4(n_f + 1)\beta_{0,Q} + 6\beta_0 \right\} \ln^3 \left( \frac{m^2}{\mu^2} \right) \\
&+ \left\{ \frac{\hat{\gamma}_{qq}^{(1),\text{PS}}}{2} \left( (n_f + 1)\beta_{0,Q} - \beta_0 \right) + \frac{\hat{\gamma}_{qg}^{(0)}}{8} \left( (n_f + 1)\hat{\gamma}_{gg}^{(1)} - \gamma_{gg}^{(1)} \right) - \frac{\gamma_{gq}^{(0)} \hat{\gamma}_{qg}^{(1)}}{8} \right\} \ln^2 \left( \frac{m^2}{\mu^2} \right) \\
&+ \left\{ \frac{\hat{\gamma}_{qq}^{(2),\text{PS}}}{2} - \zeta_2 \frac{\hat{\gamma}_{qg}^{(0)} \gamma_{gq}^{(0)}}{16} \left( \gamma_{gg}^{(0)} - \gamma_{qq}^{(0)} + 4(n_f + 1)\beta_{0,Q} + 6\beta_0 \right) - 2a_{Qq}^{(2),\text{PS}} \beta_0 \right. \\
&+ \left. \frac{n_f + 1}{2} \hat{\gamma}_{qg}^{(0)} a_{gq,Q}^{(2)} - \frac{\gamma_{gq}^{(0)}}{2} a_{Qg}^{(2)} \right\} \ln \left( \frac{m^2}{\mu^2} \right) + \zeta_3 \frac{\gamma_{gq}^{(0)} \hat{\gamma}_{qg}^{(0)}}{48} \left( \gamma_{gg}^{(0)} - \gamma_{qq}^{(0)} + 4n_f \beta_{0,Q} + 6\beta_0 \right) \\
&+ \frac{\zeta_2}{16} \left( -4n_f \beta_{0,Q} \hat{\gamma}_{qq}^{(1),\text{PS}} + \hat{\gamma}_{qg}^{(0)} \gamma_{gq}^{(1)} \right) + 4(\beta_0 + \beta_{0,Q}) \bar{a}_{Qq}^{(2),\text{PS}} + \gamma_{gq}^{(0)} \bar{a}_{Qg}^{(2)} - (n_f + 1) \hat{\gamma}_{qg}^{(0)} \bar{a}_{gq,Q}^{(2)} \\
&+ C_F \left( -\left(4 + \frac{3}{4}\zeta_2\right) \hat{\gamma}_{qg}^{(0)} \gamma_{gq}^{(0)} - 4\hat{\gamma}_{qq}^{(1),\text{PS}} + 12a_{Qq}^{(2),\text{PS}} \right) + a_{Qq}^{(3),\text{PS}} + a_{qq,Q}^{(3),\text{PS}} .
\end{aligned}$$

- $n_f$ –dependence non–trivial. Take all quantities at  $n_f$  flavors and adopt notation

$$\hat{\gamma}_{ij} \equiv \gamma_{ij}(n_f + 1) - \gamma_{ij}(n_f) , \quad \beta_{0,Q} \equiv \beta_0(n_f + 1) - \beta_0(n_f) .$$

- There are similar formulas for the remaining **OMEs**.

- Number of Diagrams to be calculated:

$$A_{Q(q)q}^{(3),PS} : 132, \quad A_{qq}^{(3),NS} : 128, \quad A_{gq}^{(3)} : 89, \quad A_{Qg}^{(3)} : 1498, \quad A_{gg,Q}^{(3)} : 865. \quad \sim 250 \text{ days CT}$$

- We calculated the terms

$$A_{Qq}^{(3),PS} + A_{qq,Q}^{(3),PS}, \quad A_{qq,Q}^{(3),NS}, \quad A_{gq,Q}^{(3)} : (2-12); \quad A_{Qg}^{(3)}, \quad A_{gg,Q}^{(3)}, \quad A_{gg,Q}^{(3)} : (2-10);$$

for  $N = 2, 4, 6, 8, 10, (12)$  using **MATAD** and find **agreement** of the pole terms with the prediction obtained from renormalization.

- An additional check is provided by the sum rule

$$A_{qq,Q}^{(3),NS} \Big|_{N=2} + A_{qq,Q}^{(3),PS} \Big|_{N=2} + A_{Qq}^{(3),PS} \Big|_{N=2} + A_{gq,Q}^{(3)} \Big|_{N=2} = 0,$$

which is fulfilled by our result.

- All terms proportional to  $\zeta_2$  cancel in the renormalized result for  $A_{Q(q)q}^{(3),NS,PS}, A_{gq,Q}^{(3)}$
- We observe the number

$$\mathbf{B4} = -4\zeta_2 \ln^2 2 + \frac{2}{3} \ln^4 2 - \frac{13}{2} \zeta_4 + 16 \text{Li}_4\left(\frac{1}{2}\right) = -8\sigma_{-3,-1} + \frac{11}{2} \zeta_4.$$

- The term **B4** appears as

$$\left(C_F - \frac{C_A}{2}\right) \mathbf{B4}.$$

Result for the renormalized **PS**-term for  $N = 4$ .

$$\begin{aligned}
A_{Qq}^{(3),\text{PS}} + A_{qq,Q}^{(3),\text{PS}} \Big|_{N=4} &= \left\{ -\frac{484}{2025} C_F T_F^2 (2n_f + 1) + \frac{4598}{3375} C_F C_A T_F - \frac{18997}{40500} C_F^2 T_F \right\} \ln^3 \left( \frac{m^2}{\mu^2} \right) \\
&+ \left\{ -\frac{16}{125} C_F T_F^2 + \frac{36751}{202500} C_F C_A T_F - \frac{697631}{405000} C_F^2 T_F \right\} \ln^2 \left( \frac{m^2}{\mu^2} \right) + \left\{ -\frac{2131169}{303750} C_F T_F^2 n_f \right. \\
&- \frac{427141}{121500} C_F T_F^2 + \left( -\frac{484}{75} \zeta_3 + \frac{24888821}{2700000} \right) C_F C_A T_F + \left( \frac{484}{75} \zeta_3 + \frac{63582197}{16200000} \right) C_F^2 T_F \left. \right\} \ln \left( \frac{m^2}{\mu^2} \right) \\
&+ \left( \frac{7744}{2025} \zeta_3 - \frac{143929913}{27337500} \right) C_F T_F^2 n_f + \left( -\frac{13552}{2025} \zeta_3 + \frac{218235943}{54675000} \right) C_F T_F^2 + \left( \frac{242}{225} \mathbf{B4} - \frac{242}{25} \zeta_4 \right. \\
&+ \left. \frac{86833}{13500} \zeta_3 + \frac{4628174}{1265625} \right) C_F C_A T_F + \left( -\frac{484}{225} \mathbf{B4} + \frac{242}{25} \zeta_4 + \frac{298363}{20250} \zeta_3 - \frac{57518389433}{2187000000} \right) C_F^2 T_F .
\end{aligned}$$

The constant terms:  $N = 12$   $\hat{a}_{gq,Q}^{(3)}$ :

$$\begin{aligned}
\hat{a}_{gq,Q}^{(3)} \Big|_{N=12} = & T_F \left( \frac{m^2}{\mu^2} \right)^{3\varepsilon/2} \left\{ C_F T_F n_f \left( -\frac{8848}{3861} \zeta_3 - \frac{14257247}{57972915} \zeta_2 + \frac{9228836319135394697}{2258497163057335200} \right) \right. \\
& + C_F C_A \left[ -\frac{1264}{1287} B_4 + \frac{2528}{715} \zeta_2^2 - \frac{999900989}{173918745} \zeta_3 - \frac{693594486209}{3798385390800} \zeta_2 \right. \\
& \left. - \frac{1515875996003174876943331}{147976734123516602304000} \right] + C_F^2 \left[ \frac{2528}{1287} B_4 - \frac{2528}{715} \zeta_2^2 + \frac{43693776149}{2260943685} \zeta_3 \right. \\
& \left. - \frac{2486481253717}{1671289571952} \zeta_2 - \frac{48679935129017185612582919}{4069360188396706563360000} \right] + C_F T_F \left[ \frac{20224}{3861} \zeta_3 \right. \\
& \left. - \frac{28514494}{57972915} \zeta_2 - \frac{2105210836073143063}{1129248581528667600} \right] \left. \right\}.
\end{aligned}$$

The constant terms:  $N = 12$   $\hat{a}_{qq,Q}^{(3),NS}$ :

$$\begin{aligned}
\hat{a}_{qq,Q}^{(3),NS} \Big|_{N=12} &= T_F \left( \frac{m^2}{\mu^2} \right)^{3\varepsilon/2} \left\{ C_F T_F n_f \left( \frac{6774784}{173745} \zeta_3 - \frac{90143221429}{7304587290} \zeta_2 - \frac{1201733391177720469772303}{7114266063630605880000} \right) \right. \\
&+ C_F C_A \left[ \frac{3387392}{135135} B_4 - \frac{6774784}{75075} \zeta_2^2 + \frac{51577729507}{158107950} \zeta_3 + \frac{2401246832561}{243486243000} \zeta_2 \right. \\
&- \left. \frac{126207343604156227942043}{2463815086971638400000} \right] + C_F^2 \left[ -\frac{6774784}{135135} B_4 + \frac{6774784}{75075} \zeta_2^2 \right. \\
&- \frac{79117185295}{243486243} \zeta_3 + \frac{108605787257580461}{1096783781593500} \zeta_2 + \left. \frac{68296027149155250557867961293}{122080805651901196900800000} \right] \\
&\left. + C_F T_F \left[ -\frac{108396544}{1216215} \zeta_3 - \frac{90143221429}{3652293645} \zeta_2 - \frac{189306988923316881320303}{3557133031815302940000} \right] \right\}.
\end{aligned}$$

The constant terms:  $N = 12$   $\hat{a}_{Qq}^{(3),PS} + \hat{a}_{qq,Q}^{(3),PS}$  :

$$\begin{aligned}
\hat{a}_{Qq}^{(3),PS} \Big|_{N=12} &= T_F \left( \frac{m^2}{\mu^2} \right)^{3\varepsilon/2} \left\{ C_F T_F n_f \left( \frac{24964}{150579} \zeta_3 - \frac{1291174013}{63306423180} \zeta_2 - \frac{6621557709293056160177}{12331394510293050192000} \right) \right. \\
&+ C_F C_A \left[ \frac{12482}{117117} B_4 - \frac{24964}{65065} \zeta_2^2 - \frac{64839185833913}{16206444334080} \zeta_3 + \frac{489403711559293}{1382612282251200} \zeta_2 \right. \\
&+ \left. \frac{968307050156826905398206547}{107727062441920086477312000} \right] + C_F^2 \left[ -\frac{24964}{117117} B_4 + \frac{24964}{65065} \zeta_2^2 \right. \\
&+ \frac{418408135384633}{8103222167040} \zeta_3 - \frac{72904483229177}{15208735104763200} \zeta_2 \\
&- \left. \frac{190211298439834685159055148289}{2962494217152802378126080000} \right] + C_F T_F \left[ -\frac{798848}{1054053} \zeta_3 \right. \\
&+ \left. \frac{11471393}{347837490} \zeta_2 + \frac{1727596215111011341}{13550982978344011200} \right] \left. \right\} . \\
\hat{a}_{qq,Q}^{(3),PS} \Big|_{N=12} &= \left( \frac{m^2}{\mu^2} \right)^{3\varepsilon/2} C_F T_F^2 n_f \left\{ \frac{24964}{150579} \zeta_3 + \frac{583767694}{15826605795} \zeta_2 - \frac{6724380801633998071}{38535607844665781850} \right\} .
\end{aligned}$$

The constant terms:  $N = 10$   $\hat{a}_{gg,Q}^{(3)}$ :

$$\begin{aligned}
\hat{a}_{gg,Q}^{(3)} \Big|_{N=10} = & T_F \left( \frac{m^2}{\mu^2} \right)^{3\varepsilon/2} \left\{ n_f T_F \left( C_A \left[ \frac{215128}{4455} \zeta_3 - \frac{81362132}{5145525} \zeta_2 - \frac{6069333056458984}{26476427525625} \right] \right. \right. \\
& + C_F \left[ -\frac{351232}{735075} \zeta_3 - \frac{799867252}{121287375} \zeta_2 - \frac{100698363899844296}{4368610541728125} \right] \Big) + C_A^2 \left[ \frac{17788828}{571725} B_4 \right. \\
& - \frac{17746492}{63525} \zeta_4 + \frac{269094476549521109}{519314374656000} \zeta_3 + \frac{1444408720649}{55468759500} \zeta_2 \\
& \left. - \frac{15434483462331661005275759}{327337774462347264000000} \right] + C_A C_F \left[ -\frac{35662328}{571725} B_4 + \frac{17831164}{63525} \zeta_4 \right. \\
& \left. - \frac{3288460968359099}{37093883904000} \zeta_3 + \frac{6078270984602}{46695639375} \zeta_2 + \frac{207095356146239371087405921}{771581896946961408000000} \right] \\
& + C_F^2 \left[ \frac{896}{3025} B_4 - \frac{4032}{3025} \zeta_4 + \frac{7140954579599}{198717235200} \zeta_3 - \frac{282148432}{4002483375} \zeta_2 \right. \\
& \left. + \frac{553777925867720521493231}{20667372239650752000000} \right] + T_F C_A \left[ -\frac{85188238297}{729907200} \zeta_3 - \frac{33330316}{735075} \zeta_2 \right. \\
& \left. - \frac{63059843481895502807}{433789788579840000} \right] + T_F C_F \left[ -\frac{71350574183}{12043468800} \zeta_3 - \frac{3517889264}{121287375} \zeta_2 \right. \\
& \left. - \frac{655690580559958774157}{35787657557836800000} \right] + \frac{64}{27} \zeta_3 T_F^2 \Big\} .
\end{aligned}$$



The constant terms:  $N = 10$   $\hat{a}_{Qg}^{(3)}$  +  $\hat{a}_{qg,Q}^{(3)}$ :

$$\begin{aligned}
\hat{a}_{Qg}^{(3)} \Big|_{N=10} &= T_F \left( \frac{m^2}{\mu^2} \right)^{3\varepsilon/2} \left\{ n_f T_F \left( C_A \left[ -\frac{1505896}{245025} \zeta_3 + \frac{189965849}{188669250} \zeta_2 + \frac{297277185134077151}{15532837481700000} \right] \right. \right. \\
&+ C_F \left[ \frac{62292104}{13476375} \zeta_3 - \frac{49652772817}{93391278750} \zeta_2 - \frac{1178560772273339822317}{107642563748181000000} \right] \left. \right) + C_A^2 \left[ -\frac{563692}{81675} B_4 \right. \\
&+ \frac{483988}{9075} \zeta_4 - \frac{103652031822049723}{415451499724800} \zeta_3 - \frac{20114890664357}{581101290000} \zeta_2 \\
&+ \left. \frac{6830363463566924692253659}{685850575063965696000000} \right] + C_A C_F \left[ \frac{1286792}{81675} B_4 - \frac{643396}{9075} \zeta_4 \right. \\
&- \frac{761897167477437907}{33236119977984000} \zeta_3 + \frac{15455008277}{660342375} \zeta_2 + \left. \frac{872201479486471797889957487}{2992802509370032128000000} \right] \\
&+ C_F^2 \left[ -\frac{11808}{3025} B_4 + \frac{53136}{3025} \zeta_4 + \frac{9636017147214304991}{7122025709568000} \zeta_3 + \frac{14699237127551}{15689734830000} \zeta_2 \right. \\
&- \left. \frac{247930147349635960148869654541}{148143724213816590336000000} \right] + T_F C_A \left[ \frac{4206955789}{377338500} \zeta_2 + \frac{123553074914173}{5755172290560} \zeta_3 \right. \\
&+ \left. \frac{23231189758106199645229}{633397356480430080000} \right] + T_F C_F \left[ -\frac{502987059528463}{113048027136000} \zeta_3 + \frac{24683221051}{46695639375} \zeta_2 \right. \\
&- \left. \frac{18319931182630444611912149}{1410892611560158003200000} \right] - \frac{896}{1485} T_F^2 \zeta_3 \left. \right\} . \\
\hat{a}_{qg,Q}^{(3)} \Big|_{N=10} &= n_f T_F^2 \left( \frac{m^2}{\mu^2} \right)^{3\varepsilon/2} \left\{ C_A \left[ -\frac{1505896}{245025} \zeta_3 + \frac{1109186999}{377338500} \zeta_2 + \frac{6542127929072987}{191763425700000} \right] \right. \\
&+ \left. C_F \left[ \frac{62292104}{13476375} \zeta_3 - \frac{83961181063}{93391278750} \zeta_2 - \frac{353813854966442889041}{21528512749636200000} \right] \right\}
\end{aligned}$$

We obtain for the **moments** of the **PS**, and  $\hat{\gamma}_{gq}^{(2)}$  **anomalous dimensions**

N	$\hat{\gamma}_{qq}^{(2),PS}/T_F/C_F$
2	$-\frac{5024}{243}T_F(1+2n_f) + \frac{256}{3}(C_F - C_A)\zeta_3 + \frac{10136}{243}C_A - \frac{14728}{243}C_F$
4	$-\frac{618673}{151875}T_F(1+2n_f) + \frac{968}{75}(C_F - C_A)\zeta_3 + \frac{2485097}{506250}C_A - \frac{2217031}{675000}C_F$
6	$-\frac{126223052}{72930375}T_F(1+2n_f) + \frac{3872}{735}(C_F - C_A)\zeta_3 + \frac{1988624681}{4084101000}C_A + \frac{11602048711}{10210252500}C_F$
8	$-\frac{13131081443}{13502538000}T_F(1+2n_f) + \frac{2738}{945}(C_F - C_A)\zeta_3 - \frac{343248329803}{648121824000}C_A + \frac{39929737384469}{22684263840000}C_F$
10	$-\frac{265847305072}{420260754375}T_F(1+2n_f) + \frac{50176}{27225}(C_F - C_A)\zeta_3 - \frac{1028766412107043}{1294403123475000}C_A + \frac{839864254987192}{485401171303125}C_F$
12	$-\frac{2566080055386457}{5703275664286200}T_F(1+2n_f) + \frac{49928}{39039}(C_F - C_A)\zeta_3 - \frac{69697489543846494691}{83039693672007072000}C_A$ $+ \frac{86033255402443256197}{54806197823524667520}C_F$
N	$\hat{\gamma}_{gq}^{(2)}/T_F/C_F$
2	$\frac{2272}{81}T_F(1+2n_f) + \frac{512}{3}(C_A - C_F)\zeta_3 + \frac{88}{9}C_A + \frac{28376}{243}C_F$
4	$\frac{109462}{10125}T_F(1+2n_f) + \frac{704}{15}(C_A - C_F)\zeta_3 - \frac{799}{12150}C_A + \frac{14606684}{759375}C_F$
6	$\frac{22667672}{3472875}T_F(1+2n_f) + \frac{2816}{105}(C_A - C_F)\zeta_3 - \frac{253841107}{145860750}C_A + \frac{20157323311}{2552563125}C_F$
8	$\frac{339184373}{75014100}T_F(1+2n_f) + \frac{1184}{63}(C_A - C_F)\zeta_3 - \frac{3105820553}{1687817250}C_A + \frac{8498139408671}{2268426384000}C_F$
10	$\frac{1218139408}{363862125}T_F(1+2n_f) + \frac{7168}{495}(C_A - C_F)\zeta_3 - \frac{18846629176433}{11767301122500}C_A + \frac{529979902254031}{323600780868750}C_F$
12	$\frac{13454024393417}{5222779912350}T_F(1+2n_f) + \frac{5056}{429}(C_A - C_F)\zeta_3 - \frac{64190493078139789}{48885219979596000}C_A + \frac{1401404001326440151}{3495293228541114000}C_F$

## The moments of the $\hat{\gamma}_{gg}^{(2)}$ and $\hat{\gamma}_{qg}^{(2)}$ anomalous dimensions

N	$\hat{\gamma}_{qg}^{(2)}/T_F$
2	$(1 + 2n_f)T_F \left( \frac{8464}{243} C_A - \frac{1384}{243} C_F \right) + \frac{\zeta_3}{3} (-416 C_A C_F + 288 C_A^2 + 128 C_F^2) - \frac{7178}{81} C_A^2 + \frac{556}{9} C_A C_F - \frac{8620}{243} C_F^2$
4	$(1 + 2n_f)T_F \left( \frac{4481539}{303750} C_A + \frac{9613841}{3037500} C_F \right) + \frac{\zeta_3}{25} (2832 C_A^2 - 3876 C_A C_F + 1044 C_F^2)$ $- \frac{295110931}{3037500} C_A^2 + \frac{278546497}{2025000} C_A C_F - \frac{757117001}{12150000} C_F^2$
6	$(1 + 2n_f)T_F \left( \frac{86617163}{11668860} C_A + \frac{1539874183}{340341750} C_F \right) + \frac{\zeta_3}{735} (69864 C_A^2 - 94664 C_A C_F + 24800 C_F^2)$ $- \frac{58595443051}{653456160} C_A^2 + \frac{1199181909343}{8168202000} C_A C_F - \frac{2933980223981}{40841010000} C_F^2$
8	$(1 + 2n_f)T_F \left( \frac{10379424541}{2755620000} C_A + \frac{7903297846481}{1620304560000} C_F \right) + \zeta_3 \left( \frac{128042}{1575} C_A^2 - \frac{515201}{4725} C_A C_F + \frac{749}{27} C_F^2 \right)$ $- \frac{24648658224523}{289340100000} C_A^2 + \frac{4896295442015177}{32406091200000} C_A C_F - \frac{4374484944665803}{56710659600000} C_F^2$
10	$(1 + 2n_f)T_F \left( \frac{1669885489}{988267500} C_A + \frac{1584713325754369}{323600780868750} C_F \right) + \zeta_3 \left( \frac{1935952}{27225} C_A^2 - \frac{2573584}{27225} C_A C_F + \frac{70848}{3025} C_F^2 \right)$ $- \frac{21025430857658971}{255684567600000} C_A^2 + \frac{926990216580622991}{6040547909550000} C_A C_F - \frac{1091980048536213833}{13591232796487500} C_F^2$
N	$\hat{\gamma}_{9g}^{(2)}/T_F$
2	$(1 + 2n_f)T_F \left( -\frac{8464}{243} C_A + \frac{1384}{243} C_F \right) + \frac{\zeta_3}{3} (-288 C_A^2 + 416 C_A C_F - 128 C_F^2) + \frac{7178}{81} C_A^2 - \frac{556}{9} C_A C_F + \frac{8620}{243} C_F^2$
4	$(1 + 2n_f)T_F \left( -\frac{757861}{30375} C_A - \frac{979774}{151875} C_F \right) + \frac{\zeta_3}{25} (-6264 C_A^2 + 6528 C_A C_F - 264 C_F^2)$ $+ \frac{53797499}{607500} C_A^2 - \frac{235535117}{1012500} C_A C_F + \frac{2557151}{759375} C_F^2$
6	$(1 + 2n_f)T_F \left( -\frac{52781896}{2083725} C_A - \frac{560828662}{72930375} C_F \right) + \zeta_3 \left( -\frac{75168}{245} C_A^2 + \frac{229024}{735} C_A C_F - \frac{704}{147} C_F^2 \right)$ $+ \frac{9763460989}{116688600} C_A^2 - \frac{9691228129}{32672808} C_A C_F - \frac{11024749151}{10210252500} C_F^2$
8	$(1 + 2n_f)T_F \left( -\frac{420970849}{16074450} C_A - \frac{6990254812}{843908625} C_F \right) + \zeta_3 \left( -\frac{325174}{945} C_A^2 + \frac{327764}{945} C_A C_F - \frac{74}{27} C_F^2 \right)$ $+ \frac{2080130771161}{25719120000} C_A^2 - \frac{220111823810087}{648121824000} C_A C_F - \frac{14058417959723}{5671065960000} C_F^2$
10	$(1 + 2n_f)T_F \left( -\frac{2752314359}{101881395} C_A - \frac{3631303571944}{420260754375} C_F \right) + \zeta_3 \left( -\frac{70985968}{190575} C_A^2 + \frac{71324656}{190575} C_A C_F - \frac{5376}{3025} C_F^2 \right)$ $+ \frac{43228502203851731}{549140719050000} C_A^2 - \frac{3374081335517123191}{9060821864325000} C_F C_A - \frac{3009386129483453}{970802342606250} C_F^2$

The moments of the NS<sup>+</sup>  $\hat{\gamma}_{qq}^{(2)}$  anomalous dimensions

N	$\hat{\gamma}_{qq}^{(2),NS,+}/T_F/C_F$
2	$-\frac{1792}{243}T_F(1+2n_f) + \frac{256}{3}(C_F - C_A)\zeta_3 - \frac{12512}{243}C_A - \frac{13648}{243}C_F$
4	$-\frac{384277}{30375}T_F(1+2n_f) + \frac{2512}{15}(C_F - C_A)\zeta_3 - \frac{8802581}{121500}C_A - \frac{165237563}{1215000}C_F$
6	$-\frac{160695142}{10418625}T_F(1+2n_f) + \frac{22688}{105}(C_F - C_A)\zeta_3 - \frac{13978373}{171500}C_A - \frac{44644018231}{243101250}C_F$
8	$-\frac{38920977797}{2250423000}T_F(1+2n_f) + \frac{79064}{315}(C_F - C_A)\zeta_3 - \frac{1578915745223}{18003384000}C_A - \frac{91675209372043}{420078960000}C_F$
10	$-\frac{27995901056887}{1497656506500}T_F(1+2n_f) + \frac{192880}{693}(C_F - C_A)\zeta_3 - \frac{9007773127403}{97250422500}C_A - \frac{75522073210471127}{307518802668000}C_F$
12	$-\frac{65155853387858071}{3290351344780500}T_F(1+2n_f) + \frac{13549568}{45045}(C_F - C_A)\zeta_3 - \frac{25478252190337435009}{263228107582440000}C_A$ $-\frac{35346062280941906036867}{131745667845011220000}C_F$

$\implies$  Agreement for the terms  $\propto T_F$  with

[Larin, Nogueira, Ritbergen, Vermaseren, 1997; Moch, Vermaseren, Vogt, 2004.]

• How far can we go ?

$N = 14$  in some cases; generally:  $N = 10$  (achieved).  $\implies$  Phenomenology

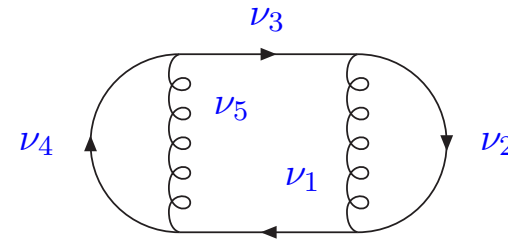
CPU time: various months; up to 64 Gb processors needed.

Unfortunately not enough to perform the automatic

fixed moments  $\rightarrow$  all moments turn. [J.B., Kauers, Klein, Schneider, 0902.4091 [hep-ph]].

## Fixed moments using Feynman–parameters

Consider e.g the **3–loop tadpole** diagram



Using Feynman–parameters, one obtains a representation in terms of a double sum

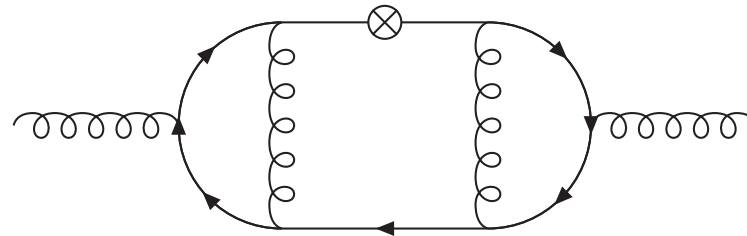
$$I = C\Gamma \left[ \begin{array}{c} 2 + \varepsilon/2 - \nu_1, 2 + \varepsilon/2 - \nu_5, \nu_{12} - 2 - \varepsilon/2, \nu_{45} - 2 - \varepsilon/2, \nu_{1345} - 4 - \varepsilon, \nu_{12345} - 6 - 3/2\varepsilon \\ \nu_1, \nu_2, \nu_4, 2 + \varepsilon/2, \nu_{345} - 2 - \varepsilon/2, \nu_{12345} - 4 - \varepsilon \end{array} \right]$$

$$\sum_{m,n=0}^{\infty} \frac{(\nu_{345} - 2 - \varepsilon/2)_{n+m} (\nu_{12345} - 6 - 3/2\varepsilon)_m (2 + \varepsilon/2 - \nu_1)_m (2 + \varepsilon/2 - \nu_5)_n (\nu_{45} - 2 - \varepsilon/2)_n}{m!n! (\nu_{12345} - 4 - \varepsilon)_{n+m} (\nu_{345} - 2 - \varepsilon/2)_m (\nu_{345} - 2 - \varepsilon/2)_n},$$

which derives from a **Appell–function of the first kind,  $F_1$**  .

## First all-N results

As in the 2-loop case, for any diagram deriving from the tadpole-ladder topology, one obtains for **fixed values of  $N$**  a finite sum over double sums of the same type. Consider e.g. the scalar diagram



For the above diagram, we obtained a result for arbitrary  $N$  using similar techniques as in the 2-loop case and the package **SIGMA**.

$$\begin{aligned}
 I_1 = & -\frac{4(N+1)S_1 + 4}{(N+1)^2(N+2)} \zeta_3 + \frac{2S_{2,1,1}}{(N+2)(N+3)} + \frac{1}{(N+1)(N+2)(N+3)} \left\{ -2(3N+5)S_{3,1} - \frac{S_1^4}{4} \right. \\
 & + \frac{4(N+1)S_1 - 4N}{N+1} S_{2,1} + 2 \left( (2N+3)S_1 + \frac{5N+6}{N+1} \right) S_3 + \frac{9+4N}{4} S_2^2 + \left( \frac{5N}{N+1} S_1 + 2 \frac{7N+11}{(N+1)(N+2)} \right. \\
 & \left. \left. - \frac{5}{2} S_1^2 \right) S_2 + \frac{N}{N+1} S_1^3 + \frac{2(3N+5)S_1^2}{(N+1)(N+2)} + \frac{4(2N+3)S_1}{(N+1)^2(N+2)} - \frac{(2N+3)S_4}{2} + 8 \frac{2N+3}{(N+1)^3(N+2)} \right\} + O(\varepsilon) .
 \end{aligned}$$

For fixed  $N$ , this formula agrees with the result we obtain using **MATAD**.

# 6. Conclusions

- The heavy flavor contributions to the structure function  $F_2$  are rather large in the region of lower values of  $x$ .
- Competitive QCD precision analyses therefore require the description of the heavy quark contributions to 3-loop order.
- The inclusive heavy flavor contributions to  $F_{2,L}(x, Q^2)$  are known to NLO (in semi-analytic form with fast numeric implementations.)
- Complete analytic results are known in the region  $Q^2 \gg m^2$  at NLO for  $F_{2,L}(x, Q^2), g_{1,2}(x, Q^2)$ . They are expressed in terms of massive operator matrix elements and the corresponding massless Wilson coefficients. Threshold resummations were performed.
- $F_L(x, Q^2)$  is known to NNLO for  $Q^2 \gg m^2$ .
- The calculation of fixed moments of the massive operator matrix elements at  $O(a_s^3)$  has been finished (Monday) for  $N = 10, 12$ , which will be followed by some first phenomenological parameterization.
- We also calculate the matrix elements necessary to transform from the **FFNS** to the **VFNS**.
- First steps towards the calculation of the massive operator matrix elements at  $O(a_s^3)$  for general values of  $N$  were performed.