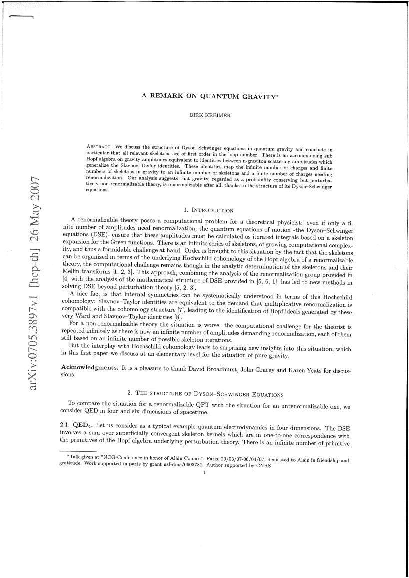
Classical Gravity at High Precision

Johannes Blümlein

Dirk Kreimer Fest, Bures sur Yvette, France, November 20, 2020

J. Blümlein, A. Maier, P. Marquard, Phys.Lett.B 800 (2020) 135100,
J. Blümlein, A. Maier, P. Marquard, A. Schäfer, Phys.Lett.B 807 (2020) 135496, Nucl.Phys.B 955 (2020) 115041,
Phys.Lett.B 801 (2020) 135157, 2010.13672 [gr-qc].



A REMARK ON QUANTUM GRAVITY*

DIRK KREIMER

Abstract. We discuss the structure of Dyson–Schwinger equations in quantum gravity and conclude in particular that all relevant skeletons are of first order in the loop number. There is an accompanying sub Hopf algebra on gravity amplitudes equivalent to identities between n-graviton scattering amplitudes which generalize the Slavnov Taylor identities. These identities map the infinite number of charges and finite numbers of skeletons in gravity to an infinite number of skeletons and a finite number of charges needing renormalization. Our analysis suggests that gravity, regarded as a probability conserving but perturbatively non-renormalizable theory, is renormalizable after all, thanks to the structure of its Dyson–Schwinger equations.

1. Introduction

A renormalizable theory poses a computational problem for a theoretical physicist: even if only a finite number of amplitudes need renormalization, the quantum equations of motion -the Dyson–Schwinger equations (DSE)- ensure that these amplitudes must be calculated as iterated integrals based on a skeleton expansion for the Green functions. There is an infinite series of skeletons, of growing computational complexity, and thus a formidable challenge at hand. Order is brought to this situation by the fact that the skeletons can be organized in terms of the underlying Hochschild cohomology of the Hopf algebra of a renormalizable theory, the computational challenge remains though in the analytic determination of the skeletons and their Mellin transforms [1, 2, 3]. This approach, combining the analysis of the renormalization group provided in [4] with the analysis of the mathematical structure of DSE provided in [5, 6, 1], has led to new methods in solving DSE beyond perturbation theory [5, 2, 3].

A nice fact is that internal symmetries can be systematically understood in terms of this Hochschild cohomology: Slavnov–Taylor identities are equivalent to the demand that multiplicative renormalization is compatible with the cohomology structure [7], leading to the identification of Hopf ideals generated by these very Ward and Slavnov–Taylor identities [8].

For a non-renormalizable theory the situation is worse: the computational challenge for the theorist is repeated infinitely as there is now an infinite number of amplitudes demanding renormalization, each of them still based on an infinite number of possible skeleton iterations.

But the interplay with Hochschild cohomology leads to surprising new insights into this situation, which in this first paper we discuss at an elementary level for the situation of pure gravity.

Acknowledgments. It is a pleasure to thank David Broadhurst, John Gracey and Karen Yeats for discussions.

2. The structure of Dyson–Schwinger Equations

To compare the situation for a renormalizable QFT with the situation for an unrenormalizable one, we consider QED in four and six dimensions of spacetime.

2.1. **QED$_4$.** Let us consider as a typical example quantum electrodynamics in four dimensions. The DSE involves a sum over superficially convergent skeleton kernels which are in one-to-one correspondence with the primitives of the Hopf algebra underlying perturbation theory. There is an infinite number of primitive

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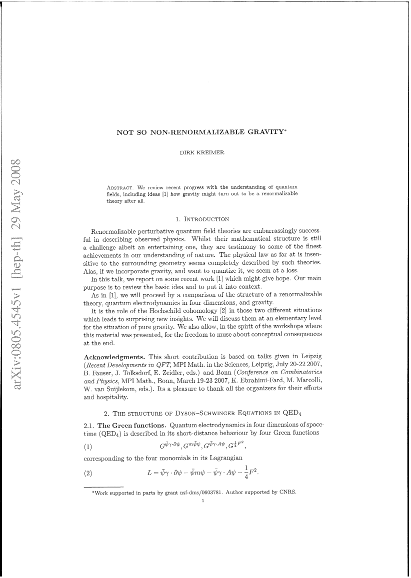

NOT SO NON-RENORMALIZABLE GRAVITY*

DIRK KREIMER

Abstract. We review recent progress with the understanding of quantum fields, including ideas [1] how gravity might turn out to be a renormalizable theory after all.

1. Introduction

Renormalizable perturbative quantum field theories are embarrassingly successful in describing observed physics. Whilst their mathematical structure is still a challenge albeit an entertaining one, they are testimony to some of the finest achievements in our understanding of nature. The physical law as far at is insensitive to the surrounding geometry seems completely described by such theories. Alas, if we incorporate gravity, and want to quantize it, we seem at a loss.

In this talk, we report on some recent work [1] which might give hope. Our main purpose is to review the basic idea and to put it into context.

As in [1], we will proceed by a comparison of the structure of a renormalizable theory, quantum electrodynamics in four dimensions, and gravity.

It is the role of the Hochschild cohomology [2] in those two different situations which leads to surprising new insights. We will discuss them at an elementary level for the situation of pure gravity. We also allow, in the spirit of the workshops where this material was presented, for the freedom to muse about conceptual consequences at the end.

Acknowledgments. This short contribution is based on talks given in Leipzig (*Recent Developments in QFT*, MPI Math. in the Sciences, Leipzig, July 20-22 2007, B. Fauser, J. Tolksdorf, E. Zeidler, eds.) and Bonn (*Conference on Combinatorics and Physics*, MPI Math., Bonn, March 19-23 2007, K. Ebrahimi-Fard, M. Marcolli, W. van Suijlekom, eds.). Its a pleasure to thank all the organizers for their efforts and hospitality.

2. The structure of Dyson–Schwinger Equations in QED$_4$

2.1. **The Green functions.** Quantum electrodynamics in four dimensions of spacetime (QED$_4$) is described in its short-distance behaviour by four Green functions

$$G^{\bar\psi\gamma\cdot\partial\psi}, G^{m\bar\psi\psi}, G^{\bar\psi\gamma\cdot A\psi}, G^{\frac{1}{4}F^2}, \tag{1}$$

corresponding to the four monomials in its Lagrangian

$$L = \bar\psi\gamma\cdot\partial\psi - \bar\psi m\psi - \bar\psi\gamma\cdot A\psi - \frac{1}{4}F^2. \tag{2}$$

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[*] Work supported in parts by grant nsf-dms/0603781. Author supported by CNRS.

Dirk has published very interesting papers on gravity. I remember excellent lectures by him on this topic at Clay Institute during Summer 2008.

Dirk regularly contributed to DESY conferences over decades: Loops & Legs, QCD09. KMPB initiated by him is an important platform for scientific exchange. We met at Boston, Linz, Les Houches, Bures, and many other places. Often, topical review books collected the achievements made.

We are very happy to have Dirk at Berlin!

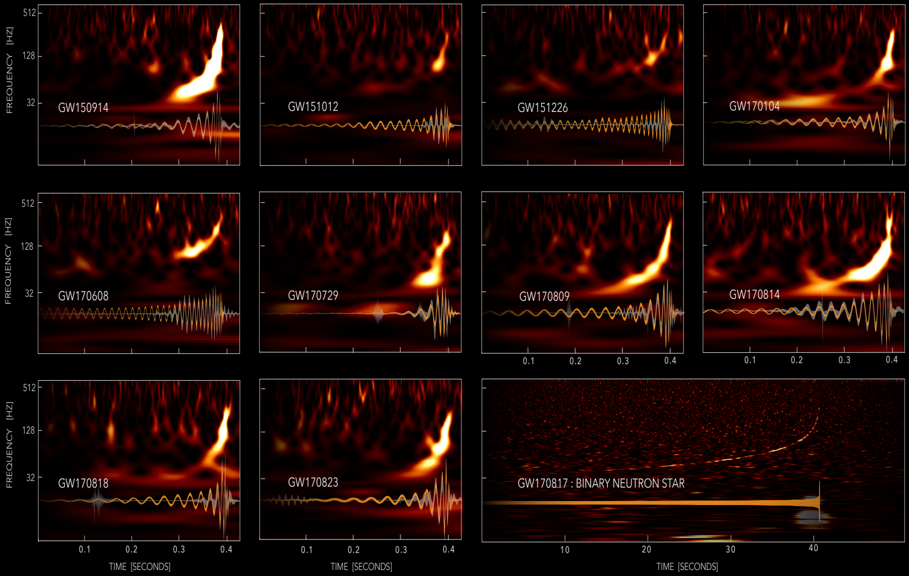
Happy Birthday, Dirk!

And many happy returns!

The motion of two gravitating masses

- Conservative part
- Radiation part
- Consider: the case of low velocities $v_i \ll c$
- The form of the orbit becomes more and more complicated, invoking higher and higher order corrections.
  - Newtonian level : elliptic motion
  - 1PN : movement of the perihelion (first observed for Mercury)
  - $\vdots$
  - current level : $\Longrightarrow$ 5PN (6PN)
  - test mass limit: all orders; $O(\nu)$ terms via self force: to 21.5 PN.
- We will consider the radiation free case of the inspiral phase in the following.
- The whole dynamics is important for the description of gravitational wave signals.
- $\Longrightarrow$ Derive the Hamiltonian Dynamics to 5PN.

GRAVITATIONAL-WAVE TRANSIENT CATALOG-1



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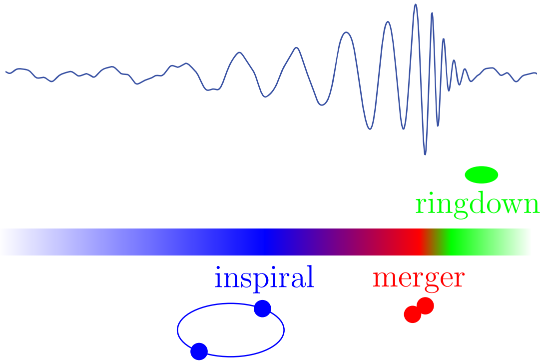
▨ **WAVELET (UNMODELED)**

🟧 **EINSTEIN'S THEORY**

S. GHONGE, K. JANI | GEORGIA TECH

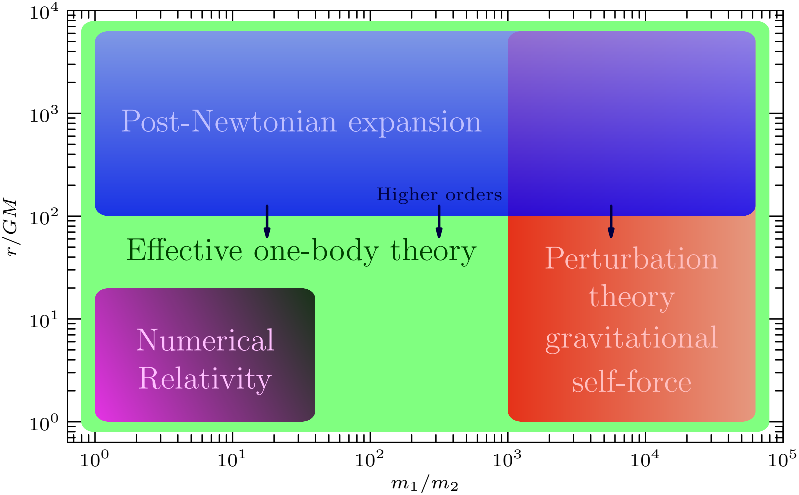
Gravitational waves

[LIGO Scientific Collaboration and Virgo Collaboration 2016]



Post-Newtonian expansion

Other approaches



adapted from [Buonanno 2018]

Post-Newtonian expansion

Theory status

- 1PN (v^2): [Lorentz, Droste 1917; Einstein, Infeld, Hoffmann 1938]
- 2PN (v^4): [Chandrasekhar, Esposito, Nutku 1969–1970; Okamura, Ohta, Kimura, Hiida 1973–1974, final form: Damour 1982]
- 2.5PN (v^5): [Damour, Deruelle 1981–1983]
- 3PN (v^6): [Damour, Jaranowski, Schäfer, 1997–2001; Andrade, Blanchet, Iyer, Faye 2000–2002]
- 3.5PN (v^7): [Iyer, Will 1993–1995]
- 4PN (v^8):
  - ADM Hamiltonian formalism [Damour, Jaranowski, Schäfer 2014-2016]
  - Fokker Lagrangian in harmonic coordinates [Bernard, Blanchet, Bohé, Faye, Marchant, Marsat 2017]
  - Non-relativistic effective field theory [Foffa, Mastrolia, Sturani, Sturm 2017–2019]
- Many more contributions, e.g. spin effects, radiation

Non-relativistic effective field theory [Goldberger, Rothstein 2004, Kol, Smolkin, 2010]

From path integral to the post-Newtonian Hamiltonians and Observables by Methods from Quantum Field Theory

General Relativity

- Here: point-like objects
  - No spin
  - No finite-size effects
- Harmonic gauge fixing: $\partial_\mu(\sqrt{-g}g^{\mu\nu} - \eta^{\mu\nu}) = 0$
  $g = \det(g^{\mu\nu})$
- Dimensional regularisation: $d = 3 - 2\epsilon$

$$
\begin{aligned}
S_{\text{GR}} &= S_{\text{EH}} + S_{\text{GF}} + S_{pp} \\
S_{\text{EH}} &= \frac{1}{16\pi G} \int d^{d+1}x \, \sqrt{-g}R \\
S_{\text{GF}} &= -\frac{1}{32\pi G} \int d^{d+1}x \, \sqrt{-g}\,\Gamma_\mu \Gamma^\mu \\
S_{\text{pp}} &= -\sum_i m_i \int d\tau_i = -\sum_i m_i \int dt \, \sqrt{-g_{\mu\nu}\frac{\partial x_i^\mu}{\partial t}\frac{\partial x_i^\nu}{\partial t}} \\
R &= g^{\mu\nu}R_{\mu\nu} \\
\Gamma^\mu &= g^{\alpha\beta}\Gamma^\mu{}_{\alpha\beta}
\end{aligned}
$$

Why can QFT methods be of help here ?

- Any dynamical physical theory is based on the Lagrange formalism and can be derived from the principle of the least action.
- Near the non-relativistic limit, Einstein gravity possesses an expansion in Newton's constant and the velocities of the involved macroscopic bodies $v_i \ll c$.
- One can consider the associated path integral representation [Feynman & Hibbs 1965, Zinn-Justin 2002] and derive the perturbative expansions from it.
- This will lead to the associated effective field theory representation, not necessarily built on the (flat space) gravitons.
- From order to order (potentially) new interaction vertices will appear.
- This systematic approach will, however, perturbatively represent the full theory.
- The key-issue is to reduce to the associated master integrals and to calculate them.
- Current level: 5–loop integrals in $d = 3 - 2\varepsilon$ (ranging to 6PN)

Non-relativistic effective theory

[Goldberger, Rothstein 2004]

Similar to non-relativistic QCD

[Caswell, Lepage 1985; Pineda, Soto 1997; Luke, Manohar, Rothstein 2000; . . .]

Full theory: General relativity		Effective theory: NRGR
$S_{\text{GR}} = S_{\text{EH}} + S_{\text{GF}} + S_{pp}$	$\longrightarrow$	$S_{\text{NRGR}} = \int dt \frac{1}{2} m_i v_i^2 + \frac{G m_1 m_2}{r} + \dots$
potential gravitons: $k_0 \sim \frac{v}{r}, \vec{k} \sim \frac{1}{r}$		classical potentials: potential terms
radiation gravitons: $k_0 \sim \frac{v}{r}, \vec{k} \sim \frac{v}{r}$		radiation gravitons: tail terms

Potential matching

Expansion of action

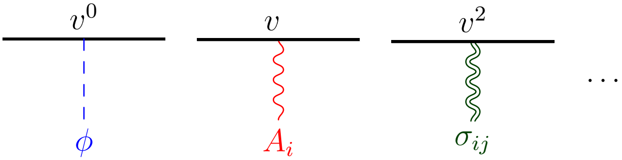
Expand S_{GR} in $v \sim \sqrt{Gm/r} \ll 1$, e.g.

$$S_{\mathrm{pp}} = -\sum_i m_i \int dt \sqrt{-g_{\mu\nu} \frac{\partial x_i^\mu}{\partial t} \frac{\partial x_i^\nu}{\partial t}} = -\sum_i m_i \int dt \sqrt{-g_{00}} + \mathcal{O}(v_i)$$

Coupling to spatial components of metric suppressed

Temporal Kaluza-Klein decomposition [Kol, Smolkin 2010]: 10 fields.

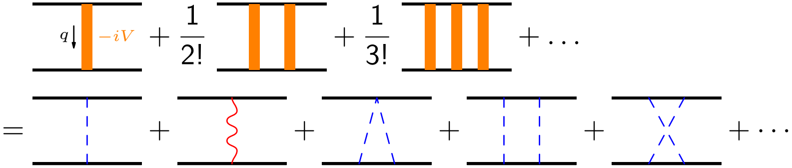
$$g^{\mu\nu} = e^{2\phi} \begin{pmatrix} -1 & A_j \\ A_i & e^{-2\frac{d-1}{d-2}\phi}(\delta_{ij} + \sigma_{ij}) - A_i A_j \end{pmatrix}$$



Potential matching

Diagrammatic expansion

Equate amplitude in effective and full theory:

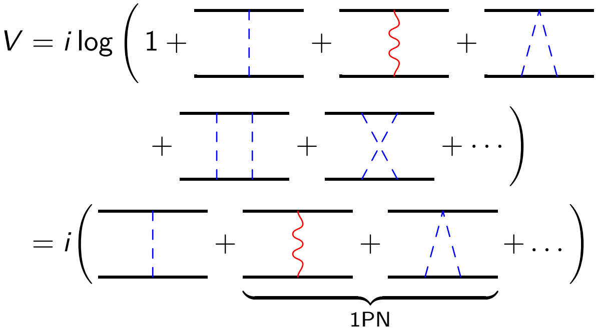


All momenta potential, $p_0 \sim \frac{v}{r} \ll p_i \sim \frac{1}{r}$
$\hookrightarrow$ expand propagators:

$$\frac{1}{\vec{p}^{\,2} - p_0^2} = \frac{1}{\vec{p}^{\,2}} + \frac{p_0^2}{\vec{p}^{\,4}} + \mathcal{O}(v^4)$$

Potential matching

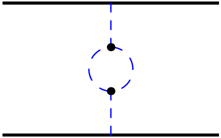
Diagrammatic expansion



Potential matching

Diagram selection

- No pure graviton loops (quantum corrections)



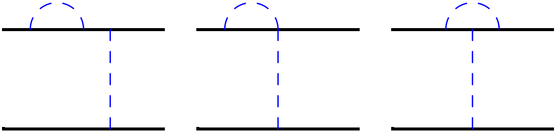
$\sim \frac{\hbar}{mvr}$

- No single-source corrections
- No source-reducible diagrams [Fischler 1977]

Potential matching

Diagram selection

- No pure graviton loops (quantum corrections)
- No single-source corrections



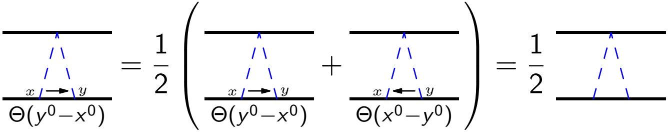
Absorbed into renormalisation of sources

- No source-reducible diagrams [Fischler 1977]

Potential matching

Diagram selection

- No pure graviton loops (quantum corrections)
- No single-source corrections
- No source-reducible diagrams [Fischler 1977]
  Initially *time-ordered* diagrams:



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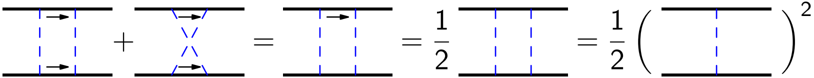

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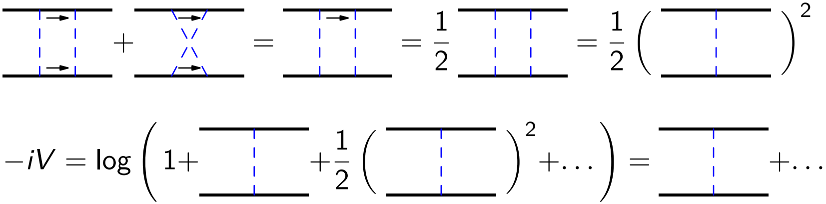


Effective Field Theory Calculations

Known results

Confirmation of previous results:

- 1PN: [1917 (1938)] [Goldberger, Rothstein 2004, Kol, Smolkin, 2010]
- 2PN: [1969/70] [Gilmore, Ross 2008]
- 3PN: [1997/2001][Foffa, Sturani 2011]
- 4PN:
  - “static” contribution $v = 0$:

    [Foffa, Mastrolia, Sturani, Sturm 2016; Damour, Jaranowski 2017]
  - $v \neq 0$: [2014] [Foffa, Sturani 2019; Foffa, Porto, Rothstein, Sturani 2019, JB, Maier, Marquard, Schäfer, 2020]

New:

- **5PN static contribution:**

  [Foffa, Mastrolia, Sturani, Sturm, Torres Bobadilla 27 Feb 2019; Blümlein, Maier, Marquard 28 Feb 2019]

Effective Field Theory Calculations

Number of diagrams at 5PN: potential terms

#loops	QGRAF	source irred.	no source loops	no tadpoles	masters
0	3	3	3	3	0
1	72	72	72	72	1
2	3286	3286	3286	2702	1
3	81526	62246	60998	41676	1
4	545812	264354	234934	116498	7
5	332020	128080	101570	27582	4

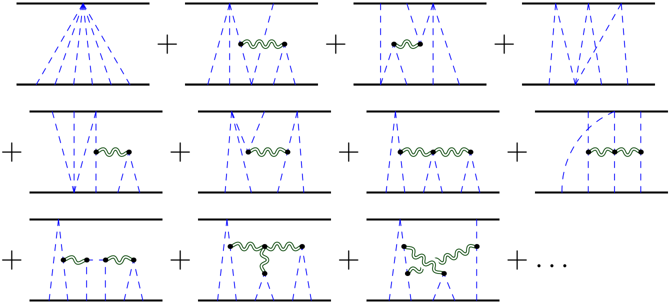
Generation of 962719 Feynman diagrams `QGRAF` [Nogueira 1993]; performing the Lorentz algebra and further steps `FORM 3.0` [Vermaseren 2001-]; Reduction to master intergrals `Crusher` [Marquard, Seidel].

188533 diagrams are finally contributing.
There are factorizing subsets [Foffa, Sturani, Torres Bobadilla 2020_v2], to which we agree.
For all the contributions up to 4PN see [JB, Maier, Marquard, Schäfer, 2020].

Effective Field Theory Calculations

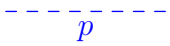
Static 5PN calculation

$$-iV^{S}_{5\mathrm{PN}} =$$

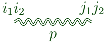


Effective Field Theory Calculations

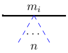
Feynman rules



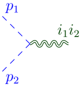
$$= -\frac{i}{2c_d\vec{p}^{\,2}}$$



$$= -\frac{i}{2\vec{p}^{\,2}}\left(\delta_{i_1 j_1}\delta_{i_2 j_2} + \delta_{i_1 j_2}\delta_{i_2 j_1} + (2-c_d)\delta_{i_1 i_2}\delta_{j_1 j_2}\right)$$

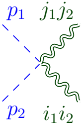


$$= -i\frac{m_i}{m_{\mathrm{Pl}}^n}$$



$$= i\frac{c_d}{2m_{\mathrm{Pl}}}(V_{\phi\phi\sigma}^{i_1 i_2} + V_{\phi\phi\sigma}^{i_2 i_1})$$

$$V_{\phi\phi\sigma}^{i_1 i_2} = \vec{p}_1\cdot\vec{p}_2\delta^{i_1 i_2} - 2p_1^{i_1}p_2^{i_2}$$

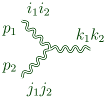


$$= i\frac{c_d}{16m_{\mathrm{Pl}}^2}(V_{\phi\phi\sigma\sigma}^{i_1 i_2, j_1 j_2} + V_{\phi\phi\sigma\sigma}^{i_2 i_1, j_1 j_2} + V_{\phi\phi\sigma\sigma}^{i_1 i_2, j_2 j_1} + V_{\phi\phi\sigma\sigma}^{i_2 i_1, j_2 j_1})$$

$$V_{\phi\phi\sigma\sigma}^{i_1 i_2, j_1 j_2} = \vec{p}_1\cdot\vec{p}_2(\delta^{i_1 i_2}\delta^{j_1 j_2} - 2\delta^{i_1 j_1}\delta^{i_2 j_2}) - 2(p_1^{i_1}p_2^{i_2}\delta^{j_1 j_2} + p_1^{j_1}p_2^{j_2}\delta^{i_1 i_2}) + 8\delta^{i_1 j_1}p_1^{i_2}p_2^{j_2}$$

Effective Field Theory Calculations

Feynman rules



$$= \frac{i}{32 m_{\mathrm{Pl}}} \left(\tilde{V}^{i_1 i_2, j_1 j_2, k_1 k_2}_{\sigma\sigma\sigma} + \tilde{V}^{i_2 i_1, j_1 j_2, k_1 k_2}_{\sigma\sigma\sigma} \right)$$

$$\tilde{V}^{i_1 i_2, j_1 j_2, k_1 k_2}_{\sigma\sigma\sigma} = V^{i_1 i_2, j_1 j_2, k_1 k_2}_{\sigma\sigma\sigma} + V^{i_1 i_2, j_2 j_1, k_1 k_2}_{\sigma\sigma\sigma} + V^{i_1 i_2, j_1 j_2, k_2 k_1}_{\sigma\sigma\sigma} + V^{i_1 i_2, j_2 j_1, k_2 k_1}_{\sigma\sigma\sigma}$$

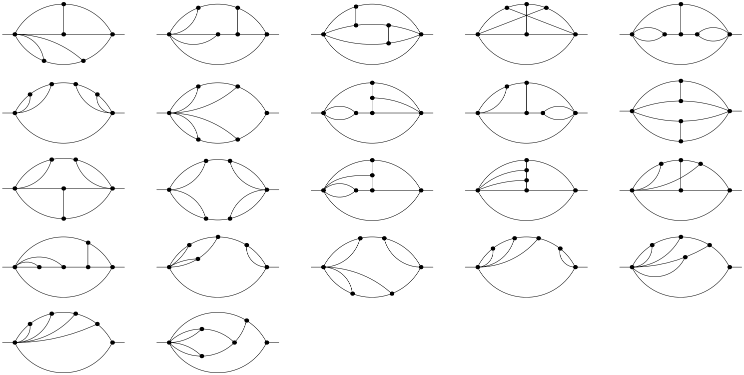
$$\begin{aligned}
V^{i_1 i_2, j_1 j_2, k_1 k_2}_{\sigma\sigma\sigma} = {} & (\vec{p}_1^{\,2} + \vec{p}_1 \cdot \vec{p}_2 + \vec{p}_2^{\,2}) \Big(- \delta^{j_1 j_2} \Big(2\delta^{i_1 k_1} \delta^{i_2 k_2} - \delta^{i_1 i_2} \delta^{k_1 k_2} \Big) \\
& \quad + 2 \Big[\delta^{i_1 j_1} \Big(4\delta^{i_2 k_1} \delta^{j_2 k_2} - \delta^{i_2 j_2} \delta^{k_1 k_2} \Big) - \delta^{i_1 i_2} \delta^{j_1 k_1} \delta^{j_2 k_2} \Big] \Big) \\
& + 2 \Big\{ 4 \Big(p_1^{k_2} p_2^{i_2} - p_1^{i_2} p_2^{k_2} \Big) \delta^{i_1 j_1} \delta^{j_2 k_1} \\
& \quad + 2 \Big[\Big(p_1^{i_1} + p_2^{i_1} \Big) p_2^{i_2} \delta^{j_1 k_1} \delta^{j_2 k_2} - p_1^{k_1} p_2^{k_2} \delta^{i_1 j_1} \delta^{i_2 j_2} \Big] \\
& \quad + \delta^{j_1 j_2} \Big[p_1^{k_1} p_2^{k_2} \delta^{i_1 i_2} + 2 \Big(p_1^{k_2} p_2^{i_2} - p_1^{i_2} p_2^{k_2} \Big) \delta^{i_1 k_1} - \Big(p_1^{i_1} + p_2^{i_1} \Big) p_2^{i_2} \delta^{k_1 k_2} \Big] \\
& \quad + p_2^{j_2} \Big(4 p_1^{i_2} \delta^{i_1 k_1} \delta^{j_1 k_2} + p_1^{j_1} \Big(2\delta^{i_1 k_1} \delta^{i_2 k_2} - \delta^{i_1 i_2} \delta^{k_1 k_2} \Big) \\
& \qquad + 2 \Big[\delta^{i_1 j_1} \Big(p_1^{i_2} \delta^{k_1 k_2} - 2 p_1^{k_2} \delta^{i_2 k_1} \Big) - p_1^{k_2} \delta^{i_1 i_2} \delta^{j_1 k_1} \Big] \Big) \\
& \quad + p_1^{j_2} \Big(p_1^{j_1} \Big(2\delta^{i_1 k_1} \delta^{i_2 k_2} - \delta^{i_1 i_2} \delta^{k_1 k_2} \Big) - 4 p_2^{i_2} \delta^{i_1 k_1} \delta^{j_1 k_2} \\
& \qquad + 2 \Big[p_2^{k_2} \delta^{i_1 i_2} \delta^{j_1 k_1} + \delta^{i_1 j_1} \Big(2 p_2^{k_2} \delta^{i_2 k_1} - p_2^{i_2} \delta^{k_1 k_2} \Big) \Big] \Big) \Big\}, \quad c_d = 2 \frac{d-1}{d-2}, \, m_{\mathrm{Pl}} = \sqrt{32\pi G}
\end{aligned}$$

Effective Field Theory Calculations

Diagram families

Massless propagators:

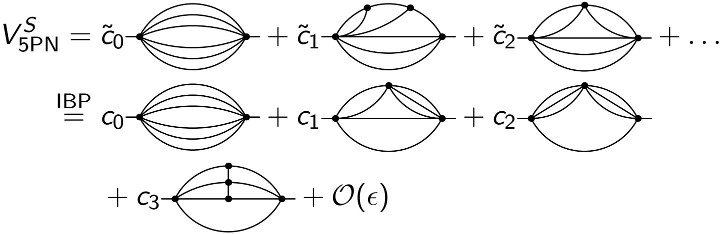
$$P_f(q) = \int \frac{d^d l_1}{\pi^{d/2}} \cdots \frac{d^d l_5}{\pi^{d/2}} \; \frac{\mathcal{N}(q, l_1, \ldots, l_5)}{\vec{p}_1^{\,2a_1} \cdots \vec{p}_{10}^{\,2a_{10}}}$$



Effective Field Theory Calculations

Reduction to master integrals

Apply integration-by-parts relations (`crusher`): [Marquard & Seidel, unpubl.], [Chetyrkin, Tkachov 1981; Laporta 2000]

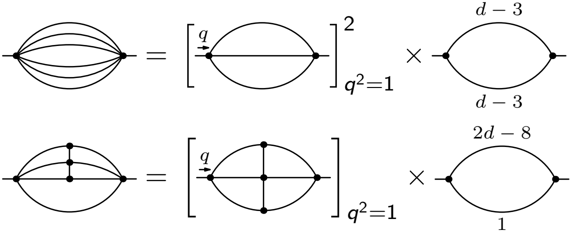


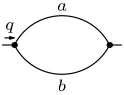
$\tilde{c}_i, c_j$: Laurent series in $\epsilon = \frac{3-d}{2}$, polynomials in m_1, m_2, r^{-1}, G^{-1}

Effective Field Theory Calculations

Calculation of master integrals

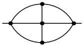
Master integrals factorise, e.g.





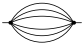
$$= \int \frac{d^d l}{\pi^{d/2}} \frac{1}{((q-l)^2)^a} \frac{1}{(l^2)^b}$$

$$= \frac{1}{(q^2)^{a+b-d/2}} \frac{\Gamma\left(\frac{d}{2}-a\right)\Gamma\left(\frac{d}{2}-b\right)\Gamma\left(a+b-\frac{d}{2}\right)}{\Gamma(a)\Gamma(b)\Gamma\left(d-a-b\right)}$$

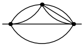
 known from 4PN [Foffa, Mastrolia, Sturani, Sturm 2016; Damour, Jaranowski 2017] up to the power in ε presently needed.

Effective Field Theory Calculations

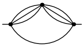
Results for master integrals



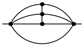
$$= e^{5\epsilon\gamma_E}\frac{\Gamma\left(6-\frac{5d}{2}\right)\Gamma^6\left(-1+\frac{d}{2}\right)}{\Gamma(-6+3d)}$$



$$= e^{5\epsilon\gamma_E}\frac{\Gamma\left(7-\frac{5d}{2}\right)\Gamma\left(3-d\right)\Gamma\left(2-\frac{d}{2}\right)\Gamma^7\left(-1+\frac{d}{2}\right)\Gamma(5-2d)}{\Gamma\left(5-\frac{3}{2}d\right)\Gamma(-2+d)\Gamma\left(-3+\frac{3}{2}d\right)\Gamma\left(-7+3d\right)}$$



$$= e^{5\epsilon\gamma_E}\frac{\Gamma\left(7-\frac{5d}{2}\right)\Gamma^2(3-d)\Gamma^7\left(-1+\frac{d}{2}\right)\Gamma\left(-6+\frac{5d}{2}\right)}{\Gamma\left(6-2d\right)\Gamma^2\left(-3+\frac{3d}{2}\right)\Gamma\left(-7+3d\right)}$$



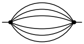
$$= 6\pi^{7/2}\left[\frac{2}{\epsilon}-4-4\ln(2)-\left(48+8\ln(2)-4\ln^2(2)-105\zeta_2\right)\epsilon+\mathcal{O}(\epsilon^2)\right]$$

$V^S_{5\text{PN}} \overset{\epsilon=0}{=} \frac{G^6}{r^6\pi^{7/2}}(m_1m_2)\Bigg\{\frac{15}{32}(m_1^5+m_2^5)\Bigg[$

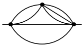
$\Bigg]_{\epsilon^0} + \frac{91}{4}m_1m_2(m_1^3+m_2^3)\Bigg[$

$\Bigg]_{\epsilon^0}$

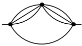
$+ m_1^2m_2^2(m_1+m_2)\Bigg(\Bigg[\frac{293}{4}$



$-\frac{45}{16}$

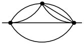


$+\frac{45}{32}$

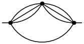


$\Bigg]_{\epsilon^0}$

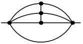
$+\Bigg[\frac{519}{16}$



$-\frac{627}{32}$



$+2$



$\Bigg]_{\epsilon^{-1}}\Bigg)\Bigg\}$

The Static Potential

Result

$$
\begin{aligned}
V^S_{\mathrm{N}} &= -\frac{G}{r} m_1 m_2 \\
V^S_{\mathrm{1PN}} &= \frac{G^2}{2r^2} m_1 m_2 (m_1 + m_2) \\
V^S_{\mathrm{2PN}} &= -\frac{G^3}{r^3} m_1 m_2 \left[\frac{1}{2}(m_1^2 + m_2^2) + 3 m_1 m_2\right] \\
V^S_{\mathrm{3PN}} &= \frac{G^4}{r^4} m_1 m_2 \left[\frac{3}{8}\left(m_1^3 + m_2^3\right) + 6 m_1 m_2 (m_1 + m_2)\right] \\
V^S_{\mathrm{4PN}} &= -\frac{G^5}{r^5} m_1 m_2 \left[\frac{3}{8}\left(m_1^4 + m_2^4\right) + \frac{31}{3} m_1 m_2 \left(m_1^2 + m_2^2\right) + \frac{141}{4} m_1^2 m_2^2\right] \\
V^S_{\mathrm{5PN}} &= \frac{G^6}{r^6} m_1 m_2 \left[\frac{5}{16}(m_1^5 + m_2^5) + \frac{91}{6} m_1 m_2 (m_1^3 + m_2^3) + \frac{653}{6} m_1^2 m_2^2 (m_1 + m_2)\right]
\end{aligned}
$$

All ζ-values cancel. This is not the case for velocity corrections, to which we turn now.

The 5PN Hamiltonian

The principal calculation steps

- $$H = H_{\rm pot} + H_{\rm tail} = H_{\rm loc} + H_{\rm nl}$$
- In the harmonic gauge $H_{\rm pot}$ are $H_{\rm tail}$ both singular.
- The potential terms $H_{\rm pot}$ are calculated as outlined before.
- The singular and logarithmic contributions to $H_{\rm tail}$ are calculated using the method of expansion by regions [Beneke, Smirnov 1997, Jantzen 2011].
- There are no overlap terms between the potential and tail contributions [JB, Maier, Marquard, Schäfer, 2020].
- One adds both terms and obtains
  - a still singular Hamiltonian
  - the non-local Hamiltonian H_{nl}.
- Perform a canonical transformtion to a pole-free Hamiltonian [then in different coordinates];
  The formulae are still rather long. [JB, Maier, Marquard, Schäfer, 2010.13672 [gr-qc].]

The 5PN Hamiltonian

The principal calculation steps

- Perform a canonical transformation of the $O(\nu^0)$ and $O(\nu^{3..5})$ terms from the Hamiltonian starting with harmonic coordinates, H_{harm}, to effective one body (EOB) coordinates, H_{EOB}.
  $\Longrightarrow$ Physical equivalence of these terms. [On an individual diagram basis even the Schwarzschild limit is not easy to obtain: very many diagrams.]
- Prediction of all terms contributing to 3PM and 6PN [JB, Maier, Marquard, Schäfer 2020 3PM, 6PN]; confirmed by [Bini, Damour, Geralico, 2020 6PN loc].
- A special treatment is needed for the O(ν) terms. $\surd$
- A series of more local tail terms contribute at $O(\nu^2)$ and have to dealt with (work in progress).
- The non-local contributions of the tail terms at 4 & 5 PN haven been calculated. Agreement with [Bini, Damour, Geralico, 2020 5PN].

Observables at 5PN

Periastron advance: local contributions

$$
\begin{aligned}
K(\hat{E},j)^{5\mathrm{PN}}_{\mathrm{loc,f}} \quad \propto \quad & \Bigg\{ \bigg[\frac{15\nu^2}{16} - \frac{15}{4}\nu^3 + 3\nu^4 \bigg] \frac{\hat{E}^4}{j^2} + \bigg[\frac{3465}{16} + \bigg(-\frac{12160657}{8400} + \frac{15829\pi^2}{256} \bigg) \nu + \bigg(-\frac{35569\pi^2}{1024} \bigg) \nu^2 \\
& + \bigg(\frac{1107\pi^2}{128} - \frac{7113}{8} \bigg) \nu^3 + 75\nu^4 \bigg] \frac{\hat{E}^3}{j^4} + \bigg[\frac{315315}{32} + \bigg(-\frac{33023719}{840} + \frac{4899565\pi^2}{4096} \bigg) \nu \\
& + \bigg(-\frac{3289285\pi^2}{1024} \bigg) \nu^2 + \bigg(\frac{35055\pi^2}{256} - \frac{240585}{32} \bigg) \nu^3 + \frac{1575}{8}\nu^4 \bigg) \frac{\hat{E}^2}{j^6} + \bigg[\frac{765765}{16} \\
& + \bigg(-\frac{30690127}{240} + \frac{16173395\pi^2}{8192} \bigg) \nu + \bigg(-\frac{77646205\pi^2}{8192} \bigg) \nu^2 + \bigg(\frac{121975\pi^2}{512} - \frac{271705}{24} \bigg) \nu^3 \\
& + \frac{2205}{16}\nu^4 \bigg] \frac{\hat{E}}{j^8} + \bigg[\frac{2909907}{64} \bigg(-\frac{61358067}{640} + \frac{1096263\pi^2}{1024} \bigg) \nu + \bigg(-\frac{87068961\pi^2}{16384} \bigg) \nu^2 \\
& + \bigg(\frac{90405\pi^2}{1024} - \frac{127995}{32} \bigg) \nu^3 + \frac{3465}{128}\nu^4 \bigg] \frac{1}{j^{10}} \Bigg\} \eta^{10} + O(\eta^{12})
\end{aligned}
$$

- Very recently the origin of the $O(\nu)$ terms has been understood within the present approach.
- Only the rational contributions to the 5PN ν^2 terms are yet open.

Observables at 5PN

Circular orbit: energy and periastron advance

$$
\begin{aligned}
\hat{E}^{\mathrm{circ}}(j) \quad = \quad & -\frac{1}{2j^2} + \biggl(-\frac{\nu}{8} - \frac{9}{8}\biggr)\frac{1}{j^4}\eta^2 + \biggl(-\frac{\nu^2}{16} + \frac{7\nu}{16} - \frac{81}{16}\biggr)\frac{1}{j^6}\eta^4 + \biggl[-\frac{5\nu^3}{128} + \frac{5\nu^2}{64} + \biggl(\frac{8833}{384} \\
& - \frac{41\pi^2}{64}\biggr)\nu - \frac{3861}{128}\biggr]\frac{1}{j^8}\eta^6 + \biggl[-\frac{7\nu^4}{256} + \frac{3\nu^3}{128} + \biggl(\frac{41\pi^2}{128} - \frac{8875}{768}\biggr)\nu^2 + \biggl(\frac{989911}{3840} \\
& - \frac{6581\pi^2}{1024}\biggr)\nu - \frac{53703}{256}\biggr]\frac{1}{j^{10}}\eta^8 + \biggl[\biggl(r^{E}_{\nu 2} + \frac{132979\pi^2}{2048}\biggr)\nu^2 - \frac{21\nu^5}{1024} + \frac{5\nu^4}{1024} + \biggl(\frac{41\pi^2}{512} \\
& - \frac{3769}{3072}\biggr)\nu^3 + \biggl(\frac{3747183493}{1612800} - \frac{31547\pi^2}{1536}\biggr)\nu - \frac{1648269}{1024}\biggr]\frac{1}{j^{12}}\eta^{10} \\
& + \frac{E^{\mathrm{circ}}_{\mathrm{nl}}(j)}{\mu c^2} + O\left(\eta^{12}\right), \\
K^{\mathrm{circ}}(j) \quad = \quad & 1 + 3\frac{1}{j^2}\eta^2 + \biggl(\frac{45}{2} - 6\nu\biggr)\frac{1}{j^4}\eta^4 + \biggl[\frac{405}{2} + \biggl(-202 + \frac{123}{32}\pi^2\biggr)\nu + 3\nu^2\biggr]\frac{1}{j^6}\eta^6 + \\
& \biggl[\frac{15795}{8} + \biggl(\frac{185767}{3072}\pi^2 - \frac{105991}{36}\biggr)\nu + \biggl(-\frac{41}{4}\pi^2 + \frac{2479}{6}\biggr)\nu^2\biggr]\frac{1}{j^8}\eta^8 + \biggl[\frac{161109}{8} \\
& + \biggl(-\frac{18144676}{525} + \frac{488373}{2048}\pi^2\biggr)\nu + \biggl(r^{K}_{\nu 2} - \frac{1379075}{1024}\pi^2\biggr)\nu^2 + \biggl(-\frac{1627}{6} + \frac{205}{32}\pi^2\biggr)\nu^3\biggr]\frac{1}{j^{10}}\eta^{10} \\
& + K^{\mathrm{nl}}_{4+5\mathrm{PN}}(j) + O\left(\eta^{12}\right). \qquad \eta^2 = 1/c^2.
\end{aligned}
$$

Observables at 5PN

Determining the missing π^2 contributions

In [Bini, Damour, Geralico, 2020] ["tutti frutti method"] two constants $\bar{d}_5$ and a_6 could not be determined. We have computed their π^2 contributions within our calculation ab inito [JB, Maier, Marquard, Schäfer, 2020]. These terms are contained in the potential terms

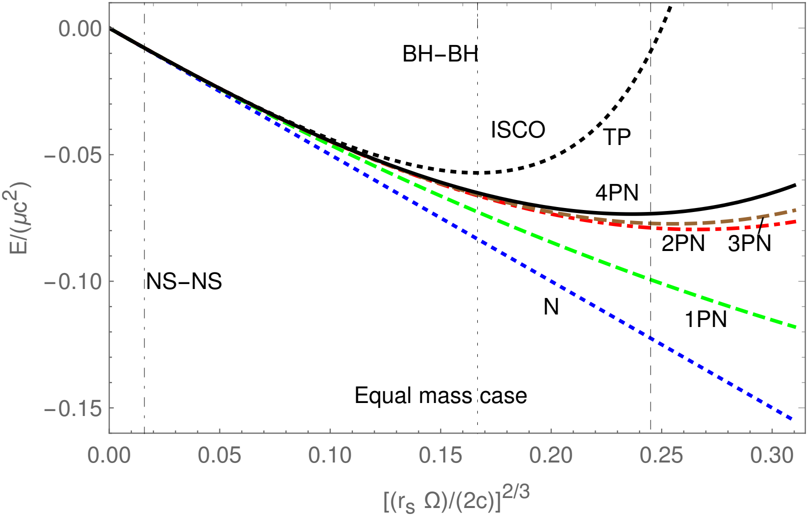
$$\begin{aligned} \bar{d}_5 &= r_{\bar{d}_5} + \frac{306545}{512}\pi^2 \\ a_6 &= r_{a_6} + \frac{25911}{256}\pi^2 \; . \end{aligned}$$

All terms are in agreement with [Bini, Damour, Geralico, 2020] and the π^2 contributions to $\bar{d}_5$ and a_6 are new.

There are also first 6PN corrections known [Bini, Damour, Geralico, 2020]. Up to the terms of $O(1/r^3)$ they were predicted in [JB, Maier, Marquard, Schäfer 2020, 6PN].

Observables up to 4PN

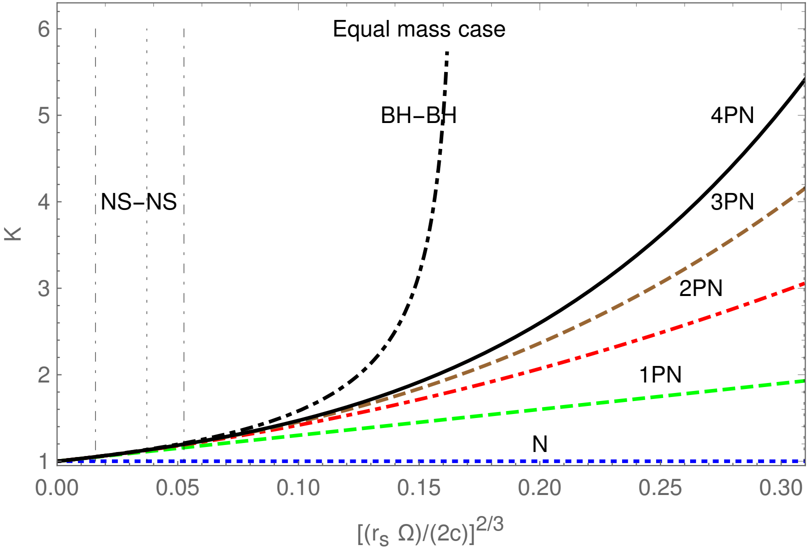
Numerical results



[JB, Maier, Marquard, Schäfer 2020, 4PN]

Observables up to 4PN

Numerical results



Conclusions and Outlook

- The inspiral phase of compact binary systems is well described by the *Post-Newtonian (PN) expansion* $v \sim \sqrt{Gm/r} \ll 1$.
- Effective field theory and calculation methods from Quantum Field Theory are very effective for high PN orders.
- The static gravitational potential is known at five loops.
- Very recently the 5PN Hamiltonian has been determined up to a small set of rational terms in $O(\nu^2)$. These results are obtained performing a 5 loop calculation of nearly 200.000 Feynman diagrams with partly very complicated vertices ab initio.
- Higher order corrections extend the area perturbatively accessible towards the region where presently only pure numerical methods are applied.
- Fully resummed velocity expressions would be welcome to have: Post-Minkowskian approach. They are currently limited to 2–loop order. [Bern et al. 2019; Kälin, Porto, 2020; Damour et al. 2019/20]