Computer-algebraic Integration of Multi-Loop Feynman Diagrams

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1. Introduction

Loops and Legs:

Feynman diagrams describe elementary scattering processes between bosons and fermions in Quantum Field Theory (QFT). Here we will thoroughly refer to renormalizable QFTs.

Where are these techniques important?

1. Perturbation Theory of the Standard Model and its renormalizable extensions.
2. String amplitude calculations
3. Perturbative calculations in Gravity
4. non-relativistic field theories in vacuum and at finite temperature and/or density

We will calculate Feynman diagrams. These are skeletons according to Feynman rules, connecting vertices with propagators.

They possess external lines: The Legs.

They possess internal closed lines: The Loops.

Introduction

Why are these calculations important ?

1. Precision extraction of coupling constants: $\alpha_s(M_Z)@1\%$
2. Do couplings unite at high scales and in which field theories?
3. Precision measurements of m_c, m_b, m_t at LHC and a future ILC
4. Precision understanding of the Higgs and top sector (at the LHC, ILC and possibly other machines)
5. Unravel the mathematical structure of microscopic processes analytically: get further with the Stueckelberg-Feynman programme as far as you can.

$\Longrightarrow$ Genetic Code of the Micro Cosmos

2. Perform all Momentum Integrals

We would like to outline an algorithm to first perform all momentum integrals in D-space. $D = 4 + \varepsilon$.

Example:

$$\int d^D k \frac{1}{[(p-k)^2 - m^2][k^2 - m^2]}$$

How to integrate this and related loop integrals of more general kind ?

$$\int d^D k_1 \ldots \int d^D k_n \prod_{i=1}^{m} \frac{1}{(p_i[L(k_j; q_a)])^2 - M_i^2}; \quad L \text{ - linear function}$$

k_j loop momenta; q_a external momenta; $m > n$

Integrate Loop by Loop.

Most of the higher loop integrals are very difficult to compute analytically.

Feynman's Trick

Combining scalar propagators:

$$\frac{1}{a \cdot b} = \int_0^1 dx \frac{1}{[xa + (1-x)b]^2}$$

$$\frac{1}{a_1 \ldots a_N} = \int_0^1 dx_1 ... dx_N \frac{\delta(1 - x_1 - ... - x_N)}{\left[\sum_{i=1}^{N} a_i x_i\right]^N}$$

$$\frac{1}{a^\alpha \cdot b^\beta} = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \int_0^1 dx \frac{x^{\alpha-1}(1-x)^{\beta-1}}{[xa + (1-x)b]^{\alpha+\beta}}; \quad \alpha, \beta \in \mathbb{R}_+$$

Momentum Integrals

The procedure is fully algorithmized to any loop order and for any number of legs.

The Feynman-Trick maps all loop integrals into a form, in which the momentum integrals can be carried out analytically in an easy manner.

Feynman-parameter integrals, however, remain to be done afterwards too.

Example:

$$\begin{aligned}\int d^Dk \frac{1}{[(p-k)^2-m^2][k^2-m^2]} &= \int d^Dk \int_0^1 dx \frac{1}{(k^2-2pkx+xp^2-m^2)^2}\\ &= \int_0^1 dx \int d^Dk' \frac{1}{((k')^2+x(1-x)p^2-m^2)^2}\\ &= \int_0^1 dx \int d^Dk \frac{1}{(k^2-M^2(x))^2}\end{aligned}$$

$k' = k - px; \qquad d^Dk = d^Dk'; \qquad M^2 = m^2 - x(1-x)p^2.$

Tensor Integrals

The Treatment of Numerators:

$$
\begin{aligned}
N &= N_0 + N_2 l_{\mu_1} l_{\mu_2} + N_4 l_{\mu_1} l_{\mu_2} l_{\mu_3} l_{\mu_4} + \dots \\
I_0 &= \int d^D\, l\; f(l^2) \\
I_{2;\,\mu_1\mu_2} &= \int d^D l l_{\mu_1} l_{\mu_2} f(l^2) = \frac{g_{\mu_1\mu_2}}{D} \int d^D l\, l^2\; f(l^2) \\
I_{4;\,\mu_1\mu_2\mu_3\mu_4} &= \int d^D l\; l_{\mu_1} l_{\mu_2} l_{\mu_3} l_{\mu_4}\; f(l^2) \\
&= \frac{g_{\mu_1\mu_2} g_{\mu_3\mu_4} + g_{\mu_1\mu_3} g_{\mu_2\mu_4} + g_{\mu_1\mu_4} g_{\mu_2\mu_3}}{D^2 + 2D} \int d^D l\, l^4\; f(l^2) \;\; \text{etc.}
\end{aligned}
$$

Higher projectors are easily constructed.

$$
\int d^D l \prod_{i=1}^{2k+1} l_{\mu_i} f(l^2) = 0
$$

Evaluating Loop Integrals

Momenta are given in Minkowski space, which has no metric.
Need to perform a Wick rotation to Euclidean space.

$$
\begin{aligned}
k &= (k_0;\ \vec{k}_{D-1}) \\
k_0 &\rightarrow ik_0 \\
\int_{-\infty}^{+\infty} dk_0^M dk_1^M \ldots dk_D^M &\rightarrow i\int_{-\infty}^{+\infty} dk_0^E dk_1^E \ldots dk_D^E \\
k_M^2 &\rightarrow -k_0^2 - \vec{k}^2 = -k_E^2 \\
I &= \int d^D k_M \frac{1}{(k_M^2 - m^2)^l} \\
&= i(-1)^{-l} \int d^D k_E \frac{1}{(k_E^2 + m^2)^l} \\
\int \frac{d^D k_M}{(2\pi)^D} \frac{(k_M^2)^n}{(k_M^2 - m^2)^l} &= i(-1)^{n-m} \int \frac{d^D k_E}{(2\pi)^D} \frac{(k_E^2)^n}{(k_E^2 + m^2)^l}
\end{aligned}
$$

The D-Dimensional Integral

$$\int_0^\infty dr \frac{r^{D-1} \quad r^{2n}}{(r^2+m^2)^l} \quad = \quad (m^2)^{n-l+D/2} \int_0^\infty dx \frac{x^{2n+D-1}}{(1+x^2)^l}$$

$$= \quad \frac{1}{2} \int_0^\infty dy \frac{y^{n+D/2-1}}{(1+y)^l} = \frac{1}{2} B\left(n+\frac{D}{2}, l-n-\frac{D}{2}\right)$$

$$= \quad \frac{1}{2} \frac{\Gamma(n+D/2)\Gamma(l-n-D/2)}{\Gamma(l)}$$

$$I \quad = \quad i(-1)^{n-l} \frac{1}{(4\pi)^{D/2}} \frac{\Gamma(n+D/2)\Gamma(l-n-D/2)}{\Gamma(D/2)\Gamma(l)} \left(M^2\right)^{n-l+D/2} .$$

$$\Gamma(1+\varepsilon)\exp(\gamma_E \varepsilon) \quad = \quad 1 + \frac{\zeta_2}{2}\varepsilon^2 - \frac{\zeta_3}{3}\varepsilon^3 + \frac{9\zeta_4}{16}\varepsilon^4 - \frac{1}{5}\left(\zeta_5 + \frac{5}{6}\zeta_2\zeta_3\right)\varepsilon^5 \ldots$$

$$S_\varepsilon \quad = \quad \exp\left\{\frac{\varepsilon}{2}\left(\gamma_E - \ln(4\pi)\right)\right\}$$

The Final Integral

$$
\begin{aligned}
I &= i(-1)^{n-l} S_\varepsilon \frac{1}{16\pi^2} \exp\left[-\frac{\varepsilon}{2}\gamma_E\right] \frac{\Gamma(n+2+\varepsilon/2)\Gamma(l-n-2-\varepsilon/2)}{(1+\varepsilon/2)\Gamma(1+\varepsilon/2)\Gamma(l)} \\
&\quad \times \left(M^2\right)^{n-l+2+\varepsilon/2} \\
&= i(-1)^{n-l} S_\varepsilon \frac{1}{16\pi^2} B(n+2+\varepsilon/2, l-n-2-\varepsilon/2) \\
&\quad \times \exp\left[-\frac{\varepsilon}{2}\gamma_E\right] \frac{\left(M^2\right)^{n-l+2+\varepsilon/2}}{(1+\varepsilon/2)\Gamma(1+\varepsilon/2)}
\end{aligned}
$$

Beta-functions possess an ε-expansion free of γ_E.

$$
g^2 \quad \rightarrow \quad g^2(-\mu^2)^{\varepsilon/2} \rightarrow g^2 \left(\frac{M^2}{\mu^2}\right)^{\varepsilon/2} (M^2)^{n-l+2}
$$

Finally, put $S_\varepsilon = 1$ after all integrals have been performed. $\Longrightarrow$ $\overline{\rm MS}$ Scheme

Remember that $M^2 = M^2(x_1, ...x_k)$ - and there are lots of nontrivial integrals ahead!

Poles in Feynman Integrals

Feynman diagrams may diverge. The degree of divergence is expressed by the pole-strength in $O(1/\varepsilon^k)$.

In case of absence of infrared singularities, k corresponds to the loop order.

Example: previous integral with $n = l = 1$

$$
\begin{aligned}
I &= iS_\varepsilon \frac{1}{16\pi^2} \exp\left[-\frac{\varepsilon}{2}\gamma_E\right] \frac{\Gamma(3+\varepsilon/2)\Gamma(-2-\varepsilon/2)}{(1+\varepsilon/2)\Gamma(1+\varepsilon/2)\Gamma(1)} \\
&\quad \times \left(M^2\right)^{2+\varepsilon/2} \\
&= iaM^4 \left(\frac{M^2}{\mu^2}\right)^{\varepsilon/2} \left\{-\frac{2}{\varepsilon} + 1 - \left(\frac{1}{2} + \frac{\zeta_2}{4}\right)\varepsilon\right\} + O(\varepsilon^2) \\
a &= \frac{g^2}{16\pi^2} \\
x^\varepsilon &= 1 + \ln(x)\varepsilon + \frac{1}{2}\ln^2(x)\varepsilon^2 \ldots
\end{aligned}
$$

Removal of Poles: Renormalization $\rightarrow$ absorption into bare constants.

3. Genuine Feynman Parameterization

Since momentum integrals can always be performed, one may derive combinatoric expressions for Feynman-parameter integrals a priori.

$$I_G = \frac{\Gamma(\nu - lD/2)}{\prod_{j=1}^n \Gamma(\nu_j)} \int_{x_j \geq 0} \left[\prod_{j=1}^n dx_j x_j^{\nu_j - 1}\right] \delta\left(1 - \sum_{i=1}^n x_i\right) \frac{\mathcal{U}^{\nu-(l+1)D/2}}{\mathcal{F}^{\nu - lD/2}}$$

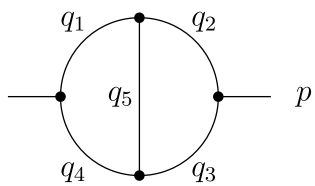
$\mathcal{U}, \mathcal{F}$ - Symanzik polynomials.

$$\begin{aligned}
\sum_{j=1}^n x_j(-q_j^2 + m_j^2) &= -\sum_{r=1}^l \sum_{s=1}^l k_r M_{rs} k_s + \sum_{r=1}^l 2k_r Q_r + J \\
\mathcal{U} &= \det[M] \\
\mathcal{F} &= \det[M](J + QM^{-1}Q)\frac{1}{\mu^2}
\end{aligned}$$

One may express the Symanzik polynomials $\mathcal{U}, \mathcal{F}$ also using the language of spanning trees and spanning 2-forests.

Genuine Feynman Parameterization

Example:



The Symanzik polynomials are given by:

$$
\begin{aligned}
\mathcal{U} &= (x_1 + x_4)(x_2 + x_3) + (x_1 + x_2 + x_3 + x + 4)x_5 \\
\mathcal{F} &= [(x_1 + x_2)(x_3 + x_4)x_5 + x_1 x_4 (x_2 + x_3) + x_2 x_3 (x_1 + x_4)] \\
&\quad \times \left(\frac{-p^2}{\mu^2}\right)
\end{aligned}
$$

Bogner, Weinzierl: 1002.3458

4. Methods to Evaluate Feynman Parameter Integrals

1. Integration by parts technique
2. Mellin-Barnes techniques
3. PSLQ: zero-dimensional integrals
4. Guessing: one-dimensional integrals
5. Generalized hypergeometric functions (and extensions)
6. Hyperlogarithms
7. Risch algorithms
8. Solution of master-integrals using difference and differential equations
9. Summation techniques: construction of difference and product fields
10. (multivalued) Almkvist-Zeilberger algorithm ... and others.

Integration by Parts Technique

Obtain a homogeneous difference equation for a Feynman-integral by using Gauß' Law.

$$\int d^D k \frac{\partial}{\partial k_\mu} p_\mu f(k,p) = 0$$

Example:

$$F(a) = \int d^D k \frac{1}{(k^2-m^2)^a} \quad Operator: \frac{\partial}{\partial k} k$$

$$0 = \int d^D k \left[\frac{D}{(k^2-m^2)^a} - \frac{2a(k^2-m^2+m^2)}{(k^2-m^2)^{a+1}}\right]$$

$$(D-2a)F(a) = 2am^2 F(a+1)$$

$$F(a) = \frac{D-2a+2}{2(a-1)m^2} F(a-1) = \frac{(-1)^a (1-D/2)_{a-1}}{\Gamma(a)\,(m^2)^{a-1}} I_1$$

$$I_1 = -i\pi^{D/2}\Gamma\left(1-\frac{D}{2}\right)\left(m^2\right)^{D/2-1}$$ [Master-Integral]

$$(\alpha)_n = \Gamma(\alpha+n)/\Gamma(\alpha)$$ [Pochhammer symbol]

$\Longrightarrow$ Difference or differential equations to be solved.

The PSLQ-Method

Seek an Integer Relation over a basis of special numbers out of a special class.

Example:

$$I = \int_0^1 dx \frac{\mathrm{Li}_3(x)}{1+x}$$

The integral is of "transcendentality" $\tau = 4$.

The expected HPL(1) basis is spanned by:

$\ln^4(2),\ \ln(2)\zeta_3,\ \ln^2(2)\zeta_2,\ \zeta_2^2,\ \mathrm{Li}_4(1/2)$.

Calculate this integral numerically to high number of digits, e.g. 40 digits.

$$I \approx 0.3395454690873598695906678484608602061388$$

The PSLQ algorithm yields:

$$I = -\frac{1}{12}\ln^4(2) + \frac{\pi^4}{60} + \frac{3}{4}\ln(2)\zeta_3 + \frac{1}{12}\ln^2(2)\pi^2 - 2\mathrm{Li}_4\left(\frac{1}{2}\right)$$

$$\zeta_{2k} = (-1)^{k-1}\frac{(2\pi)^{2k}B_{2k}}{2(2k)!}; \quad B_n$$ [Bernoulli number]

Guessing Difference Equations

It is often easier to calculate Mellin moments for a quantity for fixed values of N than to derive the relation for general values of N in the first place. If the quantity under consideration is known to be recurrent than its difference equation is of finite order and degree.

$$\exists \quad \sum_{k=0}^{O} P_k^{(l)}(N) F(k+N) = 0; \qquad \text{max}\{l\} \text{ - degree; } O \text{ - order}$$

Example:

$$-(N+1)^3 F(N) - (3N^2 - 9N - 7) F(N+1) + (N+2)^3 F(N+2) = 0$$

$$F(1) = 1; \quad F(2) = \frac{1}{8}$$

Solution: $$F(N) = \sum_{k=1}^{N} \frac{1}{k^3} = S_3(N)$$

Guessing Difference Equations

Solution of large problems.

Assume you would like to calculate the massless 3-loop Wilson coefficients in deep-inelastic scattering using this method. How many moments would you need and how do they look like ?

About 5200 moments are needed. The largest ones are ratios of #13000/#13000 digits. They are calculable in 15 min.

After 3 weeks you will find a difference equation of degree ~ 1000 and order 35, if you have a reasonable computer (100 Gbyte RAM).

After another week you have the solution as function of N.

Problem: It is sophisticated to obtain the input a priori. Combined solution-methods do work, however, to $O(1500)$ moments.

Recent result: a 3-loop anomalous dimension computed from scratch.

Generalized Hypergeometric Functions

At lower number of legs and/or loops Feynman integrals happen to be represented by these functions.

After suitable mappings these functions have compact representations in infinite (multiple) absolutely convergent sums.

This allows for the Laurent-expansion in ε under the summation operator.

Important Examples:

1. $B(a,b)$
2. ${}_pF_q(a_i;b_j;x)$; always single sums
3. Appell functions; double sums
4. Kampé de Feriet functions, Horn functions and higher; more sums

The sums may be expanded and summed using algorithms like nestedsums, xsummer, HarmonicSums, Sigma, EvaluateMultiSums

Generalized Hypergeometric Functions

Example:

Integrals of the following type emerge:

$$
\begin{aligned}
I_1(z) &= \int_0^1 dy y^\delta (1-y)^\eta \int_0^1 dx x^{\beta-1}(1-x)^{\gamma-\beta-1}(1-xyz)^{-\alpha} \\
&= B(\beta,\gamma-\beta) \int_0^1 dy y^\delta (1-y)^\eta \;\; {}_2F_1(\alpha,\beta;\gamma;yz) \\
&= B(\beta,\gamma-\beta) B(\delta,\eta-\delta) \;\; {}_3F_2(\delta,\alpha,\beta;\eta,\gamma;z)
\end{aligned}
$$

All ${}_pF_q$'s have single series representations. One series counts as one integral.

$$
{}_pF_q(a_1,...,a_p;b_1...b_q;z) = \sum_{k=0}^{\infty} \frac{(a_1)_k...(a_p)_k}{(b_1)_k...(b_q)_k} \frac{z^k}{k!}
$$

Mellin-Barnes Integrals

Used to resolve sums in denominators. In a way a counterpart to the binomial expansion of numerators.

$$\frac{1}{(A+B)^c} = \oint_{-i\infty+\gamma}^{i\infty+\gamma} d\sigma \frac{\Gamma(-\sigma)\Gamma(\sigma+c)}{\Gamma(c)} A^\sigma B^{-\sigma+c}$$

The contour integral covers the residues and closes either left or right.

One may use Beta-functions and (generalized) hypergeometric functions to perform the integrals further.

Important: One has to be able to undo the Mellin-Barnes integral(s).

1. Barnes Lemmas; map to Γ-functions, (in the most simple cases).
2. Residue theorem $\Longrightarrow$ leads to nested sums $\Longrightarrow$ use SIGMA.

The Method of Hyperlogarithms

1. Assume that a Feynman parameterization exists, which is multilinear in all parameters.
2. Assume that the integration procedure maintains this property in one order of integrations [Fubini sequence].
3. Assume, the integral has no poles in ε; or find a method to deal with it.
4. Then: the integral can be organized fully in Hyperlogarithms.
5. Hyperlogarithms are Kummer-Poincaré-Lappo-Danielevsky-Chen-(Goncharov) iterated integrals over an alphabet, the letters of which contain further integration variables in the multilinear sense.
6. Very many of them have coefficient zero in the final result.
7. In various cases the multi-linearity may not persist, but a solution can be found as well in extended function spaces.

$$L_{a_1,...a_k}(x) = \int_0^x \frac{dx_1}{x_1 - a_1} \int_0^{x_1} \frac{dx_2}{x_2 - a_2} ... \int_0^{x_{k-1}} \frac{dx_k}{x_k - a_k}, a_l \in \mathbb{C}$$

Summation Techniques

The integrals can usually be traded for a lower number of sums (finite or infinite).

Solve these sums for N and/or in terms of special constants.

Principal Idea:

1. Sums may be considered to form vector spaces, algebras, and finally fields.
2. Consider difference and product fields.
3. Implement relations due to difference equations
4. Telescoping, creative telescoping, and other principles.
5. Try to solve the recurrences; possible for most sums occurring from Feynman integrals.

Telescoping: Find a function $g(k)$ such

$$
\begin{aligned}
f(k) &= g(k+1) - g(k) \\
F(N) &= \sum_{k=1}^{N} f(k) = g(N+1) - g(1)
\end{aligned}
$$

$\Longrightarrow$ nested sums algebras $\Longrightarrow$ bases

Differential Equations

The IBPs deliver a vast amount of differential equations forming systems, which are nested hierarchically.

Provide boundary conditions [usually using other methods]

Perform uncoupling of these systems

- In case of complete 1st order uncoupling: $\exists$ complete solution algorithm in case of any basis choice for 1 parameter systems

All solutions are iterative integrals over whatsoever alphabet:
$\int_0^x dy f_a(y) H_{\vec{b}}(y)$

- Irreducible nth order systems $(n \geq 2)$: present target of research even in mathematics; good prospects in case of 2nd order systems [convergent near integer power series (CIS)]

At least one function is given by a definite integral, others iterate on.

$\Longrightarrow$ iterated integral algebras $\Longrightarrow$ bases

5. Function Spaces

Sums

Harmonic Sums

$$\sum_{k=1}^{N} \frac{1}{k} \sum_{l=1}^{k} \frac{(-1)^l}{l^3}$$

gen. Harmonic Sums

$$\sum_{k=1}^{N} \frac{(1/2)^k}{k} \sum_{l=1}^{k} \frac{(-1)^l}{l^3}$$

Cycl. Harmonic Sums

$$\sum_{k=1}^{N} \frac{1}{(2k+1)} \sum_{l=1}^{k} \frac{(-1)^l}{l^3}$$

Binomial Sums

$$\sum_{k=1}^{N} \frac{1}{k^2} \binom{2k}{k} (-1)^k$$

Integrals

Harmonic Polylogarithms

$$\int_0^x \frac{dy}{y} \int_0^y \frac{dz}{1+z}$$

gen. Harmonic Polylogarithms

$$\int_0^x \frac{dy}{y} \int_0^y \frac{dz}{z-3}$$

Cycl. Harmonic Polylogarithms

$$\int_0^x \frac{dy}{1+y^2} \int_0^y \frac{dz}{1-z+z^2}$$

root-valued iterated integrals

$$\int_0^x \frac{dy}{y} \int_0^y \frac{dz}{z\sqrt{1+z}}$$

iterated integrals on CIS fct.

$$\int_0^z dx \frac{\ln(x)}{1+x} \, {}_2F_1\left[\begin{matrix} \frac{4}{3}, \frac{5}{3} \\ 2 \end{matrix}; \frac{x^2(x^2-9)^2}{(x^2+3)^3}\right]$$

Special Numbers

multiple zeta values

$$\int_0^1 dx \frac{\mathrm{Li}_3(x)}{1+x} = -2\mathrm{Li}_4(1/2) + \ldots$$

gen. multiple zeta values

$$\int_0^1 dx \frac{\ln(x+2)}{x-3/2} = \mathrm{Li}_2(1/3) + \ldots$$

cycl. multiple zeta values

$$\mathbf{C} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2}$$

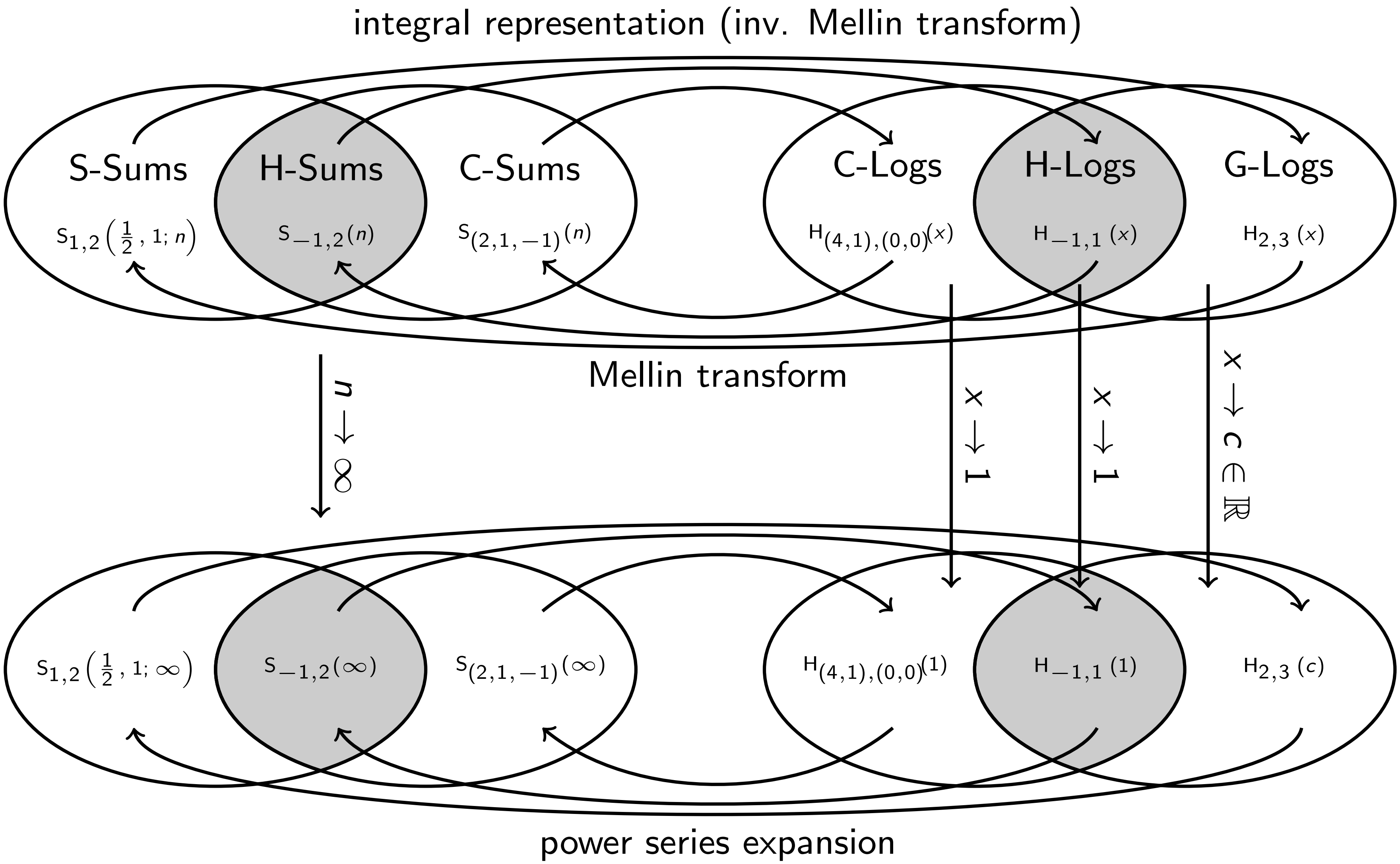
associated numbers

$$\mathrm{H}_{8,w_3} = 2\mathrm{arccot}(\sqrt{7})^2$$

associated numbers

$$\int_0^1 dx \, {}_2F_1\left[\begin{matrix} \frac{4}{3}, \frac{5}{3} \\ 2 \end{matrix}; \frac{x^2(x^2-9)^2}{(x^2+3)^3}\right]$$

shuffle, stuffle, and various structural relations $\Longrightarrow$ algebras



square-root valued letters $\Longleftrightarrow$ nested binomial sums $\binom{2i}{i}$

non-iterative iterative integrals $\Longrightarrow$ Iterate on CIS ${}_2F_1$'s: rat. argument

(special cases: complete elliptic integrals) (found last week)

6. Some Recent Results

Massless 3-loop results: anomalous dimensions & Wilson coefficients: harmonic sums only

Massive 2-loop results: Wilson coefficients: harmonic sums only

Massive 3-loop results: DIS Wilson coefficients and OMEs
all other structures above; also expected for $pp \to t\bar{t}$ and similar reactions at NNLO.

Status: 7 of 8 OMEs $\surd$; 4 of 5 Wilson Coefficients $\surd$.

$\delta m_c^{\rm DIS}$: 6.5% $\Longrightarrow 3\%$ after NNLO corrections are finished

Only then: NNLO DIS QCD analyses at larger scales Q^2 are consistently possible. Improved value of $\alpha_s(M_Z^2)$ and of $xG(x, Q^2)$

The relevance of small x predictions can only be judged then.
PS-case: leading term by CCH correct, but nowhere dominant
[found 23 years after]

Lots of new technologies and mathematical insight for QFT as a whole.

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