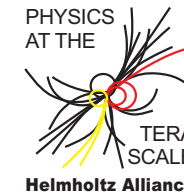
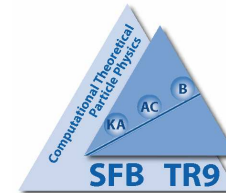


Heavy Quark Production in Deep-Inelastic Scattering

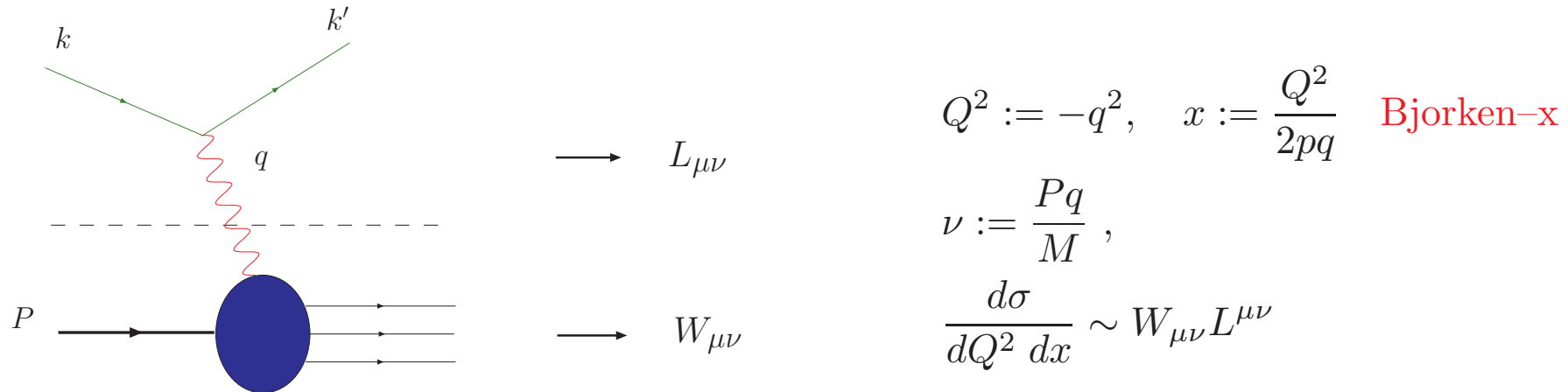
Johannes Blümlein



- Introduction
- Light and Heavy Parton Contributions
- Leading and Next-to-leading Order Contributions
- Polarized Heavy Flavor
- Trading Final- for Initial States
- Towards 3-Loop Precision
- Conclusions

1. Introduction

Deep-Inelastic Scattering (DIS):



The partonic picture of the proton at short distances:

[Feynman, 1969; Bjorken, Paschos, 1969.]

The operator picture of the proton at short distances - light cone expansion:

[Wilson, 1968; Bjorken 1969; Brandt & Preparata 1970; Frishman 1971.]

Notion of Twist (necessary): consistent renormalization [Gross and Treiman, 1971]

\implies Both concepts yield the same result at the level of Twist $\tau = 2$

- Some care is needed in the polarized case : $g_2(x, Q^2)$.

When is a parton ?

- The applicability of the **parton picture** rests on the comparison of two times.
[e.g. Drell, Yan, Levy, 1969]
- Consider the infinite momentum frame to describe deep-inelastic scattering, moving with momentum $P \gg M$.
- The 2 characteristic times are :

τ_{int} - the interaction time of the virtual gauge boson with the hadron.

τ_{life} - the life-time of individual partons.

- This is a non-covariant description, turning into a covariant one later.

$$\tau_{\text{int}} \sim \frac{1}{q_0} = \frac{4Px}{Q^2(1-x)}$$

$$\tau_{\text{life}} \sim \frac{1}{\sum_i (E_i - E)} = \frac{2P}{\sum_i (k_{\perp,i}^2 + m_i^2)/x_i - M^2}$$

- The ratio of these two times is covariant & independent of P .

- The single parton model is applicable iff:

$$\frac{\tau_{\text{life}}}{\tau_{\text{int}}} \gg 1 .$$

Light partons only: $m_i^2, M^2 \approx 0, \forall k_{\perp,i}^2 \equiv k_{\perp}^2, x_1 = x, x_2 = 1 - x$ as a model.

$$\frac{\tau_{\text{life}}}{\tau_{\text{int}}} = \frac{Q^2(1-x)^2}{2k_{\perp}^2} \gg 1$$

i.e.: $Q^2 \gg k_{\perp}^2, x$ neither close to 1 or 0.

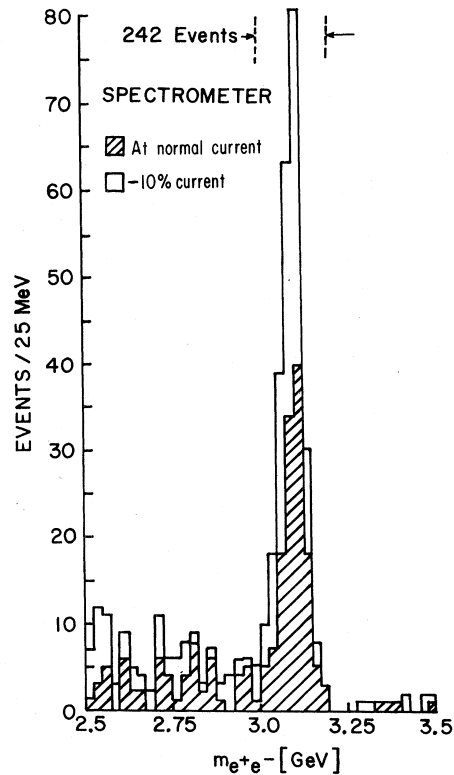
The latter condition follows from $|P_{\text{eff}}| \gg 1$.

- There is no parton model at low Q^2 . In this region also the light-cone expansion breaks down. This is the place, where vector meson dominance and other thoroughly non-perturbative phenomena live.

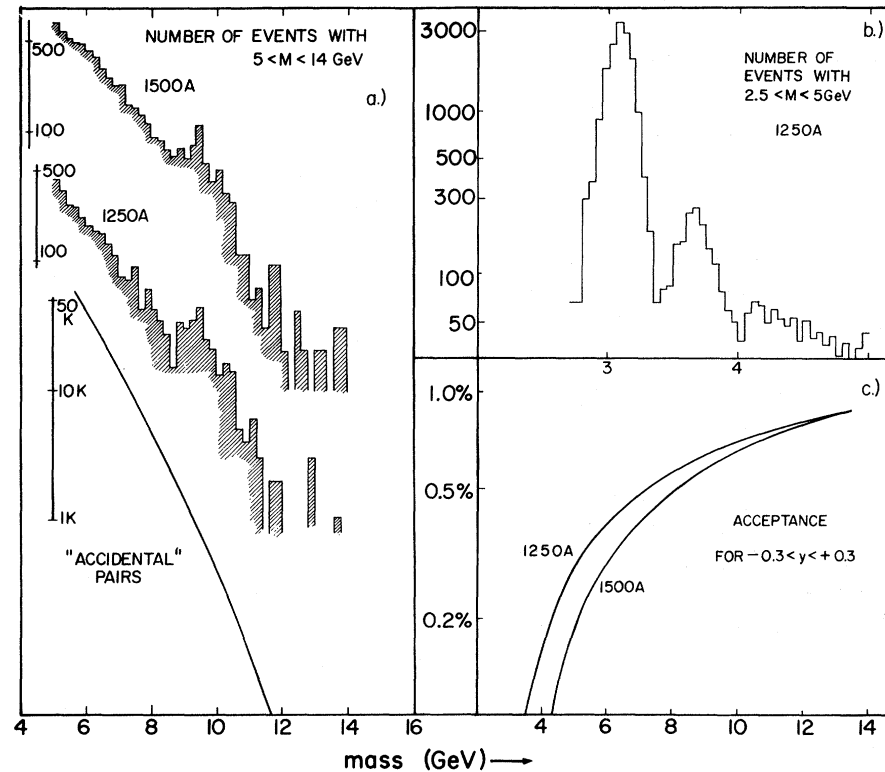
There are neither small- nor large- x parton densities.

- Light partons: $u, d, s, \bar{u}, \bar{d}, \bar{s}, g$.

The Discovery of Heavy Quarks



J/Ψ [Aubert *et al.*, 1974.]



Υ [Herb *et al.*, 1977.]

- Masses of **charm** and **bottom** [PDG, 2006.]: $m_c \approx 1.3 \text{ GeV}$, $m_b \approx 4.2 \text{ GeV}$

When is a parton ?

- strange quarks, despite $m_s \sim 100 \text{ MeV} \lesssim \Lambda_{\text{QCD}}$, are dealt with as massless.

Can heavy “partons” live in the nucleon ? [Brodsky, Hoyer, Peterson, Sakai, 1980]

- Intrinsic Charm
- Use old-fashioned perturbation theory.

$$Prob \sim \frac{1}{(M^2 - \sum_{i=1}^5 m_{\perp}^2/x_i)^2}, \quad m_{4,5} \rightarrow \infty$$

A more general question:

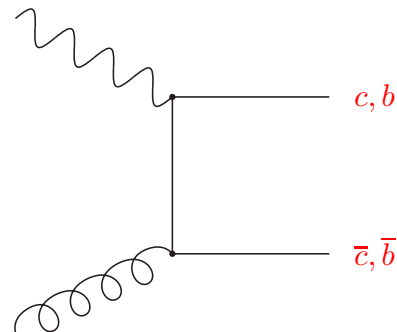
What is the heavy flavor content of DIS Structure Functions ?

Introduction

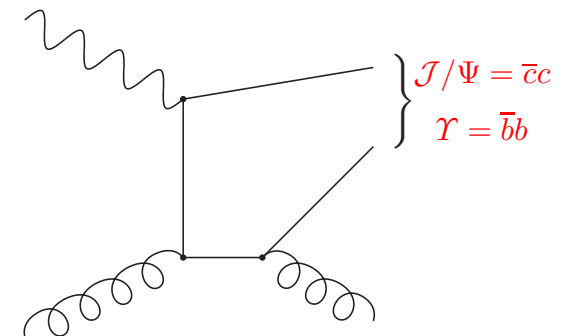
- Virtual Heavy Quark Loops: c, b, t .
- Heavy Quark final states in $F_i(x, Q^2), g_i(x, Q^2) : \theta(W^2 - (2m_H + m_p)^2)$ and stronger.
- Both effects lead to different scaling violations in wide ranges of Q^2 .
- Another Aspect: Inclusive Effects vs. Exclusive Reactions.
- Inclusive Effects: sum over all hadronic states in Fock space.
- Exclusive Reactions: tag heavy flavor in the final states.

These emerge as open heavy flavor or (in-)elastic heavy resonance production.

- open $c(b)$ production:
 $D_u = \bar{u}c, \dots$
 $B_u = \bar{u}b, \dots$

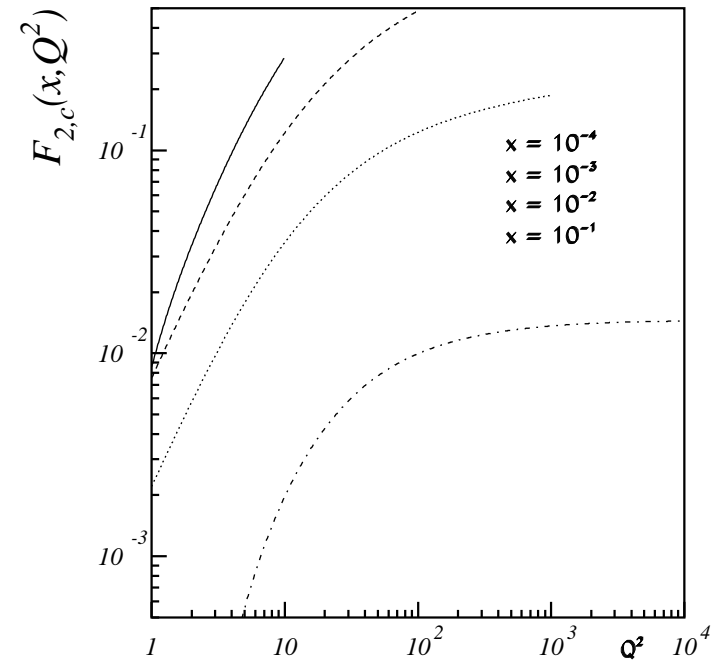
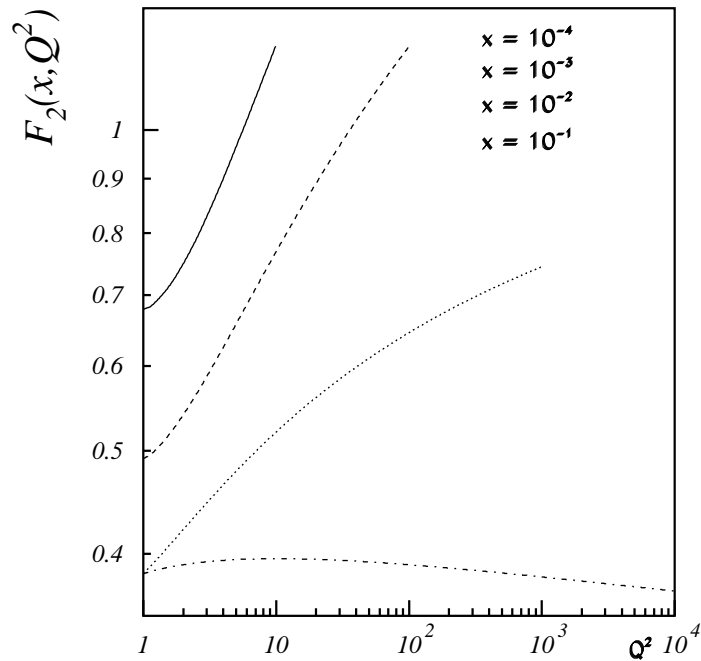


- heavy quark resonances:
 $\bar{c}c = \mathcal{J}/\Psi$
 $\bar{b}b = \Upsilon$.



Introduction

Let us compare the scaling violations in the massless and massive case.



LO charm contributions : PDFs from [Alekhin, Melnikov, Petriello, 2006.]

Introduction

- J/ψ and Υ production in the color singlet and octet contributions.

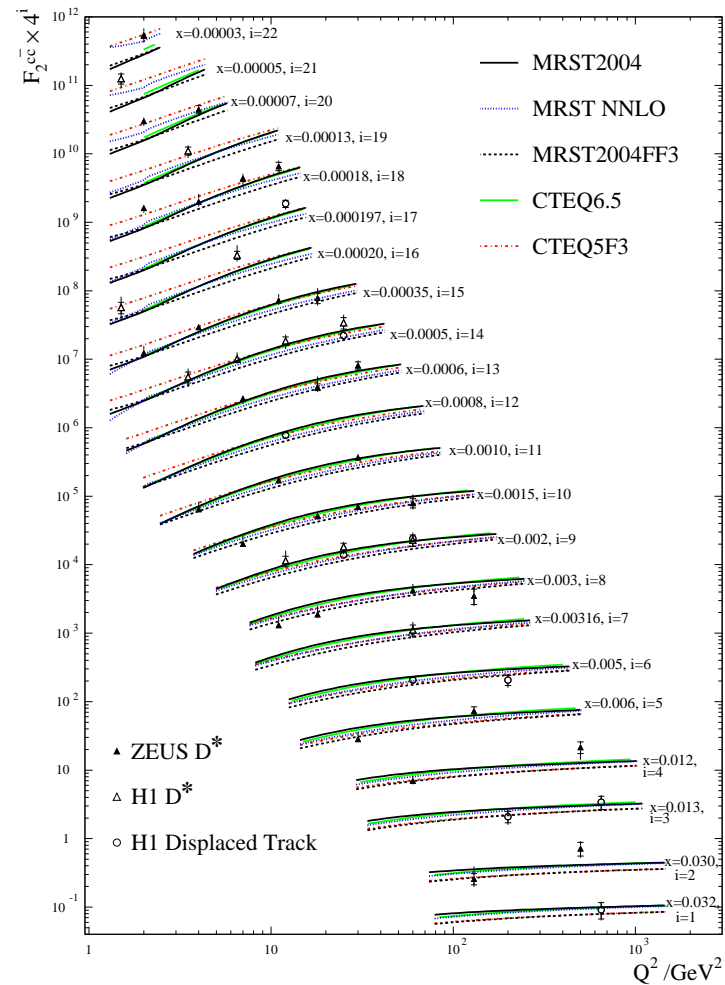
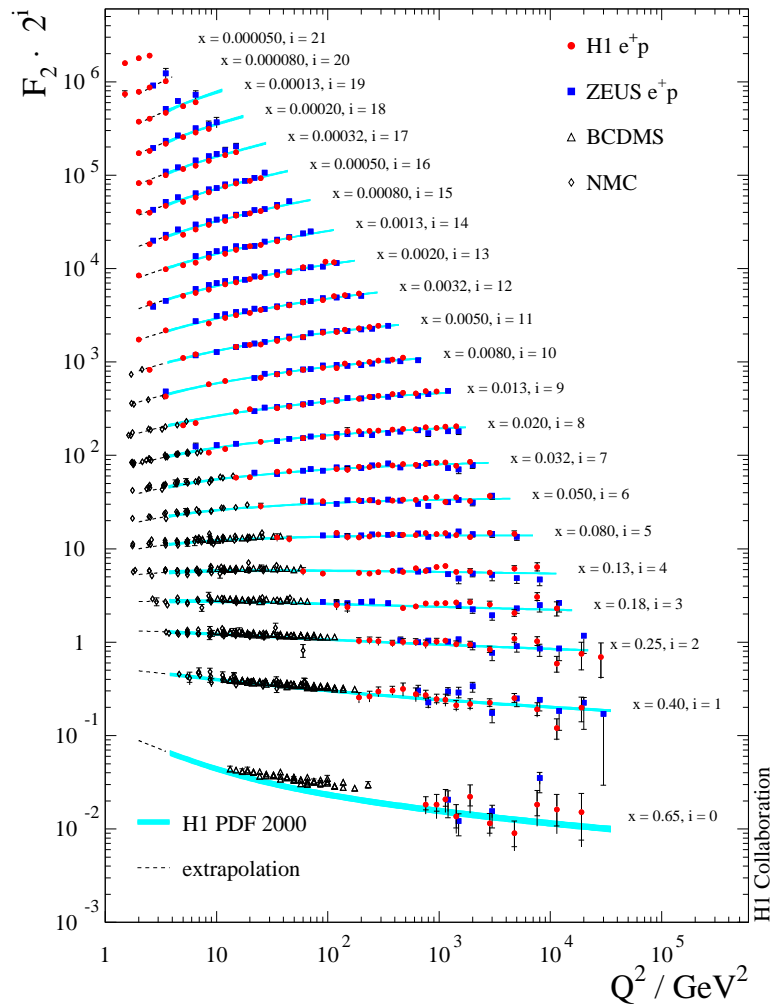
$$\frac{d\sigma}{dt} = \frac{1}{16\pi s^2} 64g_s^4 e_Q^2 M_J^2 A^2 \frac{1}{12} \frac{s^2(s - M_J^2)^2 + t^2(t - M_J^2)^2 + u^2(u - M_J^2)^2}{(s - M_J^2)^2(t - M_J^2)^2(u - M_J^2)^2}$$

[Berger & Jones, 1980; Rückl & Baier, 1983; Körner, Cleymans, Kuroda & Gounaris, 1983; Krämer, Zunft & Steegborn, 1994; Krämer, 1995, Kniehl & Kramer, 1997; Cacciari, Greco & Krämer, 1997]

- Observation of charmonium in DIS [Aubert *et al.*, 1983.]
- These processes are rather sensitive to the gluon distribution $G(x, Q^2)$.
- Can one probe m_t due to virtual excitations?

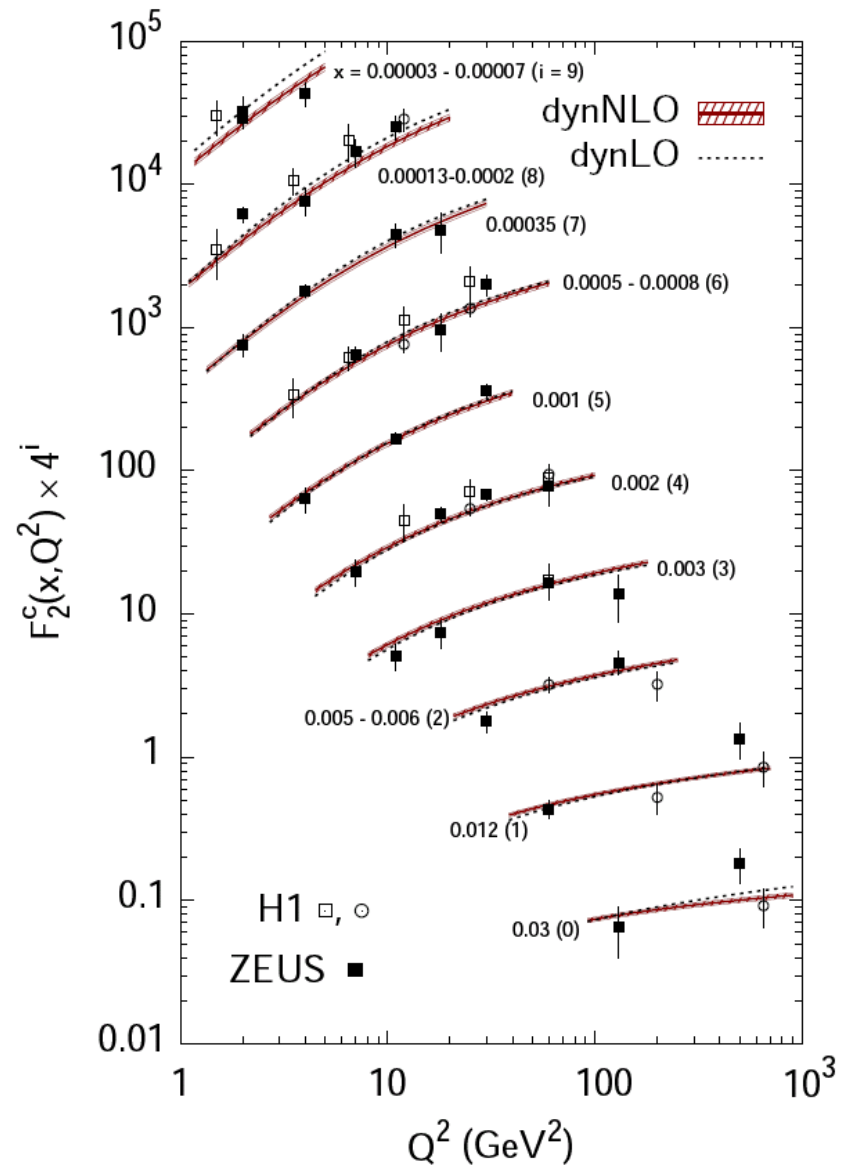
[I do not discuss diffractive, pomeron-induced or photo & VMD production of open and resonant heavy flavor. [Jung, Schuler, et al. 1990/92]]

2. Light and Heavy Quark Contributions

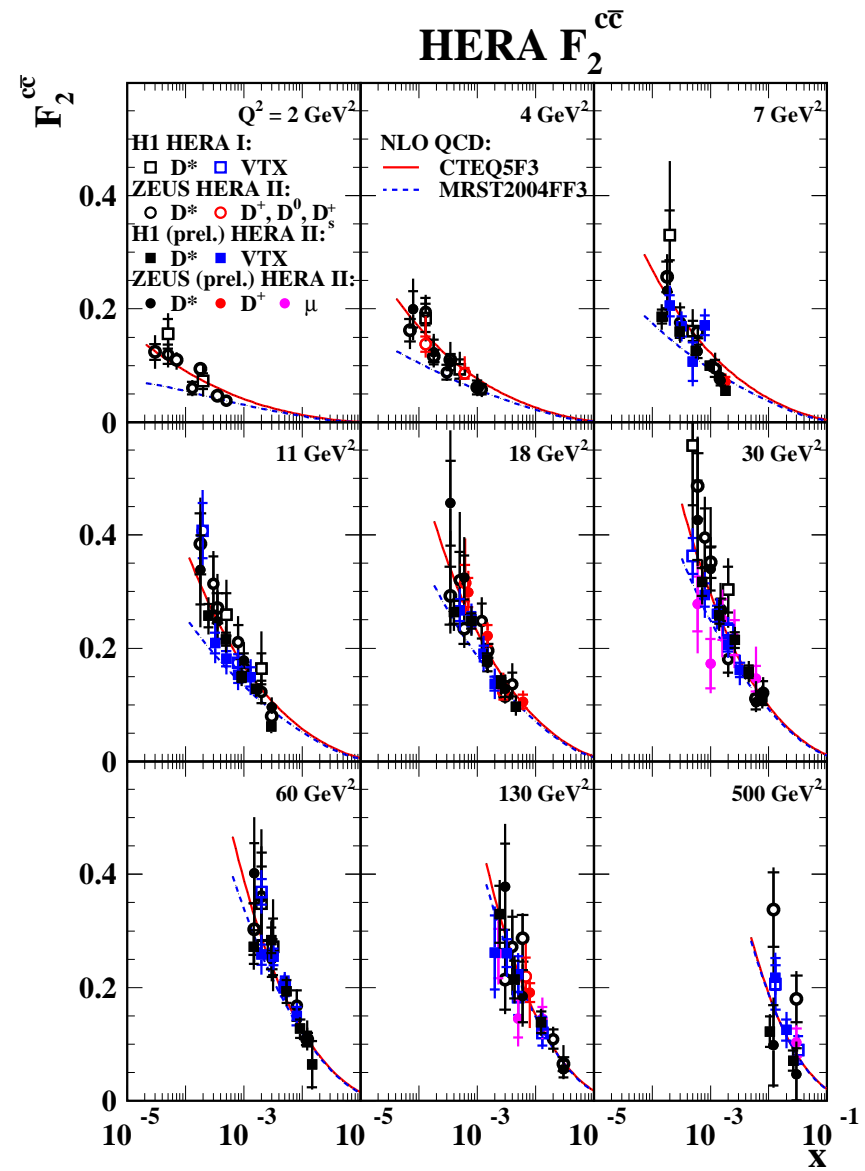


[Thompson, 2007]

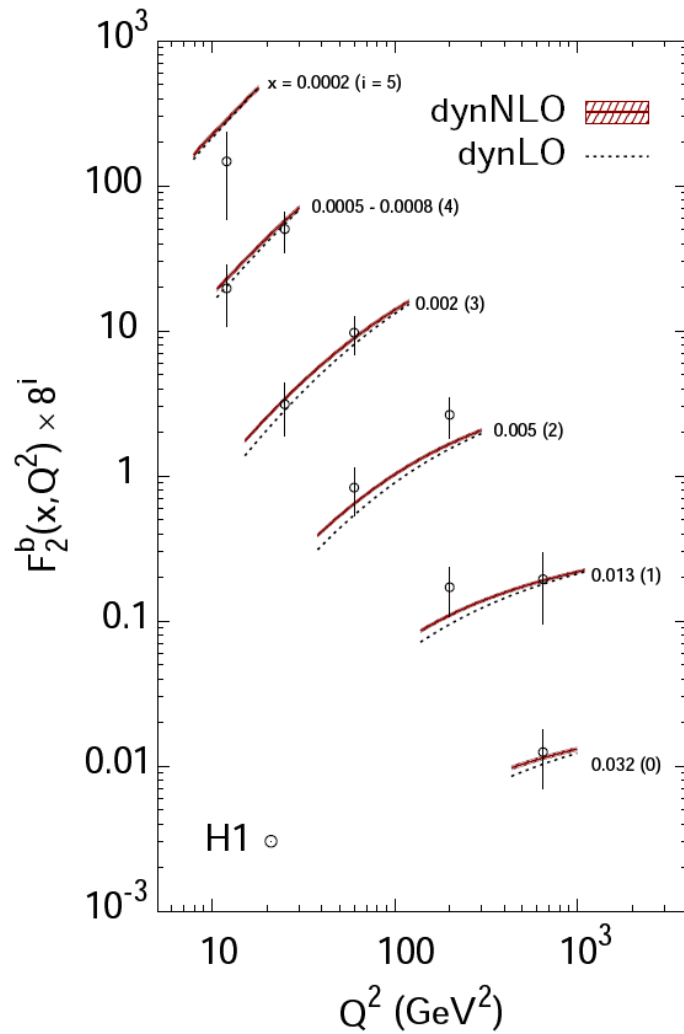
High statistics for F_2 and $F_2^{c\bar{c}}$ \implies Accuracy will increase in the future.



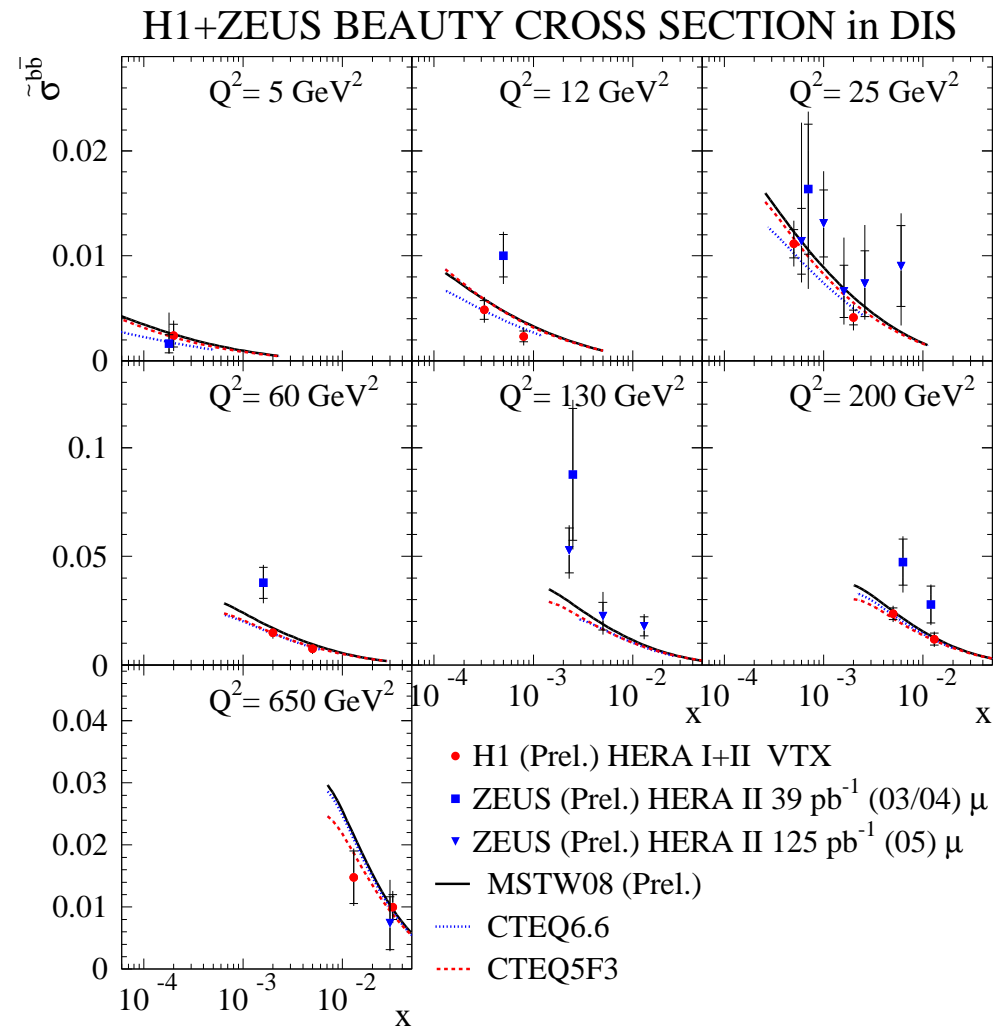
[Glück, Jimenez-Delgado, Reya, 2007.]



[Krüger (H1 and Z. Coll.), 2008.]



[Glück, Jimenez-Delgado, Reya, 2007.]



[Krüger (H1 and Z. Coll.), 2008.]

Intrinsic Charm

- Charm quarks as massive partons in the IMF in the infinite mass limit.

[Why not b and t too ?]

$$P(x_4, x_5) = N_5 \frac{x_4^n x_5^n}{(x_4 + x_5)^n} \theta(W^2 - (2m_H + m_p)^2)$$

$$P(x) = \frac{N_5}{2} x^2 \left[\frac{1}{3} (1-x)(1+10x+x^2) + 2x(1+x) \ln(x) \right] \theta(W^2 - (2m_H + m_p)^2),$$

$$n = 2, N_5 = 36$$

$n = 2$ corresponds to the usual Fock space [There may be other phenomenological choices.]; N_5 is chosen to yield a $\sim 1\%$ overall effect.

- NLO calculation : [Hoffmann and Moore, 1983]
- Phenomenological data analyzes: [Smith, Harris, R. Vogt, 1996, Gunion and R. Vogt, 1997]

Effect: $\leq 1\%$ compatible with 0 within errors.

Slow Rescaling

- Neutral or Charged currents hit massive fermion lines m_i, m_f inside a massive target M . [Georgi & Politzer 1976; Barnett 1976; K. Ellis, Parisi, Petronzio, 1976; Barbieri, J. Ellis, Gaillard, G. Ross, 1976; Brock, 1980]

$$\xi = \frac{Q^2 + m_f^2 - m_i^2 + \sqrt{(Q^2 + m_f^2 - m_i^2)^2 + 4m_i^2 Q^2}}{2 \left(\nu + \sqrt{\nu^2 + M^2 Q^2} \right)}$$

$$\xi(m_i = 0, M = 0) = x \left[1 + \frac{m_f^2}{Q^2} \right]$$

$$\xi(m_i = m_f, M = 0) = \frac{x}{2} \left[1 + \sqrt{1 + \frac{4m_f^2}{Q^2}} \right]$$

- Purely kinematic approach. Very early approach - too naive to substitute the description of heavy flavor.
- Still some numeric success for $s(d) \rightarrow \bar{c}$ inclusive charged current transitions.
- Heavy Quark parton densities cannot be coined in this way.

The **hadronic tensor** cannot be calculated perturbatively. Using symmetry considerations, it can be decomposed into several scalar **structure functions**. For **DIS** via **single photon exchange**, it is given by:

$$W_{\mu\nu}(q, P, s) = \frac{1}{4\pi} \int d^4\xi \exp(iq\xi) \langle P, s | [J_\mu^{em}(\xi), J_\nu^{em}(0)] | P, s \rangle$$

$$\text{unpol.} \left\{ \begin{aligned} &= \frac{1}{2x} \left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right) F_L(x, Q^2) + \frac{2x}{Q^2} \left(P_\mu P_\nu + \frac{q_\mu P_\nu + q_\nu P_\mu}{2x} - \frac{Q^2}{4x^2} g_{\mu\nu} \right) F_2(x, Q^2) \\ &\text{pol.} \left\{ \begin{aligned} &-\frac{M}{2Pq} \varepsilon_{\mu\nu\alpha\beta} q^\alpha \left[s^\beta g_1(x, Q^2) + \left(s^\beta - \frac{sq}{Pq} p^\beta \right) g_2(x, Q^2) \right] . \end{aligned} \right. \end{aligned} \right.$$

For **light quarks**: In the Bjorken limit, $\{Q^2, \nu\} \rightarrow \infty$, x fixed, at twist $\tau = 2$ -level:

$$\underbrace{F_i(x, Q^2)}_{\text{structure functions}} = \sum_j \underbrace{C_i^j \left(x, \frac{Q^2}{\mu^2} \right)}_{\text{Wilson coefficients, perturbative}} \otimes \underbrace{f_j(x, \mu^2)}_{\text{parton densities, non-perturbative}},$$

This representation applies both to light and heavy quarks.

Evolution of Light Quark Distributions

- The **scaling violations** are described by the **splitting functions** $P_{ij}(x, a_s)$.
- They describe the **probability** to find parton i radiated from parton j and carrying its momentum fraction x .
- They are related to the **anomalous dimensions** via a **Mellin-Transform**:

$$\mathbf{M}[f](N) := \int_0^1 dz z^{N-1} f(z) , \quad \gamma_{ij}(N, a_s) := -\mathbf{M}[P_{ij}](N, a_s) .$$

- The **splitting functions** govern the **scale-evolution** of the **parton densities**.

$$\frac{d}{d \ln Q^2} \begin{pmatrix} \Sigma(N, Q^2) \\ G(N, Q^2) \end{pmatrix} = - \begin{pmatrix} \gamma_{qq} & \gamma_{qg} \\ \gamma_{gq} & \gamma_{gg} \end{pmatrix} \otimes \begin{pmatrix} \Sigma(N, Q^2) \\ G(N, Q^2) \end{pmatrix} ,$$

$$\frac{d}{d \ln Q^2} q_{NS}(N, Q^2) = -\gamma_{qq,NS} \otimes q_{NS} .$$

- The **singlet light flavor density** is defined by

$$\Sigma(n_f, \mu^2) = \sum_{i=1}^{n_f} (f_i(n_f, \mu^2) + \bar{f}_i(n_f, \mu^2)) .$$

- The **anomalous dimensions** are presently known at NNLO [Moch, Vermaseren, Vogt, 2004] and N³LO for NS⁺, N=2 [Baikov & Chetyrkin, 2006]

Heavy Quark Contributions emerge in the Wilson Coefficients only.

⇒ This description is called : Fixed Flavor Scheme, $N_{\text{light}} = 3$.

Any fixed twist calculation of DIS heavy flavor contributions uses this frame.

The range of validity of this approach is discussed later.

For some applications one may construct the associated variable flavor number scheme.

3. Leading and Next-to-Leading Order Contributions

Leading Order : $F_{2,L}(x, Q^2)$ [Witten, 1976; Babcock, Sivers, 1978; Shifman, Vainshtein, Zakharov, 1978; Leveille, Weiler, 1979; Glück, Reya, 1979; Glück, Hoffmann, Reya, 1982.]

$$F_{2,L}(x, Q^2) = e_Q^2 a_s(Q^2) \int_{ax}^1 \frac{dy}{y} C_{F_2(F_L)}^{(1)}\left(\frac{x}{y}, m_Q^2, Q^2\right) G(y, Q^2)$$

$$C_{F_2}^{(1)}(z, m_Q^2, Q^2) = 8T_R \left\{ \beta \left[-\frac{1}{2} + 4z(1-z) + \frac{m_Q^2}{Q^2} z(2z-1) \right] + \left[-\frac{1}{2} + z - z^2 + 2\frac{m_Q^2}{Q^2} z(3z-1) - 4\frac{m_Q^4}{Q^4} z^2 \right] \ln \left(\frac{1-\beta}{1+\beta} \right) \right\}$$

$$C_{F_L}^{(1)}(z, m_Q^2, Q^2) = 16T_R \left[z(1-z)\beta - \frac{m_Q^2}{Q^2} z^2 \ln \left| \frac{1+\beta}{1-\beta} \right| \right]$$

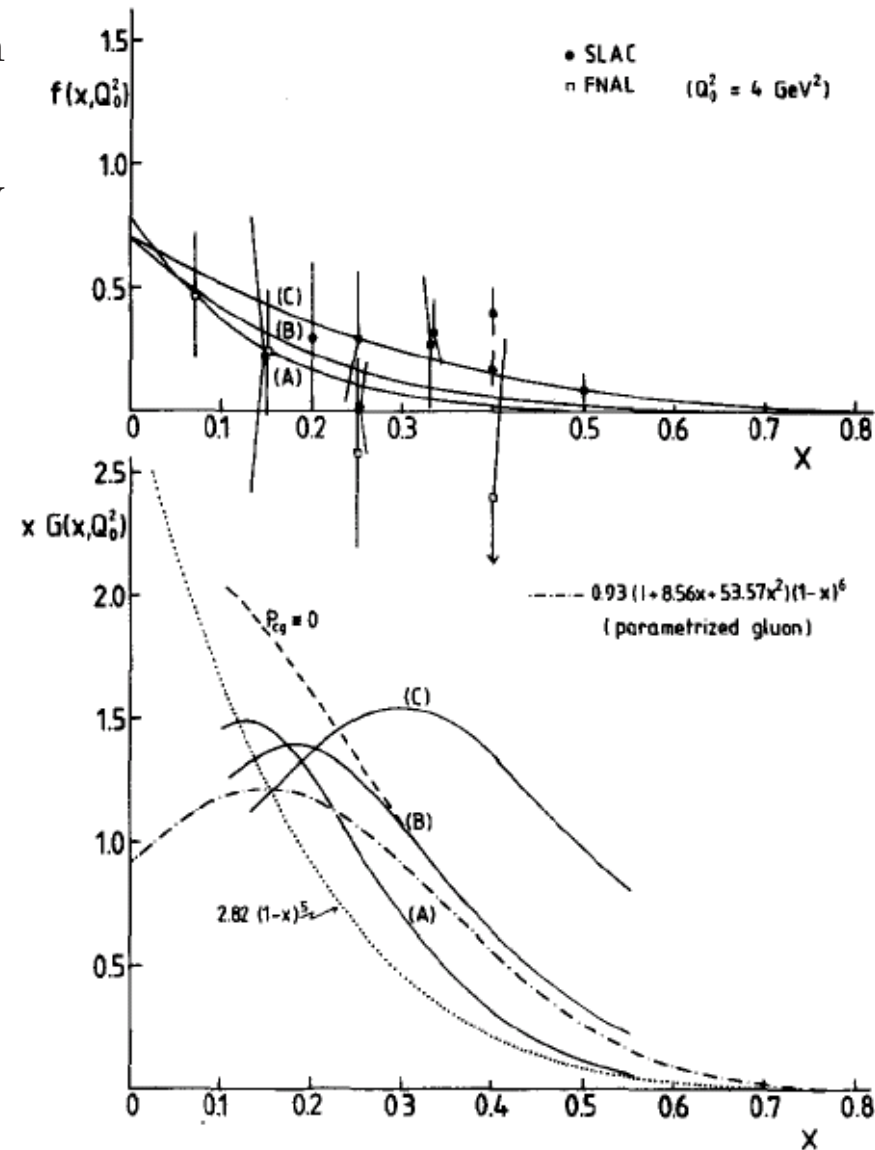
$$\beta = \sqrt{1 - \frac{4m_Q^2 z}{1-z}}, \quad a = 1 + \frac{4m_Q^2}{Q^2}$$

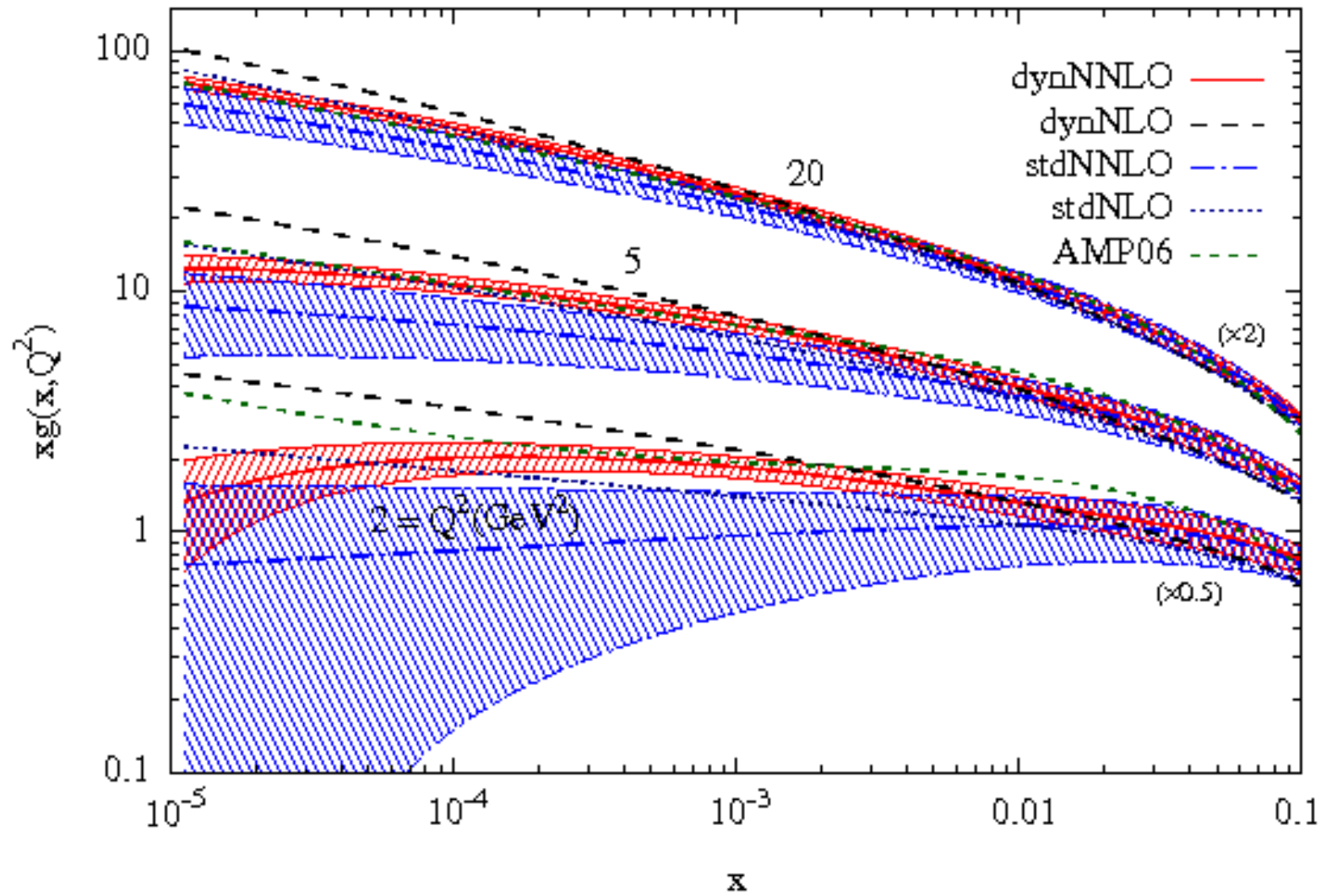
- \exists non-power and power terms m_Q^2/Q^2
- Power corrections always remain process dependent @ twist-2
- Never ever tailor on power corrections.

Unfolding the Gluon Distribution

- **Gluon** carries roughly **50%** of the proton momentum.
- **Heavy quark production** excellent way to extract **Gluon density** via measurement of
 - scaling-violations of F_2
 - $F_L^{Q\bar{Q}}$
- First extraction of the **gluon density** including **heavy quark** effects by [Glück, Hoffmann, Reya, 1982.]:
 - Unfold the **gluon density** via

$$G(x, Q^2) = P_{qg}^{-1} \otimes \left[\frac{f(x, Q^2)}{x} - \frac{2}{3} P_{cg} \otimes G(x, Q^2) \right].$$





[Jimenez-Delgado, Reya, 2008.]; other fits: MSTW, CTEQ, AMP, BB, NN-collab

Threshold Resummation

- Soft resummation may be applied in the threshold region at fixed small rapidity of the produced heavy quark pair. [Laenen & Moch, 1998; Alekhin & Moch, 2008]

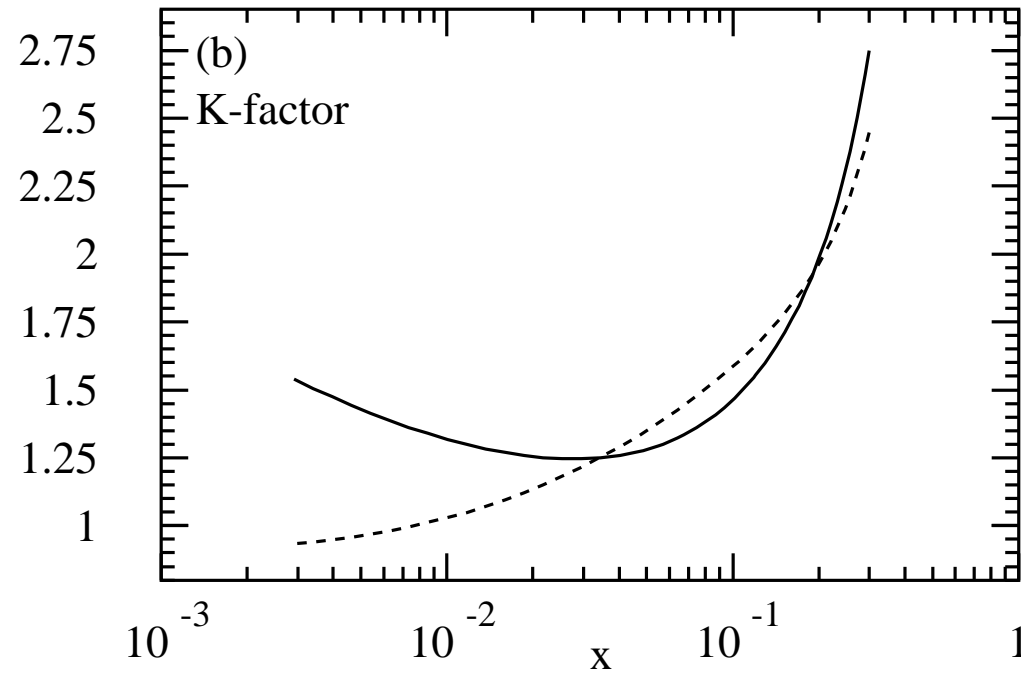
$$\begin{aligned} \frac{d^3\sigma(x, M^2, \theta, y, Q^2)}{dM^2 d\cos\theta dy} &= \frac{1}{S'^2} \int dz \int \frac{dx'}{x'} \Phi_{g/P}(x', \mu^2) \delta(y - \ln(1/x')/2) \delta(z - 4m^2 x/Q^2/(x' - x)) \\ &\times \omega(z, \theta, (Q^2/\mu^2), (M^2/\mu^2), a_s) \end{aligned}$$

$$\omega^{(k)}(s', \theta, M) = K^{(k)} \sigma^{\text{Born}}(s', \theta, M)$$

$$\begin{aligned} K^{(1)} &= 4a_s \left[2C_A \left[\frac{\ln(1-z)}{1-z} \right]_+ + \left[\frac{1}{1-z} \right]_+ \left\{ C_A \left(-2 \ln \left(\frac{1}{1-x} \right) + \text{Re}L_\beta \right. \right. \right. \\ &\quad \left. \left. \left. + \ln \left(\frac{t_1 u_1}{m^4} \right) - \ln \left(\frac{M^2}{m^2} \right) \right) - 2C_F(\text{Re}L_\beta + 1) \right\} \right] \end{aligned}$$

$$\begin{aligned} K^{(2)} &= 16a_s^2 \left[2C_A^2 \left[\frac{\ln^3(1-z)}{1-z} \right]_+ + \left[\frac{\ln^2(1-z)}{1-z} \right]_+ \left\{ 3C_A^2 \left(\ln \left(\frac{t_1 u_1}{m^4} \right) + \text{Re}L_\beta \right. \right. \right. \\ &\quad \left. \left. \left. - 6 \ln \left(\frac{1}{1-x} \right) - 2 \ln \left(\frac{M^2}{m^2} \right) \right) - 2C_A((11C_A - 2N_f)/12 + 3C_F(\text{Re}L_\beta + 1)) \right\} \right] \end{aligned}$$

Threshold Resummation



The x -dependence of the ratios $F_{2(NLO)}^{\text{charm}}/F_{2(LO)}^{\text{charm}}$ (solid line) and $F_{2(NNLO)}^{\text{charm}}/F_{2(NLO)}^{\text{charm}}$ (dashed line) with $F_{2(NNLO)}^{\text{charm}}$ in the improved NLL approximation (exact NLO result plus NLL approximate NNLO result with the damping factor $1/\sqrt{1+\eta}$).

Leading and Next-to-Leading Order Contributions

Next-to-Leading Order : $F_{2,L}(x, Q^2)$ [Laenen, Riemersma, Smith, van Neerven, 1993, 1995]
 asymptotic: [Buza, Matiounine, Smith, Migneron, van Neerven, 1996; Bierenbaum, J.B., Klein, 2007]

New variables :

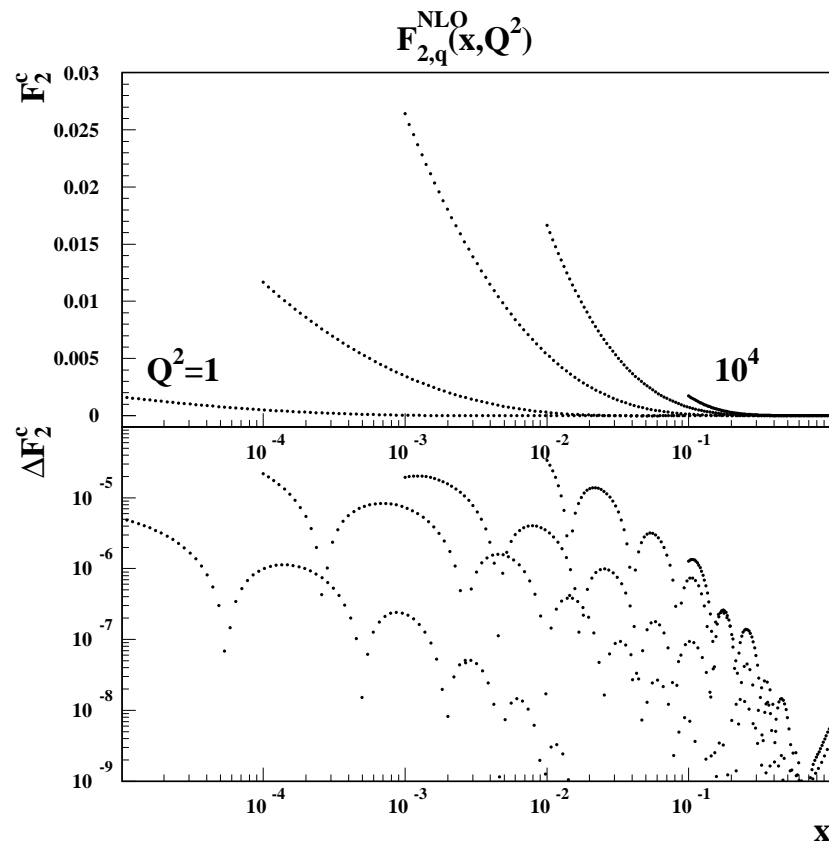
$$\xi = \frac{Q^2}{m^2}, \quad \eta = \frac{s}{4m^2} - 1 \geq 0 .$$

$$z = \frac{Q^2}{Q^2 + s} = \frac{\xi/4}{1 + \eta + \xi/4}, \quad z \in \left[x, \frac{Q^2}{Q^2 + 4m^2} \right]$$

$$L(\eta) = \ln \left[\frac{(1 + \eta)^{1/2} + \eta^{1/2}}{(1 + \eta)^{1/2} - \eta^{1/2}} \right]$$

$$\begin{aligned} F_k^{\text{NLO}}(x, Q^2, m^2) &= \frac{Q^2}{\pi m^2} \alpha_s^2(\mu^2) \int_x^{z_{\max}} \frac{dz}{z} \left\{ e_Q^2 f_g \left(\frac{x}{z}, \mu^2 \right) \left[c_{k,g}^{(1)}(\xi, \eta) + \bar{c}_{k,g}^{(1)}(\xi, \eta) \ln \left(\frac{\mu^2}{m^2} \right) \right] \right. \\ &\quad + \sum_{i=q, \bar{q}}^3 \left\{ e_Q^2 f_i \left(\frac{x}{z}, \mu^2 \right) \left[c_{k,i}^{(1)}(\xi, \eta) + \bar{c}_{k,i}^{(1)}(\xi, \eta) \ln \left(\frac{\mu^2}{m^2} \right) \right] \right. \\ &\quad \left. \left. + e_i^2 f_i \left(\frac{x}{z}, \mu^2 \right) \left[d_{k,i}^{(1)}(\xi, \eta) + \bar{d}_{k,i}^{(1)}(\xi, \eta) \right] \right\}, \quad \bar{d}_{L,q}^{(1)}(\xi, \eta) = 0. \end{aligned}$$

- Semi-analytic expressions for $c_{k,i}^{(1)}(\xi, \eta)$, $\bar{c}_{k,i}^{(1)}(\xi, \eta)$, $c_{d,i}^{(1)}(\xi, \eta)$, $\bar{d}_{k,i}^{(1)}(\xi, \eta)$; not all integrals could be done analytically.
- Fast semi-analytic representation in Mellin space: [Alekhin & J.B., 2003] This includes power corrections.



The Non-Power Contributions

- massless RGE and light-cone expansion in Bjorken-limit $\{Q^2, \nu\} \rightarrow \infty$, x fixed:

$$\lim_{\xi^2 \rightarrow 0} [J(\xi), J(0)] \propto \sum_{i,N,\tau} c_{i,\tau}^N(\xi^2, \mu^2) \xi_{\mu_1} \dots \xi_{\mu_N} O_{i,\tau}^{\mu_1 \dots \mu_N}(0, \mu^2) .$$

- Operators: flavor non-singlet (≤ 3), pure-singlet and gluon; consider leading twist.
- RGE for collinear singularities: mass factorization of the structure functions into Wilson coefficients and parton densities:

$$F_i(x, Q^2) = \sum_j \underbrace{C_i^j \left(x, \frac{Q^2}{\mu^2} \right)}_{\text{perturbative}} \otimes \underbrace{f_j(x, \mu^2)}_{\text{non-perturbative}}$$

- Light-flavor Wilson coefficients: process dependent ($O(a_s^3)$): [Moch, Vermaseren, Vogt, 2005.]

$$C_{(2,L);i}^{\text{fl}} \left(\frac{Q^2}{\mu^2} \right) = \delta_{i,q} + \sum_{l=1}^{\infty} a_s^l C_{(2,L),i}^{\text{fl},(l)}, \quad i = q, g$$

- Heavy quark contributions given by heavy quark Wilson coefficients, $H_{(2,L),i}^{\text{S,NS}} \left(\frac{Q^2}{\mu^2}, \frac{m^2}{Q^2} \right)$.

- In the limit $Q^2 \gg m_h^2$ [$Q^2 \approx 10 m^2$ for F_2, g_1]:
massive RGE, derivative $m^2 \partial / \partial m^2$ acts on Wilson coefficients only: all terms but power corrections calculable through **partonic operator matrix elements**, $\langle i | A_l | j \rangle$, which are **process independent objects!**

$$H_{(2,L),i}^{\text{S,NS}} \left(\frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2} \right) = \underbrace{A_{k,i}^{\text{S,NS}} \left(\frac{m^2}{\mu^2} \right)}_{\text{massive OMEs}} \otimes \underbrace{C_{(2,L),k}^{\text{S,NS}} \left(\frac{Q^2}{\mu^2} \right)}_{\text{light-parton-Wilson coefficients}}.$$

- holds for **polarized** and **unpolarized** case. OMEs obey expansion

$$A_{k,i}^{\text{S,NS}} \left(\frac{m^2}{\mu^2} \right) = \langle i | O_k^{\text{S,NS}} | i \rangle = \delta_{k,i} + \sum_{l=1}^{\infty} a_s^l A_{k,i}^{\text{S,NS},(l)} \left(\frac{m^2}{\mu^2} \right), \quad i = q, g$$

[Buza, Matiounine, Migneron, Smith, van Neerven, 1996; Buza, Matiounine, Smith, van Neerven, 1997.]

- **Heavy OMEs** occur as well as transition functions to define the **VFNS** starting from a **fixed flavor number scheme (FFNS)**.

[Buza, Matiounine, Smith, van Neerven, 1998; Chuvakin, Smith, van Neerven, 1998.]

One-Loop Example

Consider $F_2^{Q\bar{Q}}(x, Q^2)$:

$$\begin{aligned}
 H_{2,g}^{(1)}\left(z, \frac{m^2}{Q^2}\right) &= 8T_{Ra_s} \left\{ \beta \left[-\frac{1}{2} + 4z(1-z) + \frac{m^2}{Q^2}z(2z-1) \right] \right. \\
 &+ \left. \left[-\frac{1}{2} + z - z^2 + 2\frac{m^2}{Q^2}z(3z-1) - 4\frac{m^4}{Q^4}z^2 \right] \ln\left(\frac{1-\beta}{1+\beta}\right) \right\}, \\
 \lim_{Q^2 \gg m^2} H_{2,g}^{(1)}\left(z, \frac{m^2}{Q^2}\right) &= 4T_{Ra_s} \left\{ [z^2 + (1-z)^2] \ln\left(\frac{Q^2}{m^2} \frac{1-z}{z}\right) + 8z(1-z) - 1 \right\}.
 \end{aligned}$$

$\overline{\text{MS}}$ result for $m^2 = 0$:

$$C_{2,g}^{(1)}\left(z, \frac{Q^2}{\mu^2}\right) = 4T_{Ra_s} \left\{ [z^2 + (1-z)^2] \ln\left(\frac{Q^2}{\mu^2} \frac{1-z}{z}\right) + 8z(1-z) - 1 \right\}.$$

Massive operator matrix element:

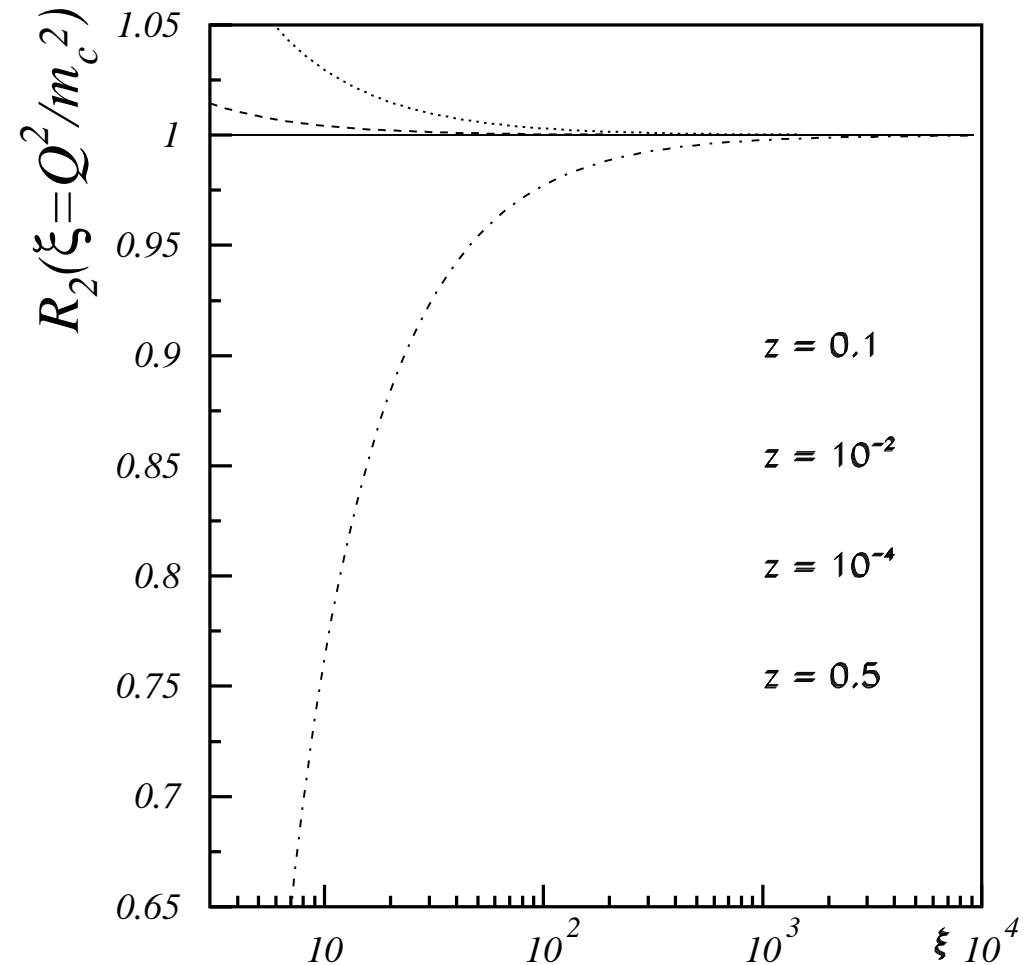
$$A_{Qg}^{(1)}\left(z, \frac{m^2}{\mu^2}\right) = -4T_{Ra_s} [z^2 + (1-z)^2] \ln\left(\frac{m^2}{\mu^2}\right) + a_{Qg}^{(1)}, \quad a_{Qg}^{(1)} = 0.$$

$$\Rightarrow \lim_{Q^2 \gg m^2} H_{2,g}^{(1)}\left(z, \frac{m^2}{Q^2}\right) = C_{2,g}^{(1)}\left(z, \frac{Q^2}{\mu^2}\right) + A_{Qg}^{(1)}\left(z, \frac{m^2}{\mu^2}\right). \quad A_{Qg}^{(1)}(z, 1) = 0$$

- Comparison for **LO**:

$$R_2\left(\xi \equiv \frac{Q^2}{m^2}\right) \equiv \frac{H_{2,g}^{(1)}}{H_{2,g,(asym)}^{(1)}} .$$

- Comparison to exact order $O(a_s^2)$ result: asymptotic formulae valid for $Q^2 \gtrsim 20$ $(\text{GeV}/c)^2$ in case of $F_2^{c\bar{c}}(x, Q^2)$ and $Q^2 \gtrsim 800$ $(\text{GeV}/c)^2$ for $F_L^{c\bar{c}}(x, Q^2)$
- **Drawbacks**:
 - **Power corrections** $(m^2/Q^2)^k$ cannot be calculated using this method.
 - The case of **two different heavy quark masses** is still too complicated \implies **2 scale** problem to be treated semi-analytically.
 - Only inclusive quantities can be calculated \implies **structure functions**.

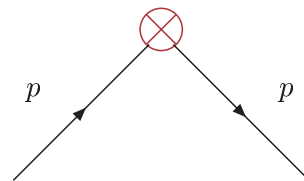


Computing Massive OMEs

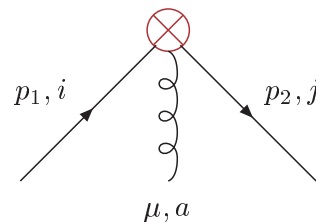
- Operator insertions in light-cone expansion

E.g. singlet heavy quark operator:

$$O_Q^{\mu_1 \dots \mu_N}(z) = \frac{1}{2} i^{N-1} S[\bar{q}(z) \gamma^{\mu_1} D^{\mu_2} \dots D^{\mu_N} q(z)] - \text{TraceTerms} .$$



$$\not{\Delta} \gamma_{\pm} (\Delta \cdot p)^{N-1} ,$$



$$g t_{ji}^a \Delta^\mu \not{\Delta} \gamma_{\pm} \sum_{j=0}^{N-2} (\Delta \cdot p_1)^j (\Delta \cdot p_2)^{N-j-2} ,$$

$$\gamma_+ = 1 , \quad \gamma_- = \gamma_5 .$$

Δ : light-like momentum, $\Delta^2 = 0$.

\implies additional vertices with 2 and more gluons at higher orders.

Renormalization

$$\hat{A}_{ij} = \delta_{ij} + \sum_{k=0}^{\infty} \hat{a}_s^k \hat{A}_{ij}^{(k)}$$

- Mass renormalization (on-mass shell scheme)

- Charge renormalization

→ use $\overline{\text{MS}}$ scheme ($D = 4 + \varepsilon$) and

decoupling formalism [Ovruv, Schnitzer, 1981; Bernreuther, Wetzel, 1982.].

- Renormalization of ultraviolet singularities

⇒ are absorbed into Z -factors given in terms of anomalous dimensions γ_{ij} .

- Factorization of collinear singularities

⇒ are factored into Γ -factors Γ_{NS} , $\Gamma_{ij,S}$ and $\Gamma_{qq,PS}$.

For massless quarks it would hold: $\Gamma = Z^{-1}$.

Here: Γ -matrices apply to parts of the diagrams with massless lines only .

Generic formula for operator renormalization and mass factorization:

$$A_{ij} = Z_{il}^{-1} \hat{A}_{lm} \Gamma_{mj}^{-1}$$

⇒ $O(\varepsilon)$ -terms of the 2-loop OMEs are needed for renormalization at 3-loops.

Calculation Techniques

- Calculation in **Mellin-space** for **space-like** q^2 , $Q^2 = -q^2$: $0 \leq x \leq 1$

$$\mathbf{M}[f](N) := \int_0^1 dz z^{N-1} f(z) .$$

- Analytic results for general value of **Mellin** N are obtained in terms of **harmonic sums** [J.B., Kurth, 1999; Vermaseren, 1999.]

$$S_{a_1, \dots, a_m}(N) = \sum_{n_1=1}^N \sum_{n_2=1}^{n_1} \dots \sum_{n_m=1}^{n_{m-1}} \frac{(\text{sign}(a_1))^{n_1}}{n_1^{|a_1|}} \frac{(\text{sign}(a_2))^{n_2}}{n_2^{|a_2|}} \dots \frac{(\text{sign}(a_m))^{n_m}}{n_m^{|a_m|}} ,$$

$$N \in \mathbb{N}, \forall l, a_l \in \mathbb{Z} \setminus 0 ,$$

$$S_{-2,1}(N) = \sum_{i=1}^N \frac{(-1)^i}{i^2} \sum_{j=1}^i \frac{1}{j} .$$

- Algebraic and structural simplification of the harmonic sums [J.B., 2003, 2007].
- Analytic continuation to **complex** N via analytic relations or integral representations, e.g.

$$\mathbf{M}\left[\frac{\text{Li}_2(x)}{1+x}\right](N+1) - \zeta_2 \beta(N+1) = (-1)^{N+1} [S_{-2,1}(N) + \frac{5}{8} \zeta_3] .$$

- Harmonic sums appear in many **single scale** higher order processes.

Str. Functions, DIS HQ, Fragn. Functions, DY, Hadr. Higgs-Prod., s+v contr. to Bhabha scatt., ...

$O(a_s^2)$ Contributions to $O(\varepsilon)$

- use of **generalized hypergeometric functions** for general analytic results
 \implies allows **feasible computation** of **higher orders in ε** & **automated check** for fixed values of N .

- use of **Mellin-Barnes integrals** for numerical checks (**MB**, [Czakon, 2006.])

- Summation of lots of **new** infinite **one-parameter sums** into **harmonic sums**. E.g.:

$$N \sum_{i,j=1}^{\infty} \frac{S_1(i)S_1(i+j+N)}{i(i+j)(j+N)} = 4S_{2,1,1} - 2S_{3,1} + S_1 \left(-3S_{2,1} + \frac{4S_3}{3} \right) - \frac{S_4}{2} \\ - S_2^2 + S_1^2 S_2 + \frac{S_1^4}{6} + 6S_1 \zeta_3 + \zeta_2 \left(2S_1^2 + S_2 \right) .$$

use of **integral techniques** and the **Mathematica package SIGMA** [Schneider, 2007.], [Bierenbaum, J.B., Klein, Schneider, 2007, 2008.]

- Partial checks for fixed values of N using **SUMMER**, [Vermaseren, 1999.]

We calculated all 2-loop $O(\varepsilon)$ -terms in the unpolarized case
and several 2-loop $O(\varepsilon)$ -terms in the polarized case:

$$\begin{aligned} & \bar{a}_{Qg}^{(2)}, \quad \bar{a}_{Qq}^{(2),\mathbf{PS}}, \quad \bar{a}_{gg,Q}^{(2)}, \quad \bar{a}_{gq,Q}^{(2)}, \quad \bar{a}_{qq,Q}^{(2),\mathbf{NS}}. \\ & \Delta\bar{a}_{Qg}^{(2)}, \quad \Delta\bar{a}_{Qq}^{(2),\mathbf{PS}}, \quad \Delta\bar{a}_{qq,Q}^{(2),\mathbf{NS}}. \end{aligned}$$

We verified all corresponding 2-loop $O(\varepsilon^0)$ -results by van Neerven et. al.

- A remark on the appearing functions:

van Neerven et al. to $O(1)$: unpolarized: 48 basic functions; polarized: 24 basic functions.

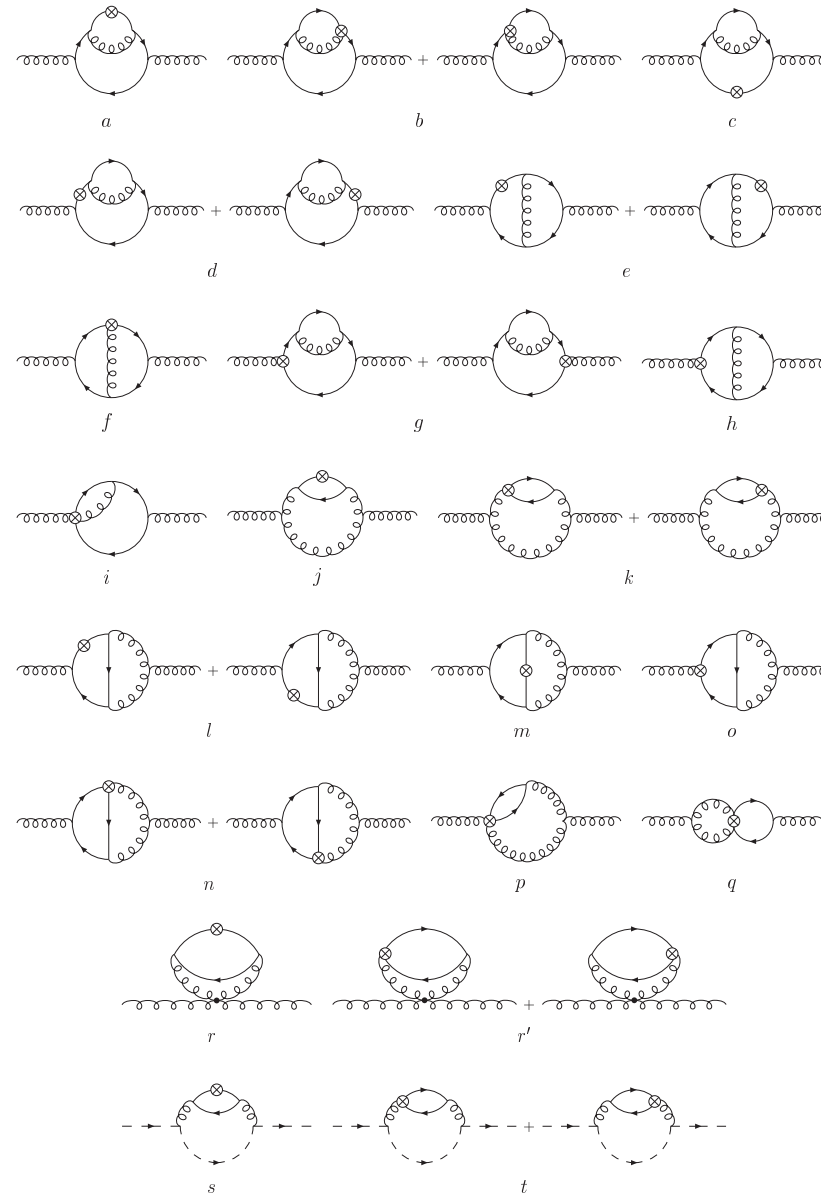
$O(1)$: $\{S_1, S_2, S_3, S_{-2}, S_{-3}\}, \quad S_{-2,1} \implies 2$ basic objects.

$O(\varepsilon)$: $\{S_1, S_2, S_3, S_4, S_{-2}, S_{-3}, S_{-4}\}, \quad S_{2,1}, \quad S_{-2,1}, \quad S_{-3,1}, \quad S_{2,1,1}, \quad S_{-2,1,1}$
 $\implies 6$ basic objects

- harmonic sums with index $\{-1\}$ cancel (holds even for each diagram)
J.B., 2004; J.B., Ravindran, 2005,2006; J.B., Klein, 2007; J.B., Moch in preparation.]
- Expectation for 3-loops: weight 5 (6) harmonic sums

- Diagrams contain **two scales**: the mass m and the Mellin-parameter N .
- **2-point functions** with on-shell external momentum, $p^2 = 0$.
 → reduce for $N = 0$ to **massive tadpoles**.
- E.g. diagrams contributing to the **gluonic OME**

$$\hat{A}_{Qg}^{(2)} \implies$$



Example: Unpolarized case, Singlet, $O(\varepsilon)$

$$\begin{aligned}
\bar{a}_{Qg}^{(2)} = & T_F C_F \left\{ \frac{2}{3} \frac{(N^2 + N + 2)(3N^2 + 3N + 2)}{N^2(N+1)^2(N+2)} \zeta_3 + \frac{P_1}{N^3(N+1)^3(N+2)} S_2 + \frac{N^4 - 5N^3 - 32N^2 - 18N - 4}{N^2(N+1)^2(N+2)} S_1^2 \right. \\
& + \frac{N^2 + N + 2}{N(N+1)(N+2)} \left(16S_{2,1,1} - 8S_{3,1} - 8S_{2,1}S_1 + 3S_4 - \frac{4}{3}S_3S_1 - \frac{1}{2}S_2^2 - S_2S_1^2 - \frac{1}{6}S_1^4 + 2\zeta_2S_2 - 2\zeta_2S_1^2 - \frac{8}{3}\zeta_3S_1 \right) \\
& - 8 \frac{N^2 - 3N - 2}{N^2(N+1)(N+2)} S_{2,1} + \frac{2}{3} \frac{3N+2}{N^2(N+2)} S_1^3 + \frac{2}{3} \frac{3N^4 + 48N^3 + 43N^2 - 22N - 8}{N^2(N+1)^2(N+2)} S_3 + 2 \frac{3N+2}{N^2(N+2)} S_2S_1 + 4 \frac{S_1}{N^2} \zeta_2 \\
& + \left. \frac{N^5 + N^4 - 8N^3 - 5N^2 - 3N - 2}{N^3(N+1)^3} \zeta_2 - 2 \frac{2N^5 - 2N^4 - 11N^3 - 19N^2 - 44N - 12}{N^2(N+1)^3(N+2)} S_1 + \frac{P_2}{N^5(N+1)^5(N+2)} \right\} \\
& + T_F C_A \left\{ \frac{N^2 + N + 2}{N(N+1)(N+2)} \left(16S_{-2,1,1} - 4S_{2,1,1} - 8S_{-3,1} - 8S_{-2,2} - 4S_{3,1} - \frac{2}{3}\beta''' + 9S_4 - 16S_{-2,1}S_1 \right. \right. \\
& + \frac{40}{3}S_1S_3 + 4\beta''S_1 - 8\beta'S_2 + \frac{1}{2}S_2^2 - 8\beta'S_1^2 + 5S_1^2S_2 + \frac{1}{6}S_1^4 - \frac{10}{3}S_1\zeta_3 - 2S_2\zeta_2 - 2S_1^2\zeta_2 - 4\beta'\zeta_2 - \frac{17}{5}\zeta_2^2 \left. \right) \\
& + \frac{4(N^2 - N - 4)}{(N+1)^2(N+2)^2} \left(-4S_{-2,1} + \beta'' - 4\beta'S_1 \right) - \frac{2}{3} \frac{N^3 + 8N^2 + 11N + 2}{N(N+1)^2(N+2)^2} S_1^3 + 8 \frac{N^4 + 2N^3 + 7N^2 + 22N + 20}{(N+1)^3(N+2)^3} \beta' \\
& + 2 \frac{3N^3 - 12N^2 - 27N - 2}{N(N+1)^2(N+2)^2} S_2S_1 - \frac{16}{3} \frac{N^5 + 10N^4 + 9N^3 + 3N^2 + 7N + 6}{(N-1)N^2(N+1)^2(N+2)^2} S_3 - 8 \frac{N^2 + N - 1}{(N+1)^2(N+2)^2} \zeta_2S_1 \\
& - \frac{2}{3} \frac{9N^5 - 10N^4 - 11N^3 + 68N^2 + 24N + 16}{(N-1)N^2(N+1)^2(N+2)^2} \zeta_3 - \frac{P_3}{(N-1)N^3(N+1)^3(N+2)^3} S_2 - \frac{2P_4}{(N-1)N^3(N+1)^3(N+2)^2} \zeta_2 \\
& - \left. \frac{P_5}{N(N+1)^3(N+2)^3} S_1^2 + \frac{2P_6}{N(N+1)^4(N+2)^4} S_1 - \frac{2P_7}{(N-1)N^5(N+1)^5(N+2)^5} \right\}.
\end{aligned}$$

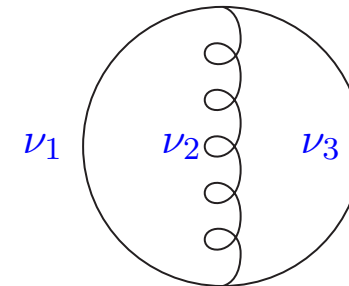
Use of hypergeometric functions for general analytic results

$${}_P F_Q \left[\begin{matrix} (a_1) \dots (a_P) \\ (b_1) \dots (b_Q) \end{matrix} ; z \right] = \sum_{i=0}^{\infty} \frac{(a_1)_i \dots (a_P)_i}{(b_1)_i \dots (b_Q)_i} \frac{z^i}{\Gamma(i+1)}, \quad {}_1 F_0[a; z] = \frac{1}{(1-z)^a}.$$

Consider the **massive 2-loop tadpole** diagram

with arbitrary exponents ν_i and

$\nu_{i\dots j} := \nu_i + \dots + \nu_j$ etc.



Using **Feynman-parameters**, this integral can be cast into the general form

$$I_1 = C_1 \int_0^1 \int_0^1 dx dy \frac{x^a (1-x)^b y^c (1-y)^d}{(1-xy)^e}.$$

Thus one obtains

$$I_1 = C_1 \Gamma \left[\begin{matrix} \nu_{123} - 4 - \varepsilon, \varepsilon/2 - \nu_2, \nu_{23} - 2 - \varepsilon/2, \nu_{12} - 2 - \varepsilon/2 \\ \nu_1, \nu_2, \nu_3, \nu_{123} - 2 - \varepsilon/2 \end{matrix} \right] {}_3 F_2 \left[\begin{matrix} \nu_{123} - 4 - \varepsilon, \varepsilon/2 + 2 - \nu_2, \nu_3 \\ \nu_3, \nu_{123} - 2 - \varepsilon/2 \end{matrix} ; 1 \right].$$

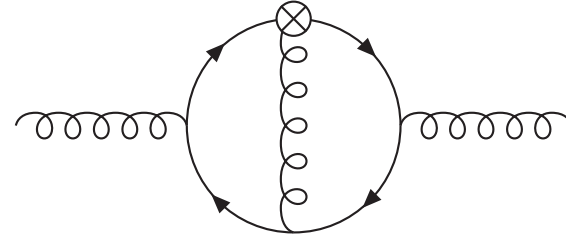
For any diagram deriving from the **2-loop tadpole** topology, one obtains as a **general integral**

$$I_2 = C_2 \int_0^1 \int_0^1 dx dy \frac{x^a (1-x)^b y^c (1-y)^d}{(1-xy)^e} \int_0^1 dz_1 \dots \int_0^1 dz_i \mathbf{P}(x, y, z_1, \dots, z_i, N) .$$

Here \mathbf{P} is a rational function of x, y and possibly more parameters $z_1 \dots z_i$. N is the Mellin-parameter and occurs in some exponents.

\implies for **fixed values of N** , one obtains for all diagrams a finite sum over integrals of the type I_1 .

Consider e.g the **scalar** Integral of the
Diagram



$$\begin{aligned} I_3 &= C_3 \exp \left\{ \sum_{l=2}^{\infty} \frac{\zeta_l}{l} \varepsilon^l \right\} \frac{2\pi}{N \sin(\frac{\pi}{2}\varepsilon)} \sum_{j=1}^N \left\{ \binom{N}{j} (-1)^j + \delta_{j,N} \right\} \\ &\times \left\{ \frac{\Gamma(j)\Gamma(j+1-\frac{\varepsilon}{2})}{\Gamma(j+2-\varepsilon)\Gamma(j+1+\frac{\varepsilon}{2})} - \frac{B(1-\frac{\varepsilon}{2}, 1+j)}{j} {}_3F_2 \left[1-\varepsilon, \frac{\varepsilon}{2}, j+1; 1, j+2-\frac{\varepsilon}{2}; 1 \right] \right\} \\ &= C_3 \left\{ \frac{4}{N} \left[S_2 - \frac{S_1}{N} \right] + \frac{\varepsilon}{N} \left[-2S_{2,1} + 2S_3 + \frac{4N+1}{N} S_2 - \frac{S_1^2}{N} - \frac{4}{N} S_1 \right] \right\} + O(\varepsilon^2) . \end{aligned}$$

4. Polarized Heavy Flavor

- DIS QCD analyzes of the polarized world data have been carried out **without reference** of heavy quark contributions so far, despite heavy flavor being produced in the final states (NMC).

Leading Order : $g_1(x, Q^2)$ [Watson, 1982; Glück, Reya, Vogelsang, 1991; Vogelsang, 1991]

$$g_1(x, Q^2) = 4e_Q^2 a_s(Q^2) \int_{ax}^1 \frac{dy}{y} C_{g_1}^{(1)}\left(\frac{x}{y}, m_Q^2, Q^2\right) \Delta G(y, Q^2)$$

$$C_{g_1}^{(1)}(z, m_Q^2, Q^2) = \frac{1}{2} \left[\beta(3 - 4z) - (1 - 2z) \ln \left| \frac{1 + \beta}{1 - \beta} \right| \right]$$

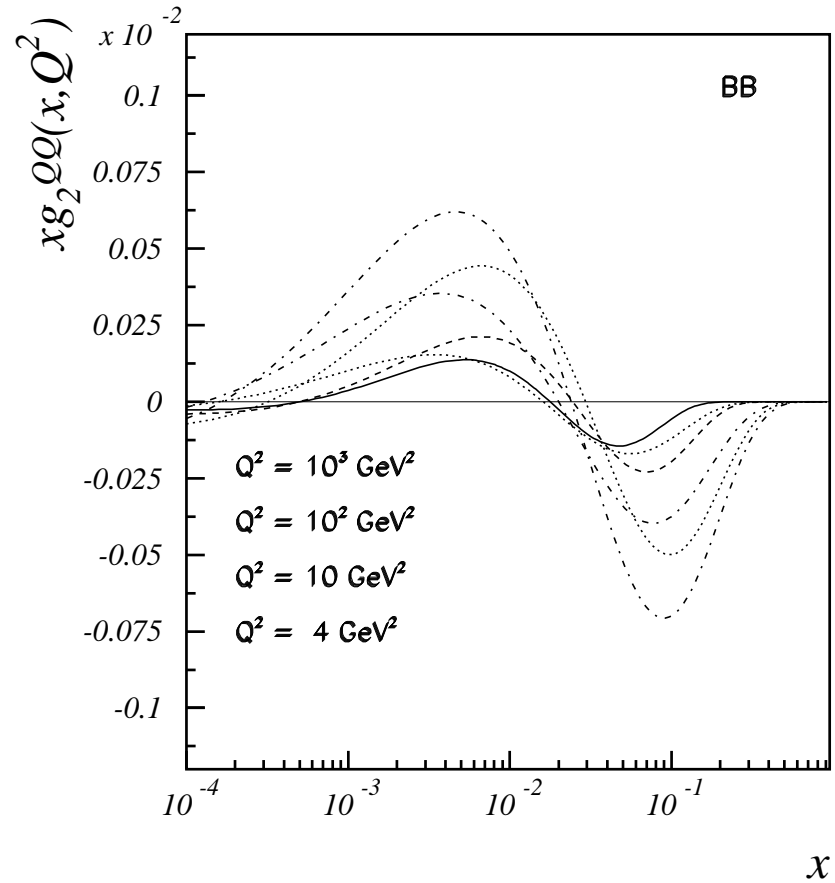
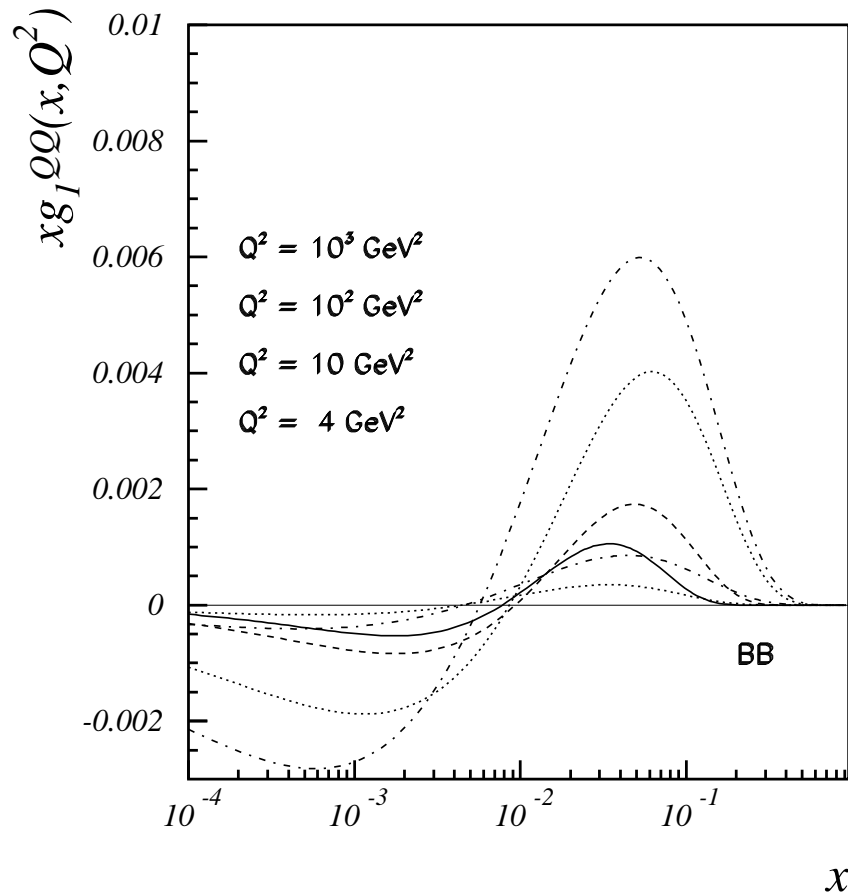
Leading Order : $g_2(x, Q^2)$ [J.B., Ravindran, van Neerven, 2003] \implies holds to all orders

$$g_2(x, Q^2) = -g_1(x, Q^2) + \int_x^1 \frac{dy}{y} g_1(y, Q^2)$$

The Wandzura-Wilczek relation follows from the covariant parton model here.

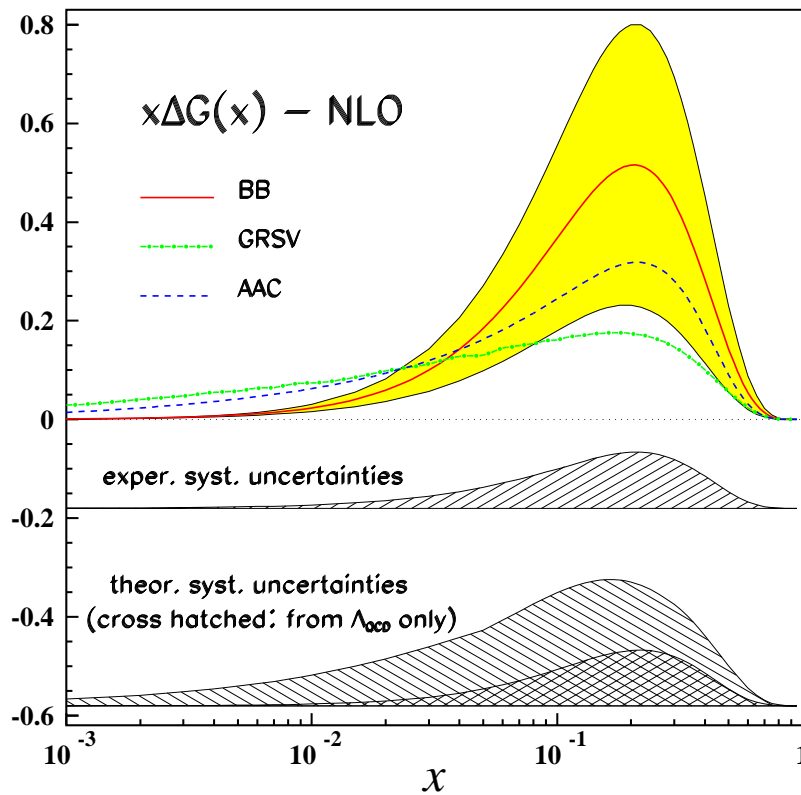
$$\int_0^{1/a} dz C_{g_1}^{(1)}(z, m_Q^2, Q^2) = 0$$

Polarized Heavy Flavor

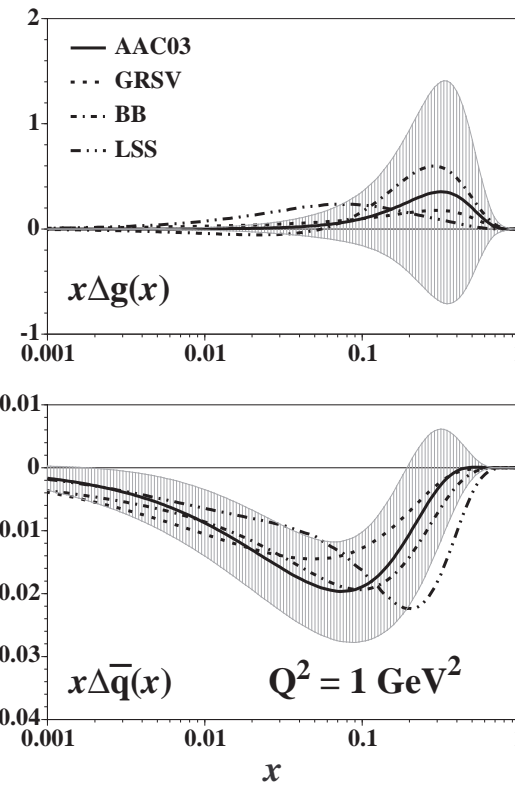


[J.B., Ravindran, van Neerven, 2003]

Polarized Gluon Density at Present



J.B., H. Böttcher (2002)



AAC

⇒ Currently slight move of ΔG towards lower values

Polarized Heavy Flavor

Next-to-Leading Order : $g_1(x, Q^2)$ only asymptotic results $Q^2 \gg m_Q^2$ i.e.

$Q^2 \gtrsim 10m_Q^2$ [M. Buza, Y. Matiounine, J. Smith and W. L. van Neerven, 1996, Bierenbaum, J.B., Klein, 2008]

$$\int_0^{1/a} dz C_{g_1}^{(2)}(z, m_Q^2, Q^2) = 0$$

Conjecture: holds for even higher orders.

$O(a_s^2 \varepsilon)$ terms: [Bierenbaum, J.B., Klein, 2008]

- NS OME's are the same as in unpolarized case (Ward Identity).

Polarized Heavy Flavor

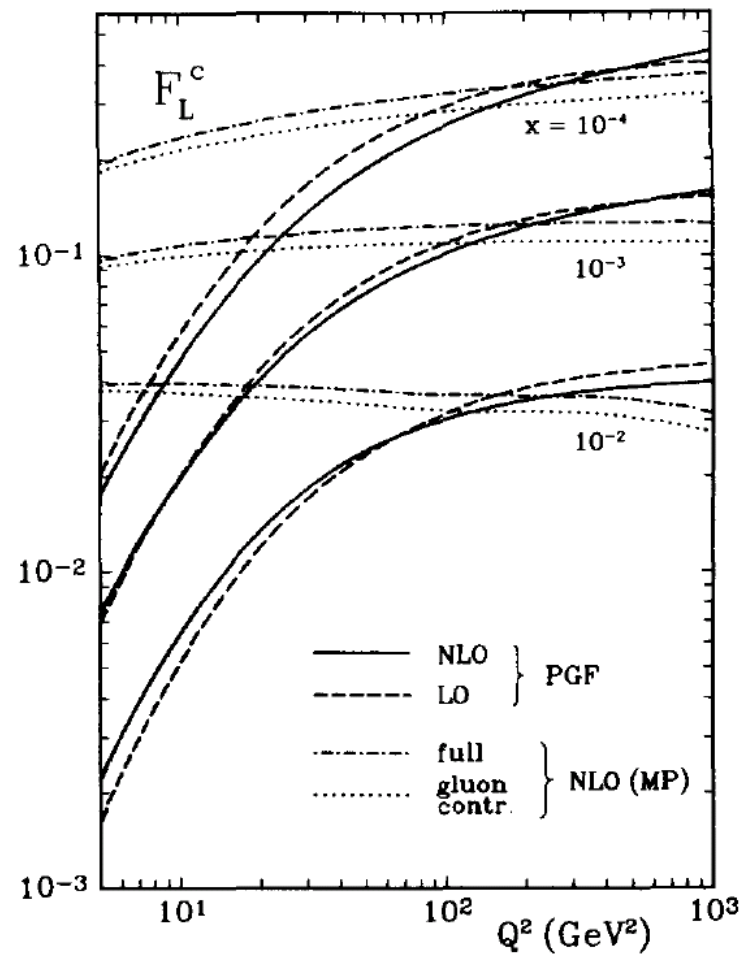
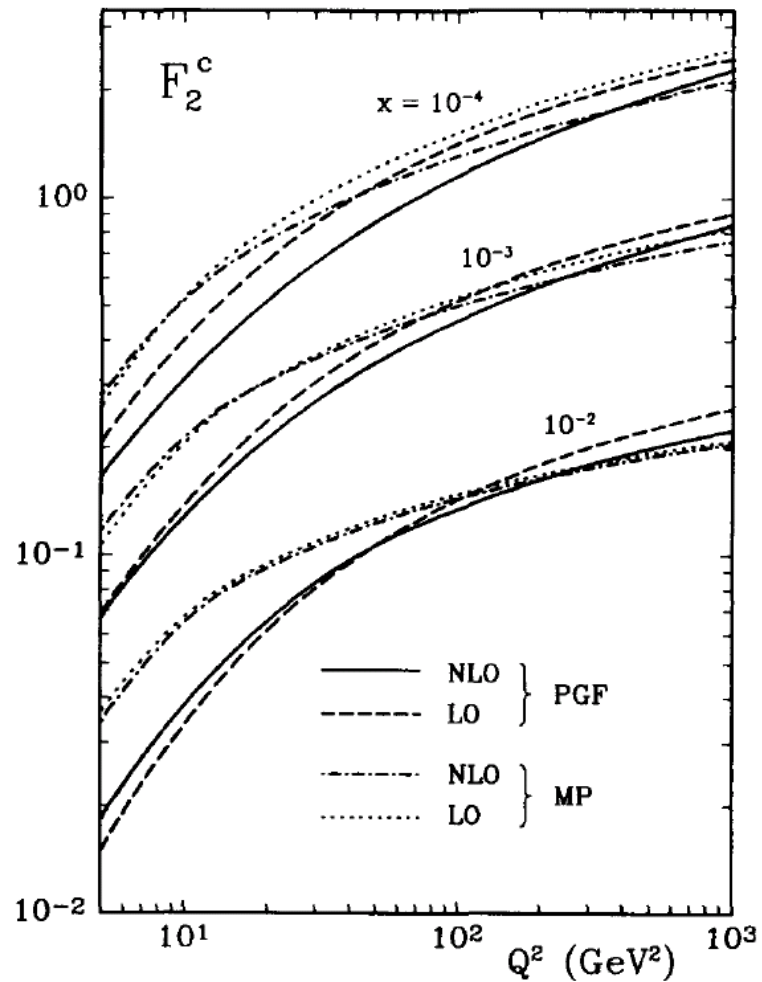
$$\begin{aligned}
\Delta a_{Q_g}^{(2)}(N) = & T_F C_F \left\{ 4 \frac{N-1}{3N(N+1)} \left(-4S_3(N) + S_1^3(N) + 3S_1(N)S_2(N) + 6S_1(N)\zeta_2 \right) \right. \\
& - 4 \frac{N^4 + 17N^3 + 43N^2 + 33N + 2}{N^2(N+1)^2(N+2)} S_2(N) - 4 \frac{3N^2 + 3N - 2}{N^2(N+1)(N+2)} S_1^2(N) \\
& - 2 \frac{(N-1)(3N^2 + 3N + 2)}{N^2(N+1)^2} \zeta_2 - 4 \frac{N^3 - 2N^2 - 22N - 36}{N^2(N+1)(N+2)} S_1(N) \\
& \left. - \frac{2P_1(N)}{N^4(N+1)^4(N+2)} \right\} \\
& + T_F C_A \left\{ 4 \frac{N-1}{3N(N+1)} \left(12M \left[\frac{\text{Li}_2(x)}{1+x} \right] (N+1) + 3\beta''(N+1) - 8S_3(N) \right. \right. \\
& - S_1^3(N) - 9S_1(N)S_2(N) - 12S_1(N)\beta'(N+1) - 12\beta(N+1)\zeta_2 - 3\zeta_3 \left. \right) \\
& - 16 \frac{N-1}{N(N+1)^2} \beta'(N+1) + 4 \frac{N^2 + 4N + 5}{N(N+1)^2(N+2)} S_1^2(N) \\
& + 4 \frac{7N^3 + 24N^2 + 15N - 16}{N^2(N+1)^2(N+2)} S_2(N) + 8 \frac{(N-1)(N+2)}{N^2(N+1)^2} \zeta_2 \\
& \left. + 4 \frac{N^4 + 4N^3 - N^2 - 10N + 2}{N(N+1)^3(N+2)} S_1(N) - \frac{4P_2(N)}{N^4(N+1)^4(N+2)} \right\}
\end{aligned}$$

+ known finite renormalization.

5. Trading Final- for Initial States

- There are no **genuine** heavy flavor parton densities.
 - ⇒ Power corrections in Wilson coefficients.
 - ⇒ Heavy quarks are not produced **collinearly**
 - ⇒ Their finite mass implies a **finite & short lifetime**.
- Do **heavy** quarks become **light** ? - will be discussed later.
- Simplifying matters with a method à la **Fermi-Williams-Weizsäcker**.
- Produce a **Heavy Quark** of large enough lifetime.
 - ⇒ Its further fate decouples (factorizes) from its past in time.
- **Heavy Quark Initial State**.
 - ⇒ Inapplicable to power corrections !
 - ⇒ Choose a physical quantity and kinematic region for which power corrections are fairly unimportant.
- Opportunities:
 - ⇒ Study of decoupling regimes in QCD.
 - ⇒ Safe one loop for certain Standard Model and “BSM” processes.

Wilson Coefficient vs Massless Parton Approach



[Glück, Reya, Stratmann, 1994]

- Even at high values of Q^2 and W^2 the massless charm approach becomes not effective; it can even lead to misleading results.

FFNS:

- Fixed order perturbation theory and Fixed number of light partons in the proton.
- The heavy quarks are produced extrinsically only.
- The large logarithmic terms in the heavy quark coefficient functions entirely determine the charm component of the structure function for large values of Q^2 .

VFNS:

- Define a threshold above which the heavy quark is treated as light, thereby obtaining a parton density.
- Phenomenological interesting to remove the mass singular terms from the asymptotic heavy quark coefficient functions and absorb them into parton densities.
- Heavy Flavor initial state parton densities for the LHC. E.g. for $c \bar{s} \rightarrow W^+$.

The VFNS is derived from the FFNS directly. A new effective density appears corresponding to the heavy quark, which is now treated as effectively massless. \implies Relations between parton densities for n_f and $n_f + 1$ flavors.

The RGE does not leave room for other scenarios.

$$\begin{aligned}
f_k(n_f + 1, \mu^2) + f_{\bar{k}}(n_f + 1, \mu^2) &= A_{qq,Q}^{\text{NS}}\left(n_f, \frac{\mu^2}{m^2}\right) \otimes \left[f_k(n_f, \mu^2) + f_{\bar{k}}(n_f, \mu^2) \right] \\
&\quad + \tilde{A}_{qq,Q}^{\text{PS}}\left(n_f, \frac{\mu^2}{m^2}\right) \otimes \Sigma(n_f, \mu^2) \\
&\quad + \tilde{A}_{qg,Q}^{\text{S}}\left(n_f, \frac{\mu^2}{m^2}\right) \otimes G(n_f, \mu^2) \\
f_{Q+\bar{Q}}(n_f + 1, \mu^2) &= \tilde{A}_{Qq}^{\text{PS}}\left(n_f, \frac{\mu^2}{m^2}\right) \otimes \Sigma(n_f, \mu^2) + \tilde{A}_{Qg}^{\text{S}}\left(n_f, \frac{\mu^2}{m^2}\right) \otimes G(n_f, \mu^2) . \\
G(n_f + 1, \mu^2) &= A_{gq,Q}^{\text{S}}\left(n_f, \frac{\mu^2}{m^2}\right) \otimes \Sigma(n_f, \mu^2) + A_{gg,Q}^{\text{S}}\left(n_f, \frac{\mu^2}{m^2}\right) \otimes G(n_f, \mu^2) . \\
\Sigma(n_f + 1, \mu^2) &= \sum_{k=1}^{n_f+1} \left[f_k(n_f + 1, \mu^2) + f_{\bar{k}}(n_f + 1, \mu^2) \right] \\
&= \left[A_{qq,Q}^{\text{NS}}\left(n_f, \frac{\mu^2}{m^2}\right) + n_f \tilde{A}_{qq,Q}^{\text{PS}}\left(n_f, \frac{\mu^2}{m^2}\right) + \tilde{A}_{Qq}^{\text{PS}}\left(n_f, \frac{\mu^2}{m^2}\right) \right] \\
&\quad \otimes \Sigma(n_f, \mu^2) \\
&\quad + \left[n_f \tilde{A}_{qg,Q}^{\text{S}}\left(n_f, \frac{\mu^2}{m^2}\right) + \tilde{A}_{Qg}^{\text{S}}\left(n_f, \frac{\mu^2}{m^2}\right) \right] \otimes G(n_f, \mu^2)
\end{aligned}$$

Polarized Case :

$$f_i \rightarrow \Delta f_i,$$

$$A_{ij} \rightarrow \Delta A_{ij}$$

6. Towards 3-Loop Precision

Need for the calculation:

- **Heavy flavor** (charm) contributions to DIS **structure functions** are rather large [20–40 % at lower values of x].
- Increase in accuracy of the perturbative description of DIS **structure functions**.
- \iff QCD analysis and determination of Λ_{QCD} , resp. $\alpha_s(M_Z^2)$, from DIS data:
 $\delta\alpha_s/\alpha_s < 1\%$.
- \iff Precise determination of the **gluon** and **sea quark** distributions.
- \iff Derivation of **variable flavor number scheme** for **heavy quark** production to $O(a_s^3)$.
 - Calculation of the **heavy flavor Wilson coefficients** to higher orders for $Q^2 \geq 25 \text{ GeV}^2$ [sufficient in many applications].
 - First recalculation of the fermionic contributions to the NNLO **anomalous dimensions**.

Goal:

$F_L(x, Q^2 \text{ for } Q^2 \gg m^2$

Mass factorization implies : [J.B., De Freitas, Klein, van Neerven, 2005/06]

(after renormalization and separation of UV and collinear singularities)

$$\begin{aligned}
 H_{L,g}^S \left(\frac{Q^2}{m^2}, \frac{m^2}{\mu^2} \right) &= a_s \widehat{C}_{L,g}^{(1)} \left(\frac{Q^2}{\mu^2} \right) + a_s^2 \left[A_{Q,g}^{(1)} \left(\frac{\mu^2}{m^2} \right) \otimes C_{L,q}^{(1)} \left(\frac{Q^2}{\mu^2} \right) + \widehat{C}_{L,g}^{(2)} \left(\frac{Q^2}{\mu^2} \right) \right] \\
 &+ a_s^3 \left[A_{Q,g}^{(2)} \left(\frac{\mu^2}{m^2} \right) \otimes C_{L,q}^{(1)} \left(\frac{Q^2}{\mu^2} \right) + A_{Q,g}^{(1)} \left(\frac{\mu^2}{m^2} \right) \otimes C_{L,q}^{(2)} \left(\frac{Q^2}{\mu^2} \right) + \widehat{C}_{L,g}^{(3)} \left(\frac{Q^2}{\mu^2} \right) \right] \\
 H_{L,q}^{\text{PS}} \left(\frac{Q^2}{m^2}, \frac{m^2}{\mu^2} \right) &= a_s^2 \widehat{C}_{L,q}^{\text{PS},(2)} \left(\frac{Q^2}{\mu^2} \right) + a_s^3 \left[A_{Qq}^{\text{PS},(2)} \left(\frac{\mu^2}{m^2} \right) \otimes C_{L,q}^{(1)} \left(\frac{Q^2}{\mu^2} \right) + \widehat{C}_{L,q}^{\text{PS},(3)} \left(\frac{Q^2}{\mu^2} \right) \right] \\
 H_{L,q}^{\text{NS}} \left(\frac{Q^2}{m^2}, \frac{m^2}{\mu^2} \right) &= a_s^2 \widehat{C}_{L,q}^{\text{NS},(2)} \left(\frac{Q^2}{\mu^2} \right) + a_s^3 \left[A_{qq,Q}^{\text{NS},(2)} \left(\frac{\mu^2}{m^2} \right) \otimes C_{L,q}^{(1)} \left(\frac{Q^2}{\mu^2} \right) + \widehat{C}_{L,q}^{\text{NS},(3)} \left(\frac{Q^2}{\mu^2} \right) \right],
 \end{aligned}$$

$$C_{L,i}^k \left(\frac{Q^2}{\mu^2}, z \right) = a_s C_{L,i}^{k,(1)} \left(\frac{Q^2}{\mu^2}, z \right) + a_s^2 C_{L,i}^{k,(2)} \left(\frac{Q^2}{\mu^2}, z \right) + a_s^3 C_{L,i}^{k,(3)} \left(\frac{Q^2}{\mu^2}, z \right)$$

Massive Operator Matrix Elements

$$\begin{aligned}
A_{Qg}^{(1)} &= -\frac{1}{2} \hat{P}_{qg}^{(0)} \ln \left(\frac{m^2}{\mu^2} \right) + a_{Qg}^{(1)} \\
A_{Qg}^{(2)} &= \frac{1}{8} \left\{ \hat{P}_{qg}^{(0)} \otimes [P_{qq}^{(0)} - P_{gg}^{(0)} + 2\beta_0] \right\} \ln^2 \left(\frac{m^2}{\mu^2} \right) \\
&\quad - \frac{1}{2} \left\{ \hat{P}_{qg}^{(1)} + a_{Qg}^{(1)} [P_{qq}^{(0)} - P_{gg}^{(0)} + 2\beta_0] \right\} \ln \left(\frac{m^2}{\mu^2} \right) \\
&\quad + \bar{a}_{Qg}^{(1)} [P_{qq}^{(0)} - P_{gg}^{(0)} + 2\beta_0] + a_{Qg}^{(2)} \\
A_{Qq}^{\text{PS},(2)} &= -\frac{1}{8} \hat{P}_{qg}^{(0)} \otimes P_{gq}^{(0)} \ln^2 \left(\frac{m^2}{\mu^2} \right) - \frac{1}{2} [\hat{P}_{qq}^{\text{PS},(1)} - a_{Qg}^{(1)} P_{gq}^{(0)}] \ln \left(\frac{m^2}{\mu^2} \right) \\
&\quad + a_{Qq}^{\text{PS},(2)} - \bar{a}_{Qg}^{(1)} \otimes P_{gq}^{(0)} \\
A_{qq,Q}^{\text{NS},(2)} &= -\frac{\beta_{0,Q}}{4} P_{qq}^{(0)} \ln^2 \left(\frac{m^2}{\mu^2} \right) - \frac{1}{2} \hat{P}_{qq}^{\text{NS},(1)} \ln \left(\frac{m^2}{\mu^2} \right) + a_{qq,Q}^{\text{NS},(2)} + \frac{1}{4} \beta_{0,Q} \zeta_2 P_{qq}^0 ,
\end{aligned}$$

with

$$\hat{f} = f(N_F + 1) - f(N_F) .$$

Expansion Coefficients

$$a_{Qg}^{(1)}(N) = 0$$

$$\bar{a}_{Qg}^{(1)}(N) = -\frac{1}{8}\zeta_2 \widehat{P}_{qg}^{(0)}(N)$$

$$a_{Qq}^{\text{PS},(2)}(N) = C_F T_R \left\{ -8 \frac{N^4 + 2N^3 + 5N^2 + 4N + 4}{(N-1)N^2(N+1)^2(N+2)} S_2(N-1) \right. \\ \left. -4 \frac{(N^2 + N + 2)^2}{(N-1)N^2(N+1)^2(N+2)} \zeta_2 + \frac{4 P_4(N)}{(N-1)N^4(N+1)^4(N+2)^3} \right\}$$

$$a_{qq,Q}^{\text{NS},(2)}(N) = C_F T_R \left\{ -\left(\frac{224}{27} + \frac{8}{3}\zeta_2\right) S_1(N-1) + \frac{40}{9} S_2(N-1) - \frac{8}{3} S_3(N-1) \right. \\ \left. + \frac{2(3N+2)(N-1)}{3N(N+1)} \zeta_2 \right. \\ \left. + \frac{(N-1)(219N^5 + 428N^4 + 517N^3 + 512N^2 + 312N + 72)}{54N^3(N+1)^3} \right\}$$

$$\begin{aligned}
a_{Qg}^{(2)}(N) = & 4C_{FT_R} \left\{ \frac{N^2 + N + 2}{N(N+1)(N+2)} \left[-\frac{1}{3}S_1^3(N-1) + \frac{4}{3}S_3(N-1) \right. \right. \\
& \left. \left. - S_1(N-1)S_2(N-1) - 2\zeta_2 S_1(N-1) \right] + \frac{2}{N(N+1)}S_1^2(N-1) \right. \\
& + \frac{N^4 + 16N^3 + 15N^2 - 8N - 4}{N^2(N+1)^2(N+2)}S_2(N-1) \\
& + \frac{3N^4 + 2N^3 + 3N^2 - 4N - 4}{2N^2(N+1)^2(N+2)}\zeta_2 \\
& \left. + \frac{N^4 - N^3 - 16N^2 + 2N + 4}{N^2(N+1)^2(N+2)}S_1(N-1) + \frac{P_2(N)}{2N^4(N+1)^4(N+2)} \right\} \\
& + 4C_{AT_R} \left\{ \frac{N^2 + N + 2}{N(N+1)(N+2)} \left[4\mathbf{M} \left[\frac{\text{Li}_2(x)}{1+x} \right] (N) + \frac{1}{3}S_1^3(N) + 3S_2(N)S_1(N) \right. \right. \\
& \left. \left. + \frac{8}{3}S_3(N) + \beta''(N+1) - 4\beta'(N+1)S_1(N) - 4\beta(N+1)\zeta_2 + \zeta_3 \right] \right. \\
& - \frac{N^3 + 8N^2 + 11N + 2}{N(N+1)^2(N+2)^2}S_1^2(N) - 2\frac{N^4 - 2N^3 + 5N^2 + 2N + 2}{(N-1)N^2(N+1)^2(N+2)}\zeta_2 \\
& - \frac{7N^5 + 21N^4 + 13N^3 + 21N^2 + 18N + 16}{(N-1)N^2(N+1)^2(N+2)^2}S_2(N) \\
& - \frac{N^6 + 8N^5 + 23N^4 + 54N^3 + 94N^2 + 72N + 8}{N(N+1)^3(N+2)^3}S_1(N) \\
& \left. - 4\frac{(N^2 - N - 4)}{(N+1)^2(N+2)^2}\beta'(N+1) + \frac{P_3(N)}{(N-1)N^4(N+1)^4(N+2)^4} \right\}
\end{aligned}$$

Final Structure of the Wilson Coefficients

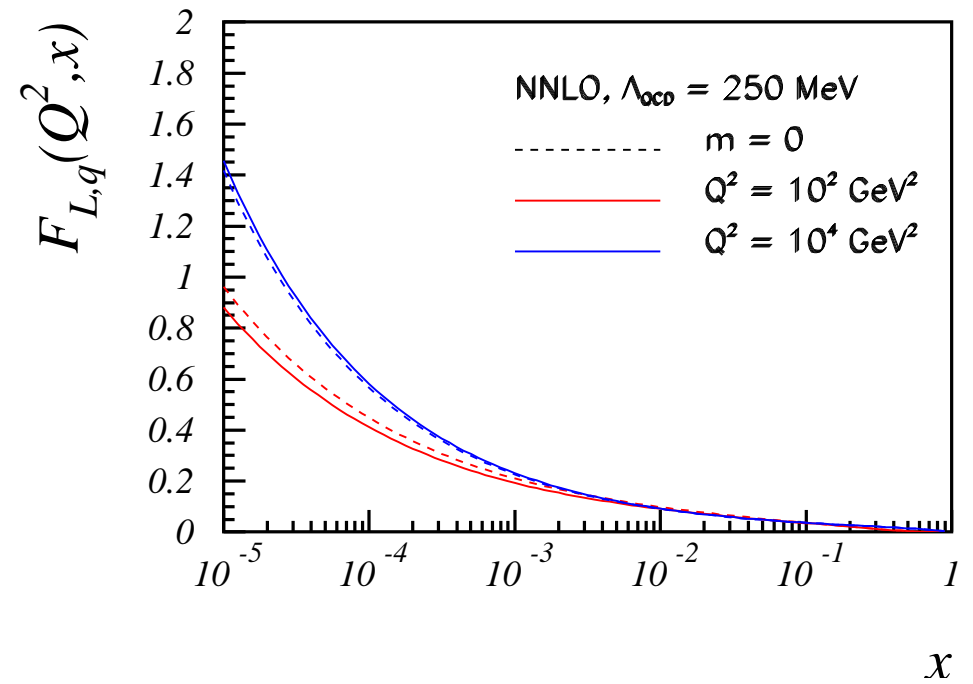
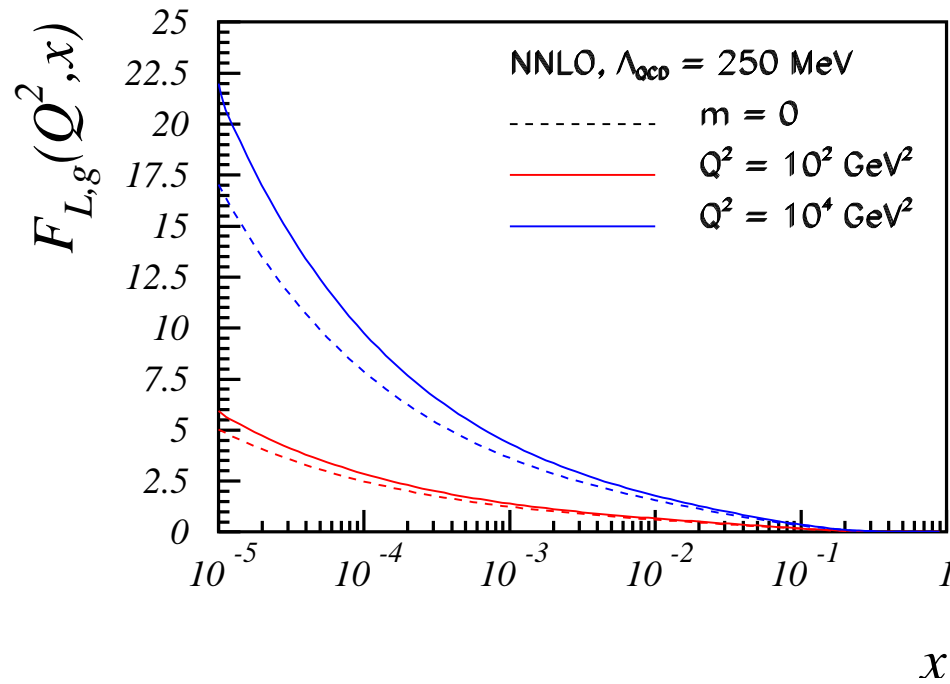
$$\mu^2 = Q^2$$

$$\begin{aligned}
 H_{L,g}^S \left(x, a_s, \frac{Q^2}{m^2} \right) &= a_s \widehat{c}_{L,g}^{(1)} + a_s^2 \left[\frac{1}{2} \widehat{P}_{qg}^{(0)} c_{L,q}^{(1)} \ln \left(\frac{Q^2}{m^2} \right) + \widehat{c}_{L,g}^{(2)} \right] \\
 &+ a_s^3 \left\{ \left[\frac{1}{8} \widehat{P}_{qg}^{(0)} \left[P_{qq}^{(0)} - P_{gg}^{(0)} + 2\beta_0 \right] \ln^2 \left(\frac{Q^2}{m^2} \right) + \frac{1}{2} \widehat{P}_{qg}^{(1)} \ln \left(\frac{Q^2}{m^2} \right) \right. \right. \\
 &\quad \left. \left. + a_{Qg}^{(2)} + \bar{a}_{Qg}^{(1)} \left[P_{qq}^{(0)} - P_{gg}^{(0)} + 2\beta_0 \right] \right] c_{L,q}^{(1)} + \frac{1}{2} \widehat{P}_{qg}^{(0)} \ln \left(\frac{Q^2}{m^2} \right) c_{L,q}^{(2)} + \widehat{c}_{L,g}^{(3)} \right\}
 \end{aligned}$$

$$\begin{aligned}
 H_{L,q}^{\text{PS}} \left(x, a_s, \frac{Q^2}{m^2} \right) &= a_s^2 \widehat{c}_{L,q}^{\text{PS},(2)} \\
 &+ a_s^3 \left\{ \left[-\frac{1}{8} \widehat{P}_{qg}^{(0)} P_{gq}^{(0)} \ln^2 \left(\frac{Q^2}{m^2} \right) + \frac{1}{2} \widehat{P}_{qq}^{\text{PS},(1)} \ln \left(\frac{Q^2}{m^2} \right) + a_{Qq}^{\text{PS},(2)} - \bar{a}_{Qg}^{(1)} P_{gq}^{(0)} \right] c_{L,q}^{(1)} + \widehat{c}_{L,q}^{\text{PS},(3)} \right\}
 \end{aligned}$$

$$\begin{aligned}
 H_{L,q}^{\text{NS}} \left(x, a_s, \frac{Q^2}{m^2} \right) &= a_s^2 \left[-\beta_{0,Q} c_{L,q}^{(1)} \ln \left(\frac{Q^2}{m^2} \right) + \widehat{c}_{L,q}^{\text{NS},(2)} \right] \\
 &+ a_s^3 \left\{ \left[-\frac{1}{4} \beta_{0,Q} P_{qq}^{(0)} \ln^2 \left(\frac{Q^2}{m^2} \right) + \frac{1}{2} \widehat{P}_{qq}^{\text{NS},(1)} \ln \left(\frac{Q^2}{m^2} \right) + a_{qq,Q}^{\text{NS},(2)} + \frac{1}{4} \beta_{0,Q} \zeta_2 P_{qq}^{(0)} \right] \right. \\
 &\quad \left. \times c_{L,q}^{(1)} + \widehat{c}_{L,q}^{\text{NS},(3)} \right\}.
 \end{aligned}$$

$O(a_s^3)$ Asymptotic Heavy Flavor Contributions to $F_L(x, Q^2)$:



Correction to NS distribution very small.

Fixed moments at 3-Loop: $F_2^{Q\bar{Q}}$

Contributing OMEs:

$$\begin{array}{l}
 \text{Singlet} \\
 \text{Pure-Singlet} \\
 \text{Non-Singlet}
 \end{array}
 \begin{array}{cccc}
 A_{Qg} & A_{qg,Q} & A_{gg,Q} & A_{gq,Q} \\
 & A_{Qq}^{\text{PS}} & A_{qq,Q}^{\text{PS}} & \\
 A_{qq,Q}^{\text{NS,+}} & A_{qq,Q}^{\text{NS,-}} & A_{qq,Q}^{\text{NS,v}} &
 \end{array}
 \left. \vphantom{\begin{array}{l} \text{Singlet} \\ \text{Pure-Singlet} \\ \text{Non-Singlet} \end{array}} \right\} \text{mixing}$$

- All 2-loop $O(\varepsilon)$ -terms in the **unpolarized** case are known:
- **Unpolarized anomalous dimensions** are known up to $O(a_s^3)$ [Moch, Vermaseren, Vogt, 2004.]
 \implies All terms needed for the renormalization of **unpolarized 3-Loop heavy OMEs** are present.
 \implies The calculation will provide first independent checks on $\gamma_{qg}^{(2)}$, $\gamma_{qq}^{(2),\text{PS}}$ and on respective color projections of $\gamma_{qq}^{(2),\text{NS}\pm,\text{v}}$, $\gamma_{gg}^{(2)}$ and $\gamma_{gq}^{(2)}$.
- The calculation proceeds in the same way in the **polarized** case.
- Independent checks provided by pole terms (**anomalous dimensions, lower order OMEs**) and **sum rules** for $N = 2$.

Fixed moments using MATAD

- three-loop “self-energy” type diagrams with an operator insertion
- **Extension:** additional scale compared to massive propagators: Mellin variable N
- Genuine tensor integrals due to

$$\Delta^{\mu_1} \dots \Delta^{\mu_n} \langle p | O_{\mu_1 \dots \mu_n} | p \rangle = \Delta^{\mu_1} \dots \Delta^{\mu_n} \langle p | S \bar{\Psi} \gamma_{\mu_1} D_{\mu_2} \dots D_{\mu_n} \Psi | p \rangle = A(N) \cdot (\Delta p)^N$$

$$D_\mu = \partial_\mu - i g t_a A_\mu^a, \quad \Delta^2 = 0.$$

- Construction of a projector to obtain the desired moment in N [undo Δ -contraction]
- 3-loop OMEs are generated with QGRAF [Nogueira, 1993.]
- Color factors are calculated with [van Ritbergen, Schellekens, Vermaseren, 1998.]
- Translation to suitable input for MATAD [Steinhauser, 2001.]

Tests performed:

- Various 2-loop calculations for $N = 2, 4, 6, \dots$ were repeated
→ agreement with our previous calculation.
- Several non-trivial scalar 3-loop diagrams were calculated using Feynman-parameters for all N
→ agreement with MATAD.

General structure of the result: the PS –case

$$\begin{aligned}
A_{Qq}^{(3),\text{PS}} + A_{qq,Q}^{(3),\text{PS}} &= \frac{\hat{\gamma}_{qg}^{(0)} \gamma_{gq}^{(0)}}{48} \left\{ \gamma_{gg}^{(0)} - \gamma_{qq}^{(0)} + 4(n_f + 1)\beta_{0,Q} + 6\beta_0 \right\} \ln^3 \left(\frac{m^2}{\mu^2} \right) \\
&+ \left\{ \frac{\hat{\gamma}_{qq}^{(1),\text{PS}}}{2} \left((n_f + 1)\beta_{0,Q} - \beta_0 \right) + \frac{\hat{\gamma}_{qg}^{(0)}}{8} \left((n_f + 1)\hat{\gamma}_{gq}^{(1)} - \gamma_{gq}^{(1)} \right) - \frac{\gamma_{gq}^{(0)} \hat{\gamma}_{qg}^{(1)}}{8} \right\} \ln^2 \left(\frac{m^2}{\mu^2} \right) \\
&+ \left\{ \frac{\hat{\gamma}_{qq}^{(2),\text{PS}}}{2} - \zeta_2 \frac{\hat{\gamma}_{qg}^{(0)} \gamma_{gq}^{(0)}}{16} \left(\gamma_{gg}^{(0)} - \gamma_{qq}^{(0)} + 4(n_f + 1)\beta_{0,Q} + 6\beta_0 \right) - 2a_{Qq}^{(2),\text{PS}} \beta_0 \right. \\
&+ \left. \frac{n_f + 1}{2} \hat{\gamma}_{qg}^{(0)} a_{gq,Q}^{(2)} - \frac{\gamma_{gq}^{(0)}}{2} a_{Qg}^{(2)} \right\} \ln \left(\frac{m^2}{\mu^2} \right) + \zeta_3 \frac{\gamma_{gq}^{(0)} \hat{\gamma}_{qg}^{(0)}}{48} \left(\gamma_{gg}^{(0)} - \gamma_{qq}^{(0)} + 4n_f \beta_{0,Q} + 6\beta_0 \right) \\
&+ \frac{\zeta_2}{16} \left(-4n_f \beta_{0,Q} \hat{\gamma}_{qq}^{(1),\text{PS}} + \hat{\gamma}_{qg}^{(0)} \gamma_{gq}^{(1)} \right) + 4(\beta_0 + \beta_{0,Q}) \bar{a}_{Qq}^{(2),\text{PS}} + \gamma_{gq}^{(0)} \bar{a}_{Qg}^{(2)} - (n_f + 1) \hat{\gamma}_{qg}^{(0)} \bar{a}_{gq,Q}^{(2)} \\
&+ C_F \left(-\left(4 + \frac{3}{4} \zeta_2 \right) \hat{\gamma}_{qg}^{(0)} \gamma_{gq}^{(0)} - 4\hat{\gamma}_{qq}^{(1),\text{PS}} + 12a_{Qq}^{(2),\text{PS}} \right) + a_{Qq}^{(3),\text{PS}} + a_{qq,Q}^{(3),\text{PS}} .
\end{aligned}$$

- n_f –dependence non–trivial. Take all quantities at n_f flavors and adopt notation

$$\hat{\gamma}_{ij} \equiv \gamma_{ij}(n_f + 1) - \gamma_{ij}(n_f) , \quad \beta_{0,Q} \equiv \beta_0(n_f + 1) - \beta_0(n_f) .$$

- There are similar formulas for the remaining **OMEs**.

- Number of Diagrams to be calculated:

$$A_{Q(q)q}^{(3),PS} : 132, \quad A_{qq}^{(3),NS} : 128, \quad A_{gq}^{(3)} : 89, \quad A_{Qg}^{(3)} : 1498, \quad A_{gg,Q}^{(3)} : 865.$$

- We calculated the terms

$$A_{Qq}^{(3),PS} + A_{qq,Q}^{(3),PS}, \quad A_{qq,Q}^{(3),NS}, \quad A_{gq,Q}^{(3)}.$$

for $N = 2, 4, 6, 8, 10, 12$ using **MATAD** and find **agreement** of the pole terms with the prediction obtained from renormalization.

- An additional check is provided by the sum rule

$$A_{qq,Q}^{(3),NS} \Big|_{N=2} + A_{qq,Q}^{(3),PS} \Big|_{N=2} + A_{Qq}^{(3),PS} \Big|_{N=2} + A_{gq,Q}^{(3)} \Big|_{N=2} = 0,$$

which is fulfilled by our result.

- All terms proportional to ζ_2 cancel in the renormalized result.
- We observe the number

$$\mathbf{B4} = -4\zeta_2 \ln^2 2 + \frac{2}{3} \ln^4 2 - \frac{13}{2} \zeta_4 + 16 \text{Li}_4\left(\frac{1}{2}\right) = -8\sigma_{-3,-1} + \frac{11}{2} \zeta_4.$$

- The term **B4** appears as

$$T_F C_F \left(C_F - \frac{C_A}{2} \right) \mathbf{B4}.$$

Result for the renormalized **PS**-term for $N = 4$.

$$\begin{aligned}
A_{Qq}^{(3),\text{PS}} + A_{qq,Q}^{(3),\text{PS}} \Big|_{N=4} &= \left\{ -\frac{484}{2025} C_F T_F^2 (2n_f + 1) + \frac{4598}{3375} C_F C_A T_F - \frac{18997}{40500} C_F^2 T_F \right\} \ln^3 \left(\frac{m^2}{\mu^2} \right) \\
&+ \left\{ -\frac{16}{125} C_F T_F^2 + \frac{36751}{202500} C_F C_A T_F - \frac{697631}{405000} C_F^2 T_F \right\} \ln^2 \left(\frac{m^2}{\mu^2} \right) + \left\{ -\frac{2131169}{303750} C_F T_F^2 n_f \right. \\
&- \frac{427141}{121500} C_F T_F^2 + \left(-\frac{484}{75} \zeta_3 + \frac{24888821}{2700000} \right) C_F C_A T_F + \left(\frac{484}{75} \zeta_3 + \frac{63582197}{16200000} \right) C_F^2 T_F \left. \right\} \ln \left(\frac{m^2}{\mu^2} \right) \\
&+ \left(\frac{7744}{2025} \zeta_3 - \frac{143929913}{27337500} \right) C_F T_F^2 n_f + \left(-\frac{13552}{2025} \zeta_3 + \frac{218235943}{54675000} \right) C_F T_F^2 + \left(\frac{242}{225} \mathbf{B4} - \frac{242}{25} \zeta_4 \right. \\
&+ \left. \frac{86833}{13500} \zeta_3 + \frac{4628174}{1265625} \right) C_F C_A T_F + \left(-\frac{484}{225} \mathbf{B4} + \frac{242}{25} \zeta_4 + \frac{298363}{20250} \zeta_3 - \frac{57518389433}{2187000000} \right) C_F^2 T_F .
\end{aligned}$$

We obtain for the **moments** of the **PS**, **NS** and $\hat{\gamma}_{gq}^{(2)}$ anomalous dimensions

N	$\hat{\gamma}_{qq}^{(2),PS}/T_F/C_F$
2	$-\frac{5024}{243}T_F(1+2n_f) + \frac{256}{3}(C_F - C_A)\zeta_3 + \frac{10136}{243}C_A - \frac{14728}{243}C_F$
4	$-\frac{618673}{151875}T_F(1+2n_f) + \frac{968}{75}(C_F - C_A)\zeta_3 + \frac{2485097}{506250}C_A - \frac{2217031}{675000}C_F$
6	$-\frac{126223052}{72930375}T_F(1+2n_f) + \frac{3872}{735}(C_F - C_A)\zeta_3 + \frac{1988624681}{4084101000}C_A + \frac{11602048711}{10210252500}C_F$
8	$-\frac{13131081443}{13502538000}T_F(1+2n_f) + \frac{2738}{945}(C_F - C_A)\zeta_3 - \frac{343248329803}{648121824000}C_A + \frac{39929737384469}{22684263840000}C_F$
10	$-\frac{265847305072}{420260754375}T_F(1+2n_f) + \frac{50176}{27225}(C_F - C_A)\zeta_3 - \frac{1028766412107043}{1294403123475000}C_A + \frac{839864254987192}{485401171303125}C_F$
12	$-\frac{2566080055386457}{5703275664286200}T_F(1+2n_f) + \frac{49928}{39039}(C_F - C_A)\zeta_3 - \frac{69697489543846494691}{83039693672007072000}C_A$ $+ \frac{86033255402443256197}{54806197823524667520}C_F$
N	$\hat{\gamma}_{gq}^{(2)}/T_F/C_F$
2	$\frac{2272}{81}T_F(1+2n_f) + \frac{512}{3}(C_A - C_F)\zeta_3 + \frac{88}{9}C_A + \frac{28376}{243}C_F$
4	$\frac{109462}{10125}T_F(1+2n_f) + \frac{704}{15}(C_A - C_F)\zeta_3 - \frac{799}{12150}C_A + \frac{14606684}{759375}C_F$
6	$\frac{22667672}{3472875}T_F(1+2n_f) + \frac{2816}{105}(C_A - C_F)\zeta_3 - \frac{253841107}{145860750}C_A + \frac{20157323311}{2552563125}C_F$
8	$\frac{339184373}{75014100}T_F(1+2n_f) + \frac{1184}{63}(C_A - C_F)\zeta_3 - \frac{3105820553}{1687817250}C_A + \frac{8498139408671}{2268426384000}C_F$
10	$\frac{1218139408}{363862125}T_F(1+2n_f) + \frac{7168}{495}(C_A - C_F)\zeta_3 - \frac{18846629176433}{11767301122500}C_A + \frac{529979902254031}{323600780868750}C_F$
12	$\frac{13454024393417}{5222779912350}T_F(1+2n_f) + \frac{5056}{429}(C_A - C_F)\zeta_3 - \frac{64190493078139789}{48885219979596000}C_A + \frac{1401404001326440151}{3495293228541114000}C_F$

N	$\hat{\gamma}_{qq}^{(2),NS,+}/T_F/C_F$
2	$-\frac{1792}{243}T_F(1+2n_f) + \frac{256}{3}(C_F - C_A)\zeta_3 - \frac{12512}{243}C_A - \frac{13648}{243}C_F$
4	$-\frac{384277}{30375}T_F(1+2n_f) + \frac{2512}{15}(C_F - C_A)\zeta_3 - \frac{8802581}{121500}C_A - \frac{165237563}{1215000}C_F$
6	$-\frac{160695142}{10418625}T_F(1+2n_f) + \frac{22688}{105}(C_F - C_A)\zeta_3 - \frac{13978373}{171500}C_A - \frac{44644018231}{243101250}C_F$
8	$-\frac{38920977797}{2250423000}T_F(1+2n_f) + \frac{79064}{315}(C_F - C_A)\zeta_3 - \frac{1578915745223}{18003384000}C_A - \frac{91675209372043}{420078960000}C_F$
10	$-\frac{27995901056887}{1497656506500}T_F(1+2n_f) + \frac{192880}{693}(C_F - C_A)\zeta_3 - \frac{9007773127403}{97250422500}C_A - \frac{75522073210471127}{307518802668000}C_F$
12	$-\frac{65155853387858071}{3290351344780500}T_F(1+2n_f) + \frac{13549568}{45045}(C_F - C_A)\zeta_3 - \frac{25478252190337435009}{263228107582440000}C_A$ $-\frac{35346062280941906036867}{131745667845011220000}C_F$

\implies Agreement for the terms $\propto T_F$ with

[Larin, Nogueira, Ritbergen, Vermaseren, 1997; Moch, Vermaseren, Vogt, 2004.]

• How far can we go ?

$N = 14$ in some cases; generally: $N = 12$. \implies Phenomenology

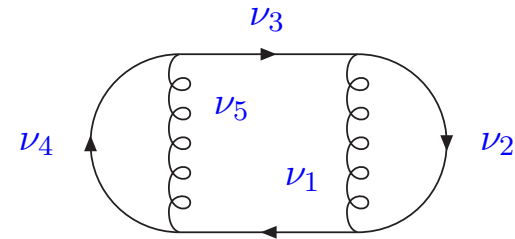
CPU time: various months; up to 64 Gb processors needed.

Unfortunately not enough to perform the automatic

fixed moments \rightarrow all moments turn. [J.B., Kauers, Klein, Schneider, 2008].

Fixed moments using Feynman–parameters

Consider e.g the **3–loop tadpole** diagram



Using Feynman–parameters, one obtains a representation in terms of a double sum

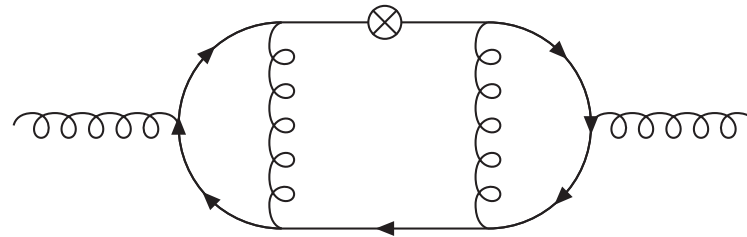
$$I = C\Gamma \left[\begin{array}{c} 2 + \varepsilon/2 - \nu_1, 2 + \varepsilon/2 - \nu_5, \nu_{12} - 2 - \varepsilon/2, \nu_{45} - 2 - \varepsilon/2, \nu_{1345} - 4 - \varepsilon, \nu_{12345} - 6 - 3/2\varepsilon \\ \nu_1, \nu_2, \nu_4, 2 + \varepsilon/2, \nu_{345} - 2 - \varepsilon/2, \nu_{12345} - 4 - \varepsilon \end{array} \right]$$

$$\sum_{m,n=0}^{\infty} \frac{(\nu_{345} - 2 - \varepsilon/2)_{n+m} (\nu_{12345} - 6 - 3/2\varepsilon)_m (2 + \varepsilon/2 - \nu_1)_m (2 + \varepsilon/2 - \nu_5)_n (\nu_{45} - 2 - \varepsilon/2)_n}{m!n! (\nu_{12345} - 4 - \varepsilon)_{n+m} (\nu_{345} - 2 - \varepsilon/2)_m (\nu_{345} - 2 - \varepsilon/2)_n},$$

which derives from a **Appell–function of the first kind, F_1** .

First all-N results

As in the 2-loop case, for any diagram deriving from the tadpole-ladder topology, one obtains for **fixed values of N** a finite sum over double sums of the same type. Consider e.g. the scalar diagram



For the above diagram, we obtained a result for arbitrary N using similar techniques as in the 2-loop case and the package **SIGMA**.

$$\begin{aligned}
 I_1 = & -\frac{4(N+1)S_1 + 4}{(N+1)^2(N+2)} \zeta_3 + \frac{2S_{2,1,1}}{(N+2)(N+3)} + \frac{1}{(N+1)(N+2)(N+3)} \left\{ -2(3N+5)S_{3,1} - \frac{S_1^4}{4} \right. \\
 & + \frac{4(N+1)S_1 - 4N}{N+1} S_{2,1} + 2 \left((2N+3)S_1 + \frac{5N+6}{N+1} \right) S_3 + \frac{9+4N}{4} S_2^2 + \left(-\frac{5}{2} S_1^2 + \frac{5N}{N+1} S_1 + \frac{N}{N+1} S_1^3 \right. \\
 & \left. \left. + 2 \frac{7N+11}{(N+1)(N+2)} \right) S_2 + \frac{2(3N+5)S_1^2}{(N+1)(N+2)} + \frac{4(2N+3)S_1}{(N+1)^2(N+2)} - \frac{(2N+3)S_4}{2} + 8 \frac{2N+3}{(N+1)^3(N+2)} \right\} + O(\varepsilon) .
 \end{aligned}$$

For fixed N , this formula agrees with the result we obtain using **MATAD**.

7. Conclusions

- The heavy flavor contributions to the structure function F_2 are rather large in the region of lower values of x .
- Competitive QCD precision analyzes therefore require the description of the heavy quark contributions to 3-loop order.
- The inclusive heavy flavor contributions to $F_{2,L}(x, Q^2)$ are known to NLO (in semi-analytic form with fast numeric implementations.)
- Complete analytic results are known in the region $Q^2 \gg m^2$ at NLO for $F_{2,L}(x, Q^2), g_{1,2}(x, Q^2)$. They are expressed in terms of massive operator matrix elements and the corresponding massless Wilson coefficients. Threshold resummations were performed.
- $F_L(x, Q^2)$ is known to NNLO for $Q^2 \gg m^2$.
- The calculation of fixed moments of the massive operator matrix elements at $O(a_s^3)$ is being performed currently to $N = 12, 14$, which will be followed by some first phenomenological parameterization.
- We also calculate the matrix elements necessary to transform from the **FFNS** to the **VFNS**.
- First steps towards the calculation of the massive operator matrix elements at $O(a_s^3)$ for general values of N were performed.