# On the Way to QCD Precision Test with Deep Inelastic Scattering

Johannes Blümlein

**DESY**

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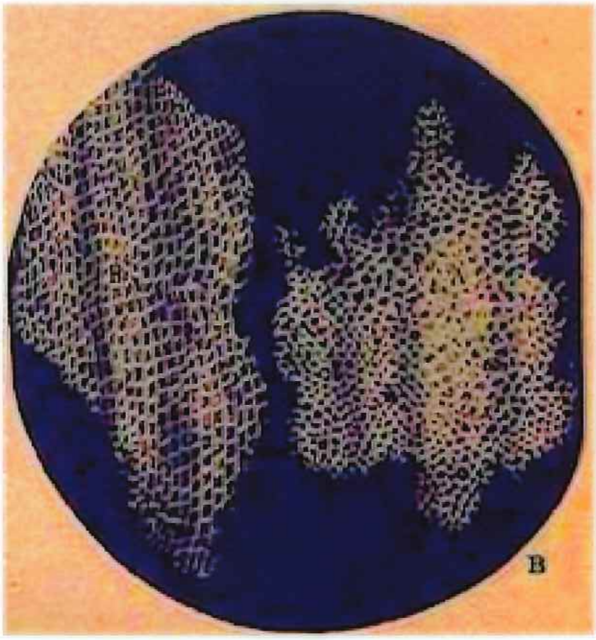
University Freiburg, October 2005

# 1. Introduction

## THE DOOR TO THE VERY SMALL IS OPENED BY MICROSCOPES.

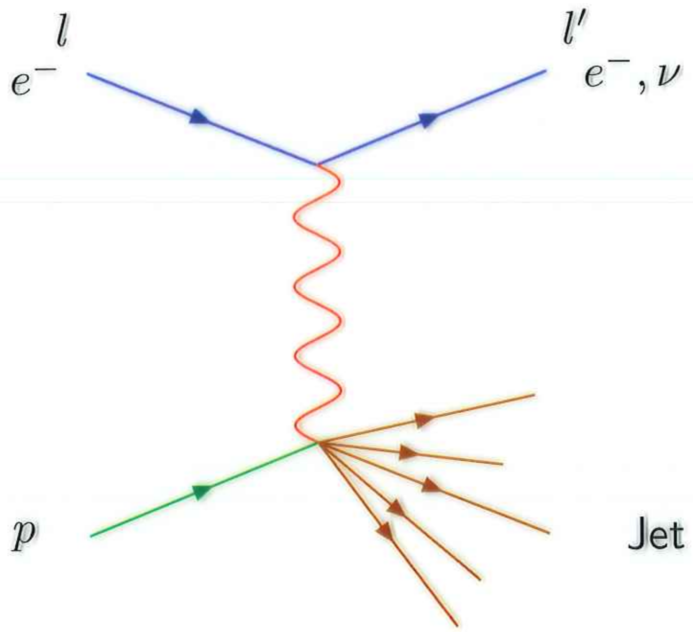
### ROBERT HOOKE (1635-1703)

Remake of the original microscope




Observation of cork cells

# Deeply Inelastic Scattering




**space-like process :**

$$
\begin{aligned}
q^2 &= (l - l')^2 = -Q^2 < 0 \\
W^2 &= (p+q)^2 \geq M_p^2
\end{aligned}
$$

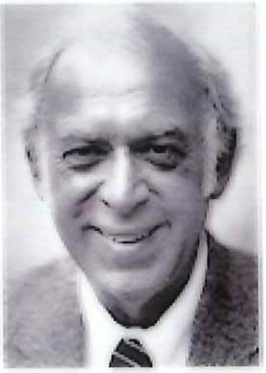
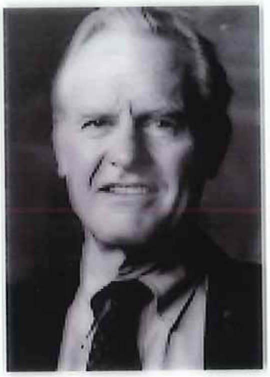
$$x = \frac{Q^2}{2p.q}, \qquad y = \frac{p.q}{p.l}$$

$$0 \leq x, y \leq 1$$

# Study of the Nucleon Structure

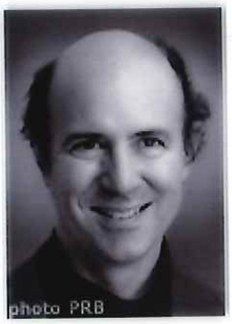



Rutherford Chadwick Stern Hofstadter

Friedman Kendall Taylor Bjorken
Dirac Medal 2004





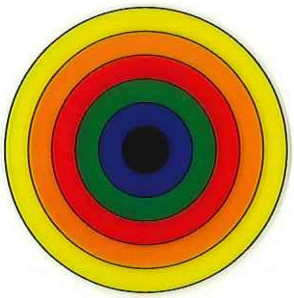

Feynman Gross (LL2004: April DESY) Politzer Wilczek
Nobel Laureates 2004

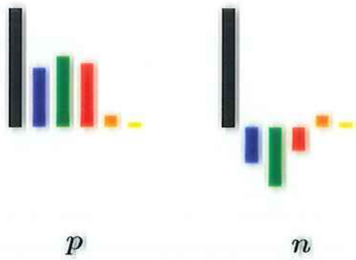
# The Resolution of the Nucleon Microscope

$$\Delta x \sim \frac{1}{|Q|} = \frac{1}{\sqrt{-q^2}}$$

**Examples :**

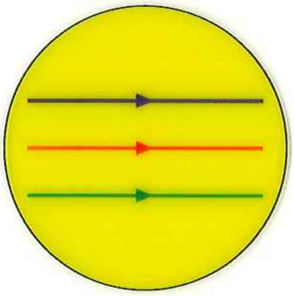
$Q^2 \sim 0.5 \cdot M_p^2$





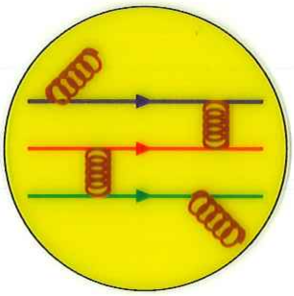

Charge Distribution

$Q^2 \sim 3 \cdot M_p^2$



Scaling

$Q^2 \sim 10...500 \cdot M_p^2$



Violation of Scaling

If there are New Compositeness Scales, one may find them in the Future.

$$Q^2 > 10^4\,\mathrm{GeV}^2, \qquad 1\,\mathrm{GeV}^2 \sim M_p^2$$

# THE EXPERIMENTAL FACTS

HOFSTADTER et al.
1950'ies

OLSON, SCHOPPER, WILSON
1961.

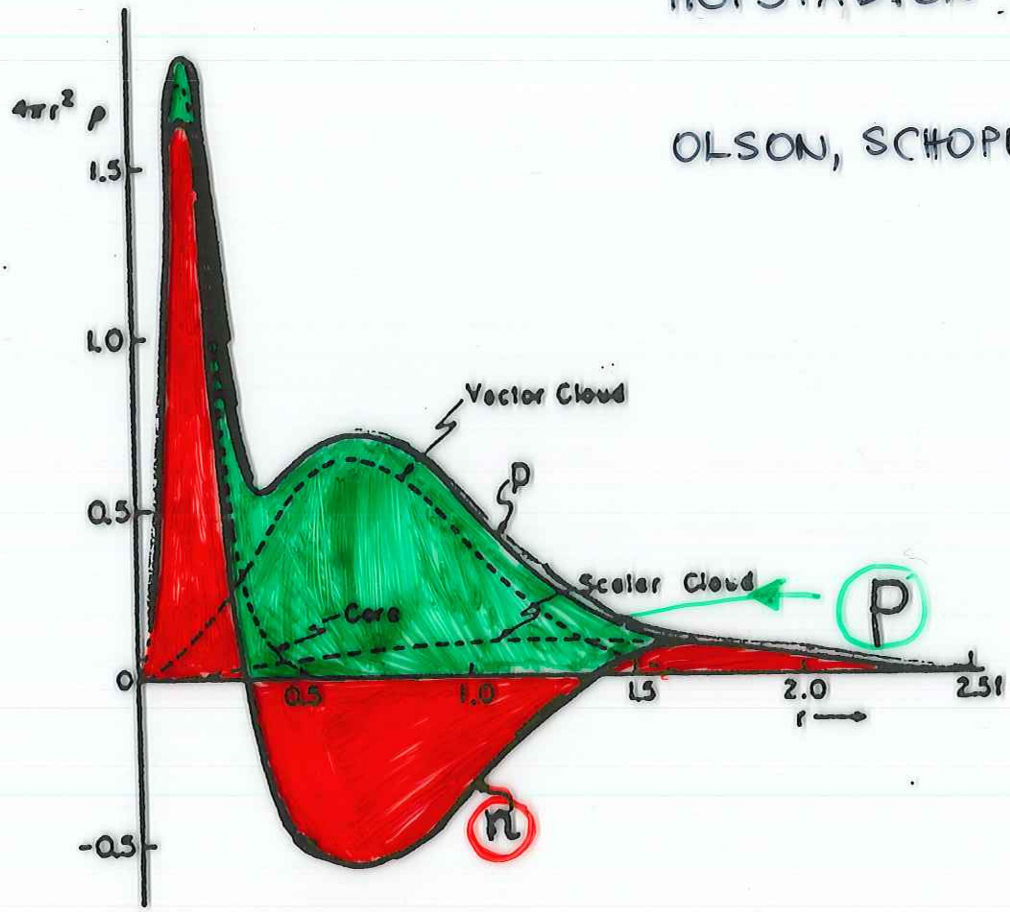



FIG. 4. Charge distribution for the proton and the neutron implied by the form factors shown for the fit (b) in Fig. 2(b).

NUCLEONS ARE COMPOSITE !

# When is a Parton ?

S. DRELL: **Infinite Momentum Frame:** $P$ - large

$$\tau_{\rm int} \ll \tau_{\rm life}$$

$$\tau_{\rm int} \sim \frac{1}{q_0} = \frac{4Px}{Q^2(1-x)}$$

$$\tau_{\rm life} \sim \frac{1}{\sum_i E_i - E} = \frac{2P}{\sum_i (k_{\perp i}^2 + M_i^2)/x_i - M^2} \simeq \frac{2Px(1-x)}{k_\perp^2}$$

$$\frac{\tau_{\rm int}}{\tau_{\rm life}} = \frac{2k_\perp^2}{Q^2(1-x)^2}$$

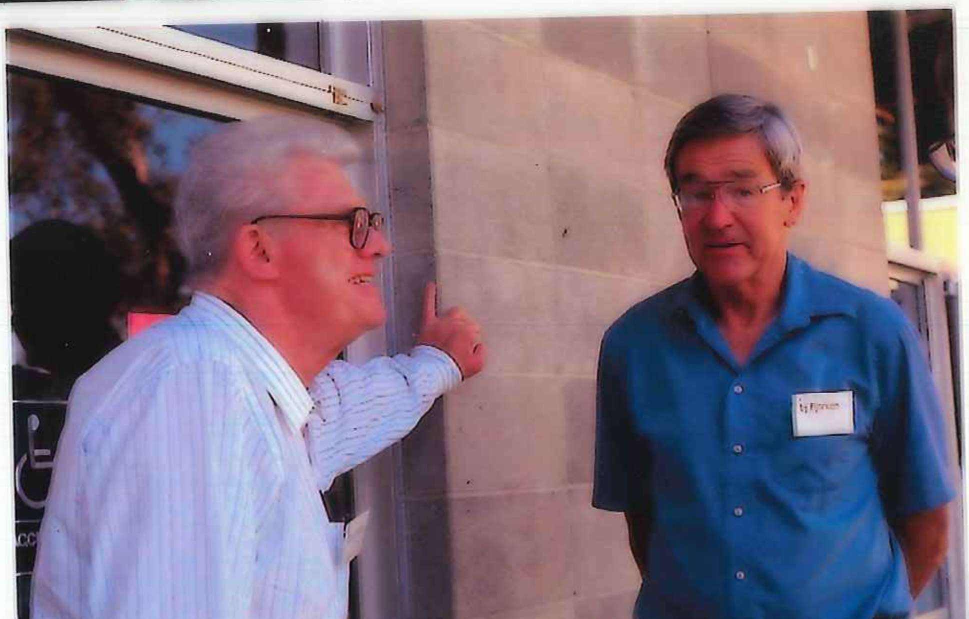
Stay away from $x \rightarrow 0$, since $xP$ becomes too small.

Stay away from $x \rightarrow 1$.

$$Q^2 \gg k_\perp^2.$$

S.DRELL T.M.YAN

J.D. BJORKEN



R.TAYLOR.

## Main Research Objectives :

- ☞ Precise Measurement of $\alpha_s(M_Z^2)$
- ☞ Reveal polarized and unpolarized parton densities at highest precision
- ☞ Precision tests of QCD
- ☞ Find novel sub-structures

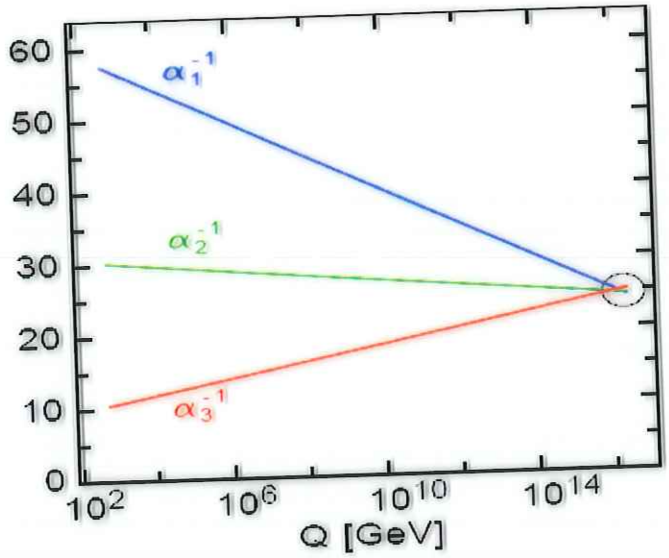
$\Longrightarrow$ Perturbative QCD :

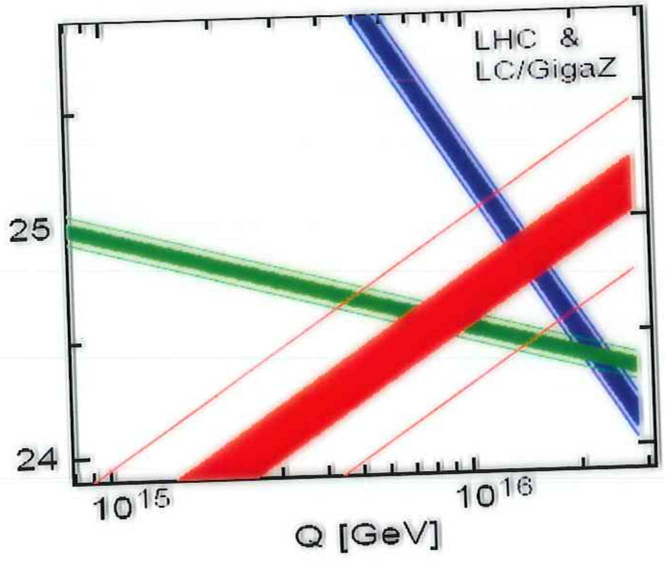
NNLO calculations using new technologies

$\Longrightarrow$ Lattice QCD :

Calculation of certain non-perturbative quantities a priori

# UNIFICATION OF FORCES AND $\alpha_s$




P. Zerwas, 2004

$$\frac{\delta\alpha(0)}{\alpha(0)} \sim 3 \cdot 10^{-11} \qquad \frac{\delta\alpha_w}{\alpha_w} \sim 7 \cdot 10^{-4} \qquad \frac{\delta\alpha_s(M_Z^2)}{\alpha_s(M_Z^2)} \sim 2 \cdot 10^{-2}$$

# 2. Basic Techniques

$$\frac{d\sigma^{\rm DIS}}{dxdy} \propto \sum_{s'} \overline{|M|^2} = \frac{1}{Q^4} \quad L_{\mu\nu} \quad W^{\mu\nu}, \qquad \text{pure } \gamma \text{ exchange.}$$

| | | |
|---|---|---|
| $L_{\mu\nu}$ | – | calculable |
| $W^{\mu\nu}$ | – | not calculable |

**Parameterize:** according to the symmetries *P, T, C, etc.*

$$W^{\mu\nu} = \left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}\right) W_1(x,Q^2) + \frac{1}{M_p^2} \widehat{P}_\mu \widehat{P}_\nu W_2(x,Q^2) + \ldots .$$

$$\widehat{P}_\mu = p_\mu - \frac{q.p}{q^2} q_\mu .$$

# The Parton Model :

R.P. Feynman, 1969; J.D. Bjorken, E.A. Paschos, 1969

**ANSATZ:**
$W_i(x, Q^2)$ is obtained as an integral over the momentum distributions of Local Sub-Components, the Partons.

$$W_2(x, Q^2) = \sum_i \int_0^1 dx_i f(x_i) x_i e_i^2 \delta\left(\frac{q.p_i}{M^2} - \frac{Q^2}{2M}\right)$$

$\Longrightarrow$ Strong correlation between $p.q$ and $Q^2$

$\Longrightarrow$ "Micro Canonical Ensemble"

$f_i(x)$ - Distribution Function

$$q.p_i = x_i p.q, \quad 2p.q = Q^2/x, \quad M\nu = p.q$$

$$\nu W_2(x, Q^2) = \sum_i e_i^2 x f_i(x) \equiv F_2(x) \ .$$

**Bjorken Limit :**

$$Q^2 \to \infty, \qquad \nu \to \infty$$

$$x = \text{const.}$$

**Scaling :**

$$MW_1(\nu, Q^2) \rightarrow F_1(x)$$
$$\nu W_2(\nu, Q^2) \rightarrow F_2(x)$$

# THE LIGHT CONE EXPANSION :

## Non-forward scattering:

$$\gamma^* + p_1 \rightarrow \gamma'^* + p_2$$

$$T_{\mu\nu}(p_+, p_-, q) = i \int d^4x\, e^{iqx} \langle p_2, S_2 | T(J_\mu(x/2) J_\nu(-x/2)) | p_1, S_1 \rangle \ .$$

$$p_+ = p_2 + p_1, \qquad p_- = p_2 - p_1 = q_1 - q_2,$$
$$q = \frac{1}{2}(q_1 + q_2), \qquad p_1 + q_1 = p_2 + q_2 \ ,$$

## Generalized Bjorken Limit:

$$\nu = qp_+ \longrightarrow \infty, \qquad Q^2 = -q^2 \longrightarrow \infty \ ,$$

## Scaling Variables:

$$\xi = \frac{Q^2}{qp_+}, \qquad \eta = \frac{qp_-}{qp_+} = \frac{q_1^2 - q_2^2}{2\nu}$$

$$\Longrightarrow \qquad RT\,(J_\mu(x/2) J_\nu(-x/2) S)$$

$$J_\mu(x) = \overline{\psi}(x) \gamma_\mu \lambda^{\rm em} \psi(x) \ ,$$

$$RT(J_\mu(x/2)J_\nu(-x/2)S)$$
$$= i : \overline{\psi}(x/2)\{\hat{S}_{\mu\nu\rho\sigma} - i\varepsilon_{\mu\nu\rho\sigma}\gamma_5\}\gamma^\sigma(i\partial_x^\rho D^c(x))c^a\lambda_f^a\psi(-x/2) :$$
$$- [(x/2,\ \mu) \leftrightarrow (-x/2,\ \nu)]$$
$$+ \text{higher order terms}$$

$$\hat{S}_{\mu\nu\rho\sigma} \equiv -S_{\mu\rho\nu\sigma} = g_{\mu\nu}g_{\rho\sigma} - g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\rho\nu}$$

$\varepsilon_{\mu\nu\rho\sigma}$ Levi – –Civita symbol

$$RT\,(J_\mu(x/2)J_\nu(-x/2)S) =$$
$$\frac{1}{2\pi^2}\frac{c^a x^\rho}{(x^2 - i\varepsilon)^2}\int_{-\infty}^{+\infty} d\kappa_1 \int_{-\infty}^{+\infty} d\kappa_2$$
$$\times \left\{\frac{i}{2}\left[\overline{\psi}(\kappa_1 x)\lambda_f^a\gamma^\sigma\psi(\kappa_2 x) - \overline{\psi}(\kappa_2 x)\lambda_f^a\gamma^\sigma\psi(\kappa_1 x)\right](-\hat{S}_{\mu\nu\rho\sigma})\Delta_-(\kappa_1,\kappa_2)\right.$$
$$+\frac{i}{2}\left[\overline{\psi}(\kappa_1 x)\gamma_5\lambda_f^a\gamma^\sigma\psi(\kappa_2 x) + \overline{\psi}(\kappa_2 x)\gamma_5\lambda_f^a\gamma^\sigma\psi(\kappa_1 x)\right] i\varepsilon_{\mu\nu\rho\sigma}\Delta_+(\kappa_1,\kappa_2)$$
$$+ \ \ldots\ ,$$

Wilson coefficient :

$$\Delta_\pm(\kappa_1,\kappa_2) = \left[\delta(\kappa_1 - \tfrac{1}{2})\delta(\kappa_2 + \tfrac{1}{2}) \pm \delta(\kappa_2 - \tfrac{1}{2})\delta(\kappa_1 + \tfrac{1}{2})\right] .$$

Light-cone vector :

$$\tilde{x} = x + \frac{\zeta}{\zeta^2}\left(\sqrt{(x\zeta)^2 - x^2\,\zeta^2} - (x\zeta)\right), \qquad \tilde{x}^2 = 0$$

$$RT\left(J_\mu(x/2)J_\nu(-x/2)S\right) \approx \int_{-1}^{+1} d^2\underline{\kappa}\, C_\Gamma(x^2,\underline{\kappa};\mu^2) RT\Big(O^\Gamma(\kappa_1\tilde{x},\kappa_2\tilde{x})S\Big)$$

$$O^\Gamma(\kappa_1\tilde{x},\kappa_2\tilde{x}) = \int \frac{dp_1}{(2\pi)^4}\frac{dp_2}{(2\pi)^4} e^{i\kappa_1\tilde{x}p_1 + i\kappa_2\tilde{x}p_2} : \overline{\psi}(p_1)\Gamma\psi(p_2):$$

$$\begin{aligned} O^\Gamma(\kappa_1\tilde{x},\kappa_2\tilde{x}) &= \sum_{n_1 n_2} \frac{\kappa_1^{n_1}}{n_1!}\frac{\kappa_2^{n_2}}{n_2!} O^\Gamma_{n_1 n_2}(\tilde{x}), \\ O^\Gamma_{n_1 n_2}(\tilde{x}) &= \frac{\partial^{n_1}}{\partial\kappa_1^{n_1}}\frac{\partial^{n_2}}{\partial\kappa_2^{n_2}} O^\Gamma(\kappa_1\tilde{x},\kappa_2\tilde{x})\Big|_{\kappa_1=\kappa_2=0} \end{aligned}$$

$$(\tilde{x}\vec{\partial})^n \equiv \tilde{x}^{\mu_1}...\tilde{x}^{\mu_n}\vec{\partial}_{\mu_1}...\vec{\partial}_{\mu_n} \quad \text{(axial gauge)}.$$

$$O^\Gamma_{n_1 n_2}(\tilde{x}) = \overline{\psi}(0)(\overleftarrow{\partial}\,\tilde{x})^{n_1}\Gamma(\tilde{x}\,\overrightarrow{\partial})^{n_2}\psi(0)\,.$$

**Current–Current Operator:**

$$\begin{aligned} &RT(J_\mu(x/2)J_\nu(-x/2)S) \approx \frac{1}{2}\int D\kappa \\ &\times\Big[C_a(x^2,\kappa_+,\kappa_-,\mu^2)\Big(g_{\mu\nu}O^a(\kappa_+\tilde{x},\kappa_-\tilde{x},\mu^2) - \tilde{x}_\mu O^a_\nu(\kappa_+\tilde{x},\kappa_-\tilde{x},\mu^2) \\ &\qquad -\tilde{x}_\nu O^a_\mu(\kappa_+\tilde{x},\kappa_-\tilde{x},\mu^2)\Big) \\ &+ i\,C_{a,5}(x^2,\kappa_+,\kappa_-,\mu^2)\varepsilon_{\mu\nu}{}^{\rho\sigma}\tilde{x}_\rho O^a_{5,\sigma}(\kappa_+\tilde{x},\kappa_-\tilde{x},\mu^2)\Big] + ... \end{aligned}$$

**Measure:**

$$\begin{aligned} D\kappa &= d\kappa_1 d\kappa_2 \theta(1-\kappa_1)\theta(1+\kappa_1)\theta(1-\kappa_2)\theta(1+\kappa_2) \\ &= 2d\kappa_+ d\kappa_- \theta(1+\kappa_+ + \kappa_-)\theta(1+\kappa_+ - \kappa_-) \\ &\quad \theta(1-\kappa_+ + \kappa_-)\theta(1-\kappa_+ - \kappa_-) \end{aligned}$$

$$C_a(x^2,\kappa_+,\kappa_-) = \frac{1}{2\pi^2}\frac{c_a}{(x^2-i\varepsilon)^2}\delta(\kappa_+)\left[\delta(\kappa_- - \tfrac{1}{2}) - \delta(\kappa_- + \tfrac{1}{2})\right],$$

$$C_{a,5}(x^2,\kappa_+,\kappa_-) = \frac{1}{2\pi^2}\frac{c_a}{(x^2-i\varepsilon)^2}\delta(\kappa_+)\left[\delta(\kappa_- - \tfrac{1}{2}) + \delta(\kappa_- + \tfrac{1}{2})\right]$$

$$\begin{aligned} O_\sigma^a(\kappa_1,\kappa_2) &= \frac{i}{2}RT\Big[\overline{\psi}(\kappa_1\tilde{x})\lambda_f^a\gamma_\sigma U(\kappa_1,\kappa_2)\psi(\kappa_2\tilde{x}) \\ &\quad -\overline{\psi}(\kappa_2\tilde{x})\lambda_f^a\gamma_\sigma U(\kappa_2,\kappa_1)\psi(\kappa_1\tilde{x})\Big]S, \\ O_{5,\sigma}^a(\kappa_1,\kappa_2) &= \frac{i}{2}RT\Big[\overline{\psi}(\kappa_1\tilde{x})\lambda_f^a\gamma_5\gamma_\sigma U(\kappa_1,\kappa_2)\psi(\kappa_2\tilde{x}) \\ &\quad +\overline{\psi}(\kappa_2\tilde{x})\lambda_f^a\gamma_5\gamma_\sigma U(\kappa_2,\kappa_1)\psi(\kappa_1\tilde{x})\Big]S, \end{aligned}$$

$$\begin{aligned} O_{\mu\nu}^G(\kappa_1,\kappa_2) &= \frac{1}{2}RT\big[F_\mu^{a\lambda}(\kappa_1\tilde{x})U^{ab}(\kappa_1,\kappa_2)F_{\nu\lambda}^b(\kappa_2\tilde{x}) \\ &\quad +F_\mu^{a\lambda}(\kappa_2\tilde{x})U^{ab}(\kappa_2,\kappa_1)F_{\nu\lambda}^b(\kappa_2\tilde{x})\big]S, \\ O_{5\mu\nu}^G(\kappa_1,\kappa_2) &= \frac{1}{2}RT\big[F_\mu^{a\lambda}(\kappa_1\tilde{x})U^{ab}(\kappa_1,\kappa_2)\widetilde{F}_{\nu\lambda}^b(\kappa_2\tilde{x}) \\ &\quad -F_\mu^{a\lambda}(\kappa_2\tilde{x})U^{ab}(\kappa_2,\kappa_1)\widetilde{F}_{\nu\lambda}^b(\kappa_1\tilde{x})\big]S, \end{aligned}$$

Gauge link:

$$U(\kappa_1,\kappa_2) \equiv U(\kappa_1\tilde{x},\kappa_2\tilde{x}) = \mathcal{P}\exp\left\{-ig\int_{\kappa_2}^{\kappa_1} d\tau\, \tilde{x}^\mu A_\mu(\tau\tilde{x})\right\},$$

$$F_{\mu\nu} \equiv F_{\mu\nu}^a t^a \quad \text{and} \quad \widetilde{F}_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$$

# Twist Decomposition

$$O_\sigma^a = O_\sigma^{a,\,\rm twist2} + O_\sigma^{a,\,\rm twist3} + O_\sigma^{a,\,\rm twist4},$$
$$O_{5,\sigma}^a = O_{5,\sigma}^{a,\,\rm twist2} + O_{5,\sigma}^{a,\,\rm twist3} + O_{5,\sigma}^{a,\,\rm twist4}.$$

$$O^{q,\,\rm traceless}(-\kappa x, \kappa x) = \frac{i}{2}\left[\overline{\psi}(-\kappa x)(x\gamma)\psi(\kappa x) - \overline{\psi}(\kappa x)(x\gamma)\psi(-\kappa x)\right]$$
$$+ \sum_{k=1}^{\infty}\int_0^1 dt \left(\frac{1-t}{t}\right)^{k-1}\left(\frac{-x^2}{4}\right)^k \frac{\Box^k}{k!(k-1)!}$$
$$\times\frac{i}{2}\left[\overline{\psi}(-\kappa t x)(x\gamma)\psi(\kappa t x) - \overline{\psi}(\kappa t x)(x\gamma)\psi(-\kappa t x)\right],$$

$$O^{q,\,\rm twist2}(-\kappa\tilde{x}, \kappa\tilde{x}) = O^{q,\,\rm traceless}(-\kappa\tilde{x}, \kappa\tilde{x}) \equiv \tilde{x}^\sigma O_\sigma^q(-\kappa\tilde{x}, \kappa\tilde{x})$$

$$\Box O^{q,\,\rm traceless}(-\kappa x, \kappa x) = 0$$

$$\tilde{x}^\sigma O_\sigma^{q,\,\rm twist2} = O^{q,\,\rm twist2}, \qquad \tilde{x}^\sigma O_\sigma^{q,\,\rm twist3} = 0, \qquad \tilde{x}^\sigma O_\sigma^{q,\,\rm twist4} = 0.$$

# FORWARD SCATTERING :

Brandt, Preparata, Zimmermann, Frishman, Christ et al.

$$W_{\mu\nu}(p.q) = \int d^4 x e^{iqx} \langle p \| [j_\mu(x), j_\nu(0)] \| p \rangle$$

$$\begin{aligned} T\left[j_\mu(x), j_\nu(0)\right] &= \frac{x^2 g_{\mu\nu} - 2x_\mu x_\nu}{\pi^4 (x^2 - i\varepsilon)^4} + O_{\mu\nu} \\ &\quad - i \frac{x^\lambda \sigma_{\mu\lambda\nu\rho} O_V^\rho(x,0)}{2\pi^2(x^2 - i\varepsilon)} - i \frac{x^\lambda \varepsilon_{\mu\lambda\nu\rho} O_{V5}^\rho(x,0)}{2\pi^2(x^2 - i\varepsilon)} \end{aligned}$$

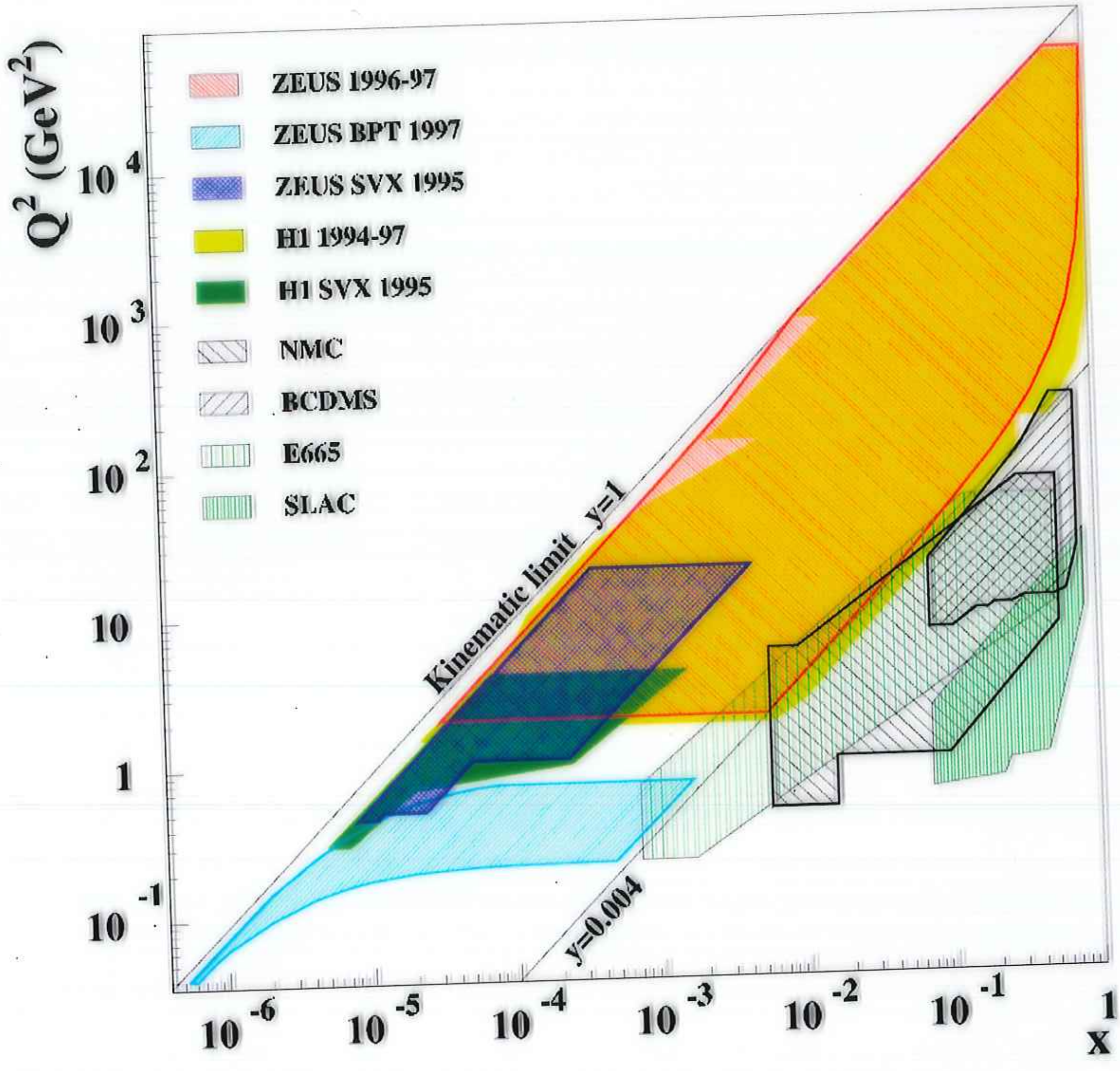
$$\begin{aligned} O_V^\mu(x,y) &= :\overline{\psi(x)}\gamma^\mu\psi(y) - \overline{\psi(y)}\gamma^\mu\psi(x): \\ O_{V5}^\mu(x,y) &= :\overline{\psi(x)}\gamma^\mu\gamma_5\psi(y) - \overline{\psi(y)}\gamma^\mu\gamma_5\psi(x): \\ O^{\mu\nu}(x,y) &= :\overline{\psi(x)}\gamma^\mu\psi(x)\overline{\psi}(y)\gamma^\nu\psi(x): \\ \psi(x) &= \psi(0) + x^\mu \left[\partial_\mu \psi(x)\right]_{x=0} + \frac{1}{2!} x^\mu x^\nu \left[\partial_\mu \partial_\nu \psi(x)\right]_{x=0} \\ &\quad + \ldots \\ O_{V,V5}^\mu(x,0) &= \sum_{n=0}^{\infty} \frac{1}{n!} x^{\mu_1} \ldots x^{\mu_n} O_{V,V5,\mu_1,\ldots,\mu_n}^\mu(0) \end{aligned}$$

$\Longrightarrow$ Calculate anomalous dimensions for Operators.

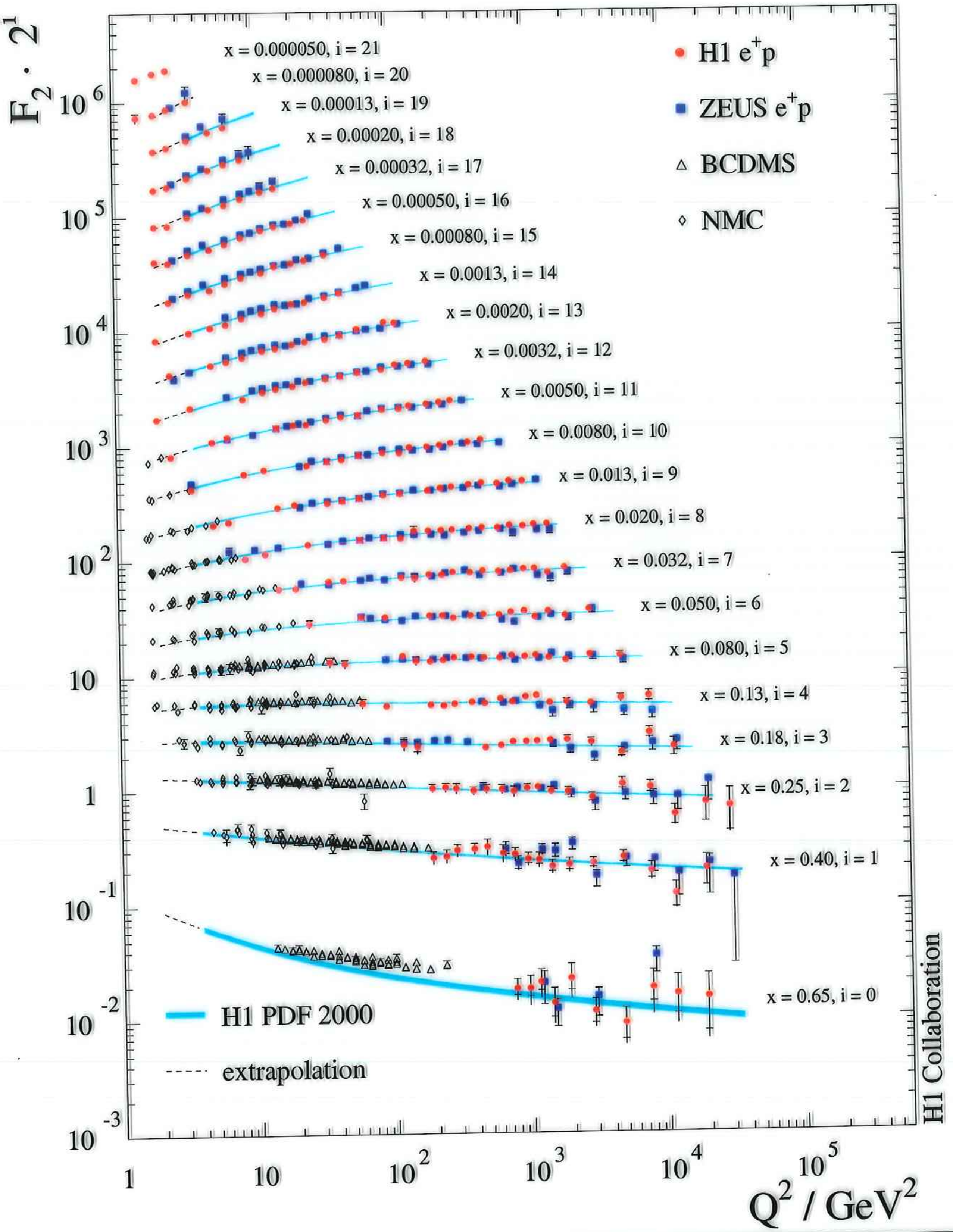
$\Longrightarrow$ Only safe way to Higher Twists

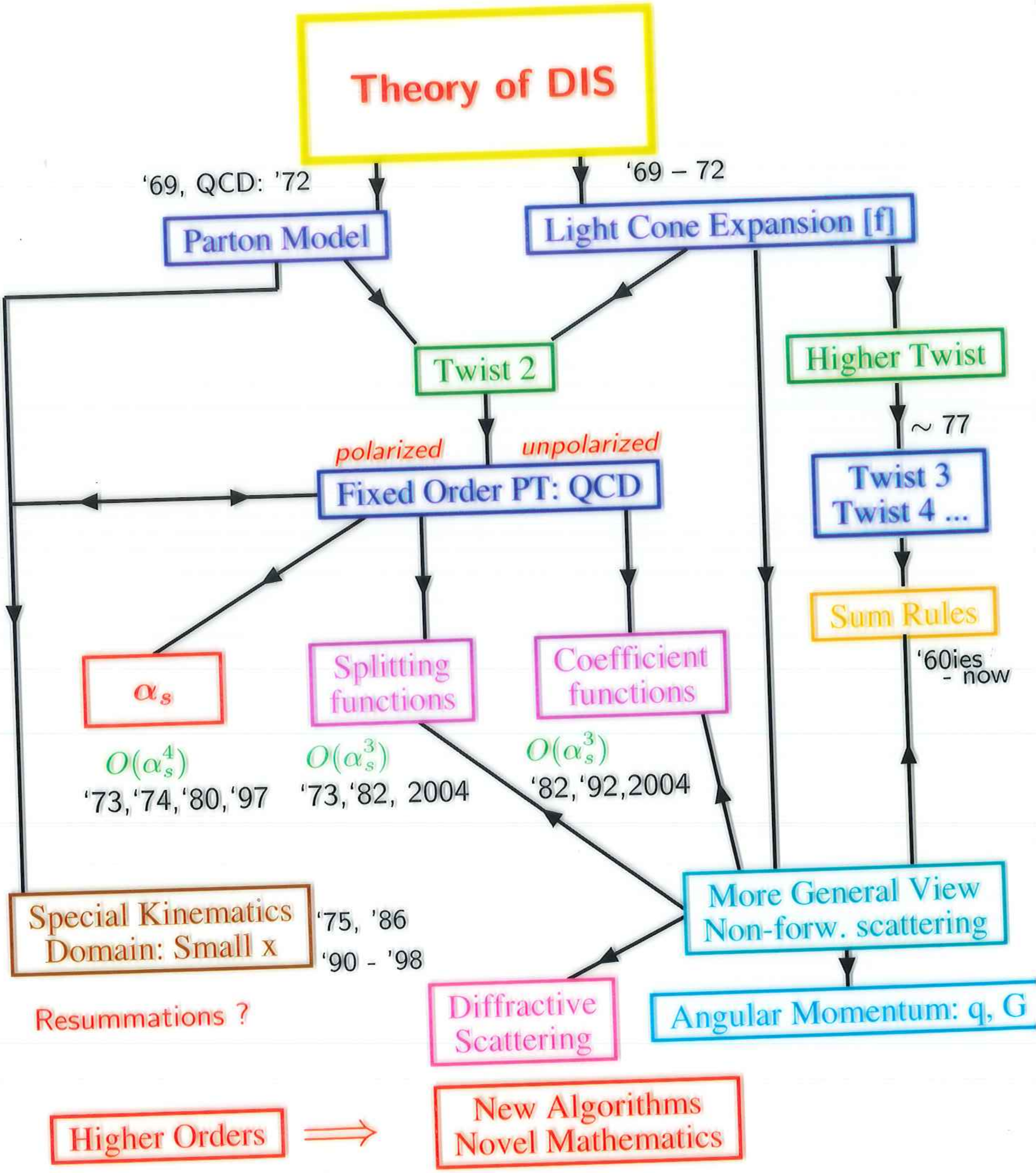
Twist 2: LCE $\simeq$ PARTON MODEL

# Kinematic Domain



# H1, ZEUS + fixed target data




Theory of DIS
'69, QCD: '72
'69 – 72
Parton Model
Light Cone Expansion [f]
Twist 2
Higher Twist
~ 77
polarized
unpolarized
Fixed Order PT: QCD
Twist 3
Twist 4 ...
Sum Rules
'60ies - now
$\alpha_s$
Splitting functions
Coefficient functions
$O(\alpha_s^4)$
'73,'74,'80,'97
$O(\alpha_s^3)$
'73,'82, 2004
$O(\alpha_s^3)$
'82,'92,2004
Special Kinematics Domain: Small x
'75, '86
'90 - '98
More General View Non-forw. scattering
Resummations ?
Diffractive Scattering
Angular Momentum: q, G
Higher Orders
$\Longrightarrow$
New Algorithms Novel Mathematics

# 3. QCD Perturbation Theory to $O(\alpha_s^3)$, $\Lambda_{QCD}$ and the PDF's

How can we measure $\alpha_s(Q^2)$ from the scaling violations of Structure Functions?

$$
\begin{aligned}
F_j(x,Q^2) &= \hat{f}_i(x,\mu^2) \otimes \sigma_j^i\left(\alpha_s, \frac{Q^2}{\mu^2}, x\right) \\
& \quad \uparrow \text{bare pdf} \qquad \uparrow \text{sub}-\text{system cross}-\text{sect.} \\
&= \underbrace{\hat{f}_i(x,\mu^2) \otimes \Gamma_k^i\left(\alpha_s(R^2), \frac{M^2}{\mu^2}, \frac{M^2}{R^2}\right)}_{\text{finite pdf} \equiv \mathrm{f_k}} \\
& \quad \otimes \underbrace{C_j^k\left(\alpha_s(R^2), \frac{Q^2}{\mu^2}, \frac{M^2}{R^2}, x\right)}_{\text{finite Wilson coefficient}}
\end{aligned}
$$

**Move to Mellin space :**

$$
F_j(N) = \int_0^1 dx x^{N-1} F_j(x)
$$

Diagonalization of the convolutions $\otimes$ into ordinary products.

## Renormalization Group Equations :

$$\left[M\frac{\partial}{\partial M}+\beta(g)\frac{\partial}{\partial g}-2\gamma_\psi(g)\right]F_i(N) = 0$$

$$\left[M\frac{\partial}{\partial M}+\beta(g)\frac{\partial}{\partial g}+\gamma_\kappa^N(g)-2\gamma_\psi(g)\right]f_k(N) = 0$$

$$\left[M\frac{\partial}{\partial M}+\beta(g)\frac{\partial}{\partial g}-\gamma_\kappa^N(g)\right]C_j^k(N) = 0$$

Callan–Symnanzik equations for mass factorization

≡ Altarelli–Parisi evolution equations

**x-space :**

$$\frac{d}{d\log(\mu^2)}\begin{pmatrix} q^+(x,Q^2) \\ G(x,Q^2)\end{pmatrix} = \frac{\alpha_s}{2\pi}\boldsymbol{P}(x,\alpha_s)\otimes\begin{pmatrix} q^+(x,Q^2) \\ G(x,Q^2)\end{pmatrix}$$

$$\boldsymbol{P}(x,\alpha_s) = \boldsymbol{P}^{(0)}(x)+\frac{\alpha_s}{2\pi}\boldsymbol{P}^{(1)}(x)+\left(\frac{\alpha_s}{2\pi}\right)^2\boldsymbol{P}^{(2)}(x)+\dots$$

Evolution equs.: 3 non-singlet, 1 singlet

Separation of non-singlet and singlet quark contributions is **essential.**

## 3.1. Running Coupling Constant

$$\frac{\partial a_s(\mu^2)}{\partial \log \mu^2} = -\beta_0 a_s^2 - \beta_1 a_s^3 - \beta_2 a_s^4 - \beta_3 a_s^5 + O(a_s^6)$$

$$a_s \equiv \frac{g_{\rm ren}^2}{(4\pi)^2} = \frac{\alpha_s}{2\pi}$$

**The values of the $\beta_k$ :**

$\beta_0 = 11 - \frac{2}{3} N_f$ Gross, Politzer, Wilczek, t'Hooft, 1973

Discovery of asymptotic freedom :

Nobel Laureates 2004

$\beta_1 = 102 - \frac{38}{3} N_f$ Caswell(†11.9.01), Jones, 1974

$\beta_2 = \frac{2857}{2} - \frac{5033}{18} N_f + \frac{325}{54} N_f^2$

Tarasov, Vladimirov, Zharkov, 1981

Larin, Vermaseren, 1992

$$\beta_3 = \left(\frac{149753}{6} + 3564\zeta_3\right) - \left(\frac{1078361}{162} + \frac{6508}{27}\zeta_3\right) N_f + \left(\frac{50065}{162} + \frac{6472}{81}\zeta_3\right) N_f^2 + \frac{1093}{729} N_f^3$$

van Ritbergen, Vermaseren, Larin, 1997

The solution of the RGE leads to a falling coupling constant as scales increase.

## SOLUTION OF THE RGE :

$$\frac{1}{\alpha_s(Q^2)} = \frac{1}{\alpha_s(Q_0^2)} + \frac{\beta_0}{4\pi}\log\left(\frac{Q^2}{Q_0^2}\right)$$

$$+ \phi^{(n)}(\alpha_s(Q^2);\beta_i) - \phi^{(n)}(\alpha_s(Q_0^2);\beta_i)$$

$$\alpha_s^{LO}(Q^2) = \frac{\alpha_s^{LO}(Q_0^2)}{1+\frac{\beta_0}{4\pi}\alpha_s(Q_0^2)\log\left(\frac{Q^2}{Q_0^2}\right)}$$

$$\phi^{(2)}(x;\beta_i) = -\frac{\beta_1}{8\pi\beta_0}\log\left|\frac{16\pi^2x^2}{16\pi^2\beta_0+4\beta_1\pi x+\beta_2x^2}\right|$$

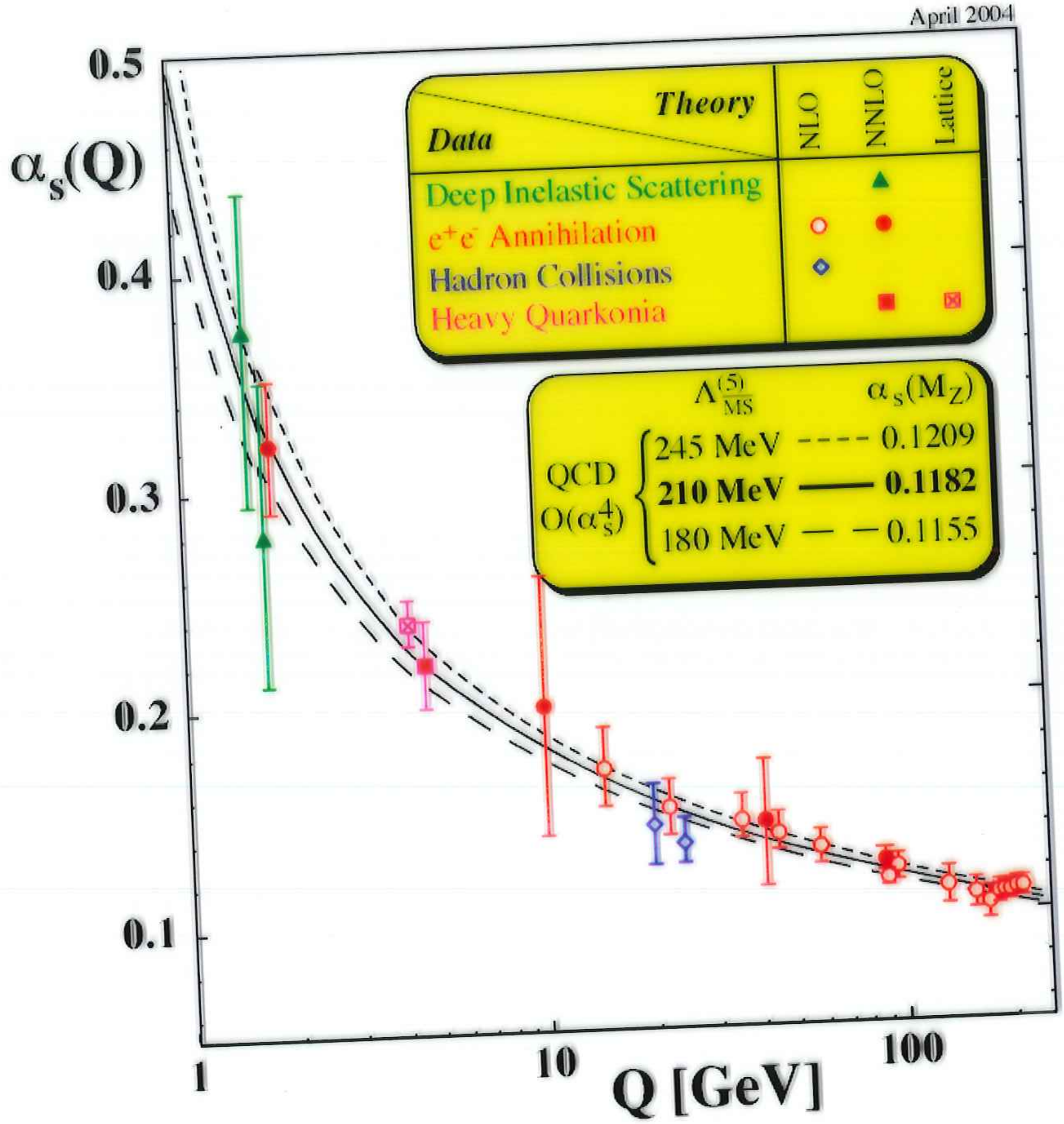
$$+\frac{\beta_1^2-2\beta_0\beta_2}{8\pi\beta_0\sqrt{4\beta_2\beta_0-\beta_1^2}}\arctan\left(\frac{2\pi\beta_1+\beta_2x}{2\pi\sqrt{4\beta_0\beta_2-\beta_1^2}}\right)$$

... etc.

ITERATIVE SOLUTION.

$N_f \leq 5 \qquad 4\beta_0\beta_2-\beta_1^2 > 0$

$N_f = 6 \qquad 4\beta_0\beta_2-\beta_1^2 < 0$.




S. Bethke, LL2004.

## 3.2. Splitting Functions

**$O(\alpha_s)$ unpolarized:**

$$
\begin{aligned}
P_{\rm NS}^{(0)}(z) \equiv P_{qq}^{(0)}(z) &= C_F \left[\frac{1+z^2}{1-z}\right]_+ \\
P_{qg}^{(0)}(z) &= T_f \left[(1-z)^2 + z^2\right] \\
P_{gq}^{(0)}(z) &= C_F \frac{1+(1-z)^2}{z} \\
P_{gg}^{(0)}(z) &= 2C_A \left[\frac{1-z}{z} + \frac{z}{(1-z)_+} + z(1-z)\right] + \frac{1}{2}\beta_0 \delta(1-z)
\end{aligned}
$$

QED : $P_{qq}$ Fermi, 1924 $P_{gq}$ Williams, 1933; Weizsäcker, 1934

Gross, Wilczek; Georgi, Politzer, 1973;

further: Lipatov, 1975; Altarelli, Parisi, 1977; Kim, Schilcher, 1977; Dokshitser, 1977

**$O(\alpha_s)$ polarized:**

$$
\begin{aligned}
\Delta P_{qq}^{(0)}(z) &= P_{qq}^{(0)}(z) \\
\Delta P_{qg}^{(0)}(z) &= T_f \left[(1-z)^2 - z^2\right] \\
\Delta P_{gq}^{(0)}(z) &= C_F \frac{1-(1-z)^2}{z} \\
\Delta P_{gg}^{(0)}(z) &= 2C_A \left[\left(\frac{1}{1-z}\right)_+ + 1 - 2z\right] + \frac{1}{2}\beta_0 \delta(1-z)
\end{aligned}
$$

Ito, 1975; K. Sasaki, 1975; Ahmed & Ross 1975,1976;

correct: Altarelli, Parisi, 1977.

**no terms $\propto 1/z$.**

# SINGLET:

## $P_{ij}^{(1)}(x)$ :

$$
\begin{aligned}
p_{FF}^{(1,S)} = {} & C_F^2 [-1 + x + (\tfrac{1}{2} - \tfrac{3}{2}x)\ln x - \tfrac{1}{2}(1+x)\ln^2 x - (\tfrac{3}{2}\ln x + 2\ln x \ln(1-x))p_{FF}(x) + 2p_{FF}(-x)S_2(x)] \\
& + C_F C_G [\tfrac{14}{3}(1-x) + (\tfrac{11}{6}\ln x + \tfrac{1}{2}\ln^2 x + \tfrac{67}{18} - \tfrac{1}{6}\pi^2)p_{FF}(x) - p_{FF}(-x)S_2(x)] \\
& + C_F T_R N_F [-\tfrac{16}{3} + \tfrac{40}{3}x + (10x + \tfrac{16}{3}x^2 + 2)\ln x - \tfrac{112}{9}x^2 + \tfrac{40}{9}x^{-1} - 2(1+x)\ln^2 x - (\tfrac{10}{9} + \tfrac{2}{3}\ln x)p_{FF}(x)] ,
\end{aligned}
$$

$$
\begin{aligned}
p_{FG}^{(1,S)} = {} & C_F^2 [-\tfrac{5}{2} - \tfrac{7}{2}x + (2 + \tfrac{7}{2}x)\ln x + (-1 + \tfrac{1}{2}x)\ln^2 x - 2x\ln(1-x) + (-3\ln(1-x) - \ln^2(1-x))p_{FG}(x)] \\
& + C_F C_G [\tfrac{28}{9} + \tfrac{65}{18}x + \tfrac{44}{9}x^2 + (-12 - 5x - \tfrac{8}{3}x^2)\ln x + (4+x)\ln^2 x + 2x\ln(1-x) + (-2\ln x\ln(1-x) \\
& + \tfrac{1}{2}\ln^2 x + \tfrac{11}{3}\ln(1-x) + \ln^2(1-x) - \tfrac{1}{6}\pi^2 + \tfrac{1}{2})p_{FG}(x) + p_{FG}(-x)S_2(x)] \\
& + C_F T_R N_F [-\tfrac{4}{3}x - (\tfrac{20}{9} + \tfrac{4}{3}\ln(1-x))p_{FG}(x)] ,
\end{aligned}
$$

$$
\begin{aligned}
p_{GF}^{(1,S)} = {} & C_F T_R N_F [4 - 9x + (-1 + 4x)\ln x + (-1 + 2x)\ln^2 x + 4\ln(1-x) \\
& + (-4\ln x\ln(1-x) + 4\ln x + 2\ln^2 x - 4\ln(1-x) + 2\ln^2(1-x) - \tfrac{2}{3}\pi^2 + 10)p_{GF}(x)] \\
& + C_G T_R N_F [\tfrac{182}{9} + \tfrac{14}{9}x + \tfrac{40}{9}x^{-1} + (\tfrac{136}{3}x - \tfrac{38}{3})\ln x - 4\ln(1-x) - (2 + 8x)\ln^2 x + (-\ln^2 x \\
& + \tfrac{44}{3}\ln x - 2\ln^2(1-x) + 4\ln(1-x) + \tfrac{1}{3}\pi^2 - \tfrac{218}{9})p_{GF}(x) + 2p_{GF}(-x)S_2(x)] ,
\end{aligned}
$$

$$
\begin{aligned}
p_{GG}^{(1,S)} = {} & C_F T_R N_F [-16 + 8x + \tfrac{20}{3}x^2 + \tfrac{4}{3}x^{-1} + (-6 - 10x)\ln x + (-2 - 2x)\ln^2 x] \\
& + C_G T_R N_F [2 - 2x + \tfrac{26}{9}x^2 - \tfrac{26}{9}x^{-1} - \tfrac{4}{3}(1+x)\ln x - \tfrac{20}{9}p_{GG}(x)] \\
& + C_G^2 [\tfrac{27}{2}(1-x) + \tfrac{67}{9}(x^2 - x^{-1}) + (-\tfrac{25}{3} + \tfrac{11}{3}x - \tfrac{44}{3}x^2)\ln x + 4(1+x)\ln^2 x + (\tfrac{67}{9} - 4\ln x\ln(1-x) \\
& + \ln^2 x - \tfrac{1}{3}\pi^2)p_{GG}(x) + 2p_{GG}(-x)S_2(x)] .
\end{aligned}
$$

$$
S_2(x) \equiv \int_{(1+x)/x}^{1/(1+x)} \frac{dz}{z} \ln\left(\frac{1-z}{z}\right); \quad S_1(x) \equiv \int_0^{1-x} \frac{dz}{z} \ln(1-z) .
$$

## 2 Loop :

**UNPOLARIZED:**

Floratos, D. Ross, Sachraida, 1977-79; Curci, Furmanski, Pertonzio, 1980; Furmanski, Petronzio, 1980; Gonzalez-Arroyo, Lopez, Yndurain, 1979, 1980; Floratos, Kounnas, Lacaze, 1981abc; van Neerven, Hamberg, 1982;

**POLARIZED:**

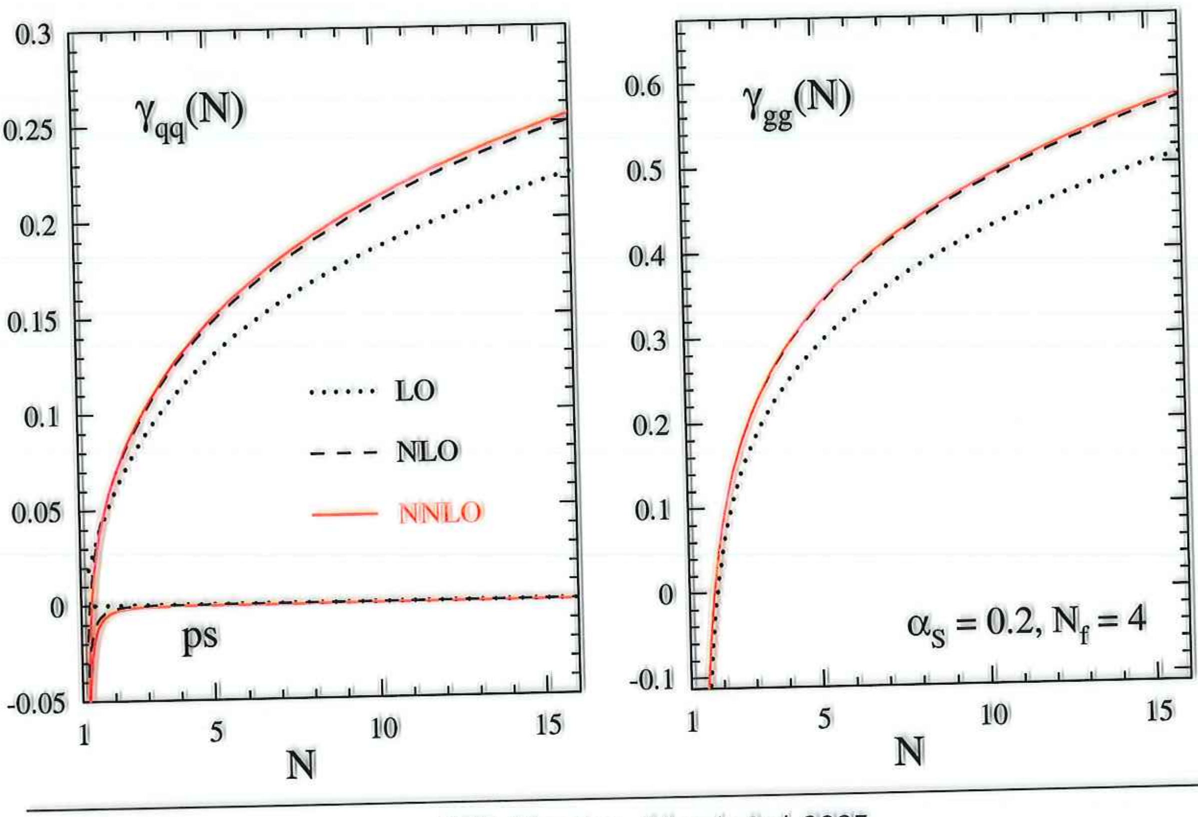
Zijlstra, van Neerven, 1994; Mertig, van Neerven, 1995; Vogelsang 1995.

## 3 Loop :

**UNPOLARIZED:**

Moments : Larin, Nogueira, van Ritbergen, Vermaseren, 1994, 1997; Retey, Vermaseren, 2001; J.B., Vermaseren, 2004.

Complete : Moch, Vermaseren, Vogt, 2004.



## 3.3. Coefficient Functions

**$O(\alpha_s)$ unpolarized:**

$$C^{(1)}_{F_2^q}(z) = C_F\left\{\frac{1+z^2}{1-z}\left[\ln\left(\frac{1-z}{z}\right)-\frac{3}{4}\right]+\frac{1}{4}(9+5z)\right\}_+$$

$$C^{(1)}_{F_2^g}(z) = 2N_fT_f\left\{\left[z^2+(1-z)^2\right]\ln\left(\frac{1-z}{z}\right)-1+8z(1-z)\right\}$$

$$C^{(1)}_{F_1^q}(z) = C^{(1)}_{F_2^q}(z)-C_F\cdot 2z$$

$$C^{(1)}_{F_1^g}(z) = C^{(1)}_{F_2^g}(z)-8N_fT_fz(1-z)$$

$$C^{(1)}_{F_3^q}(z) = C^{(1)}_{F_2^q}(z)-C_F(1+z)$$

FURMANSKI, PETRONZIO, 1982: correct form.

**$O(\alpha_s)$ polarized:**

$$C^{(1)}_{g_1^q}(z) = C^{(1)}_{F_1^q}(z)$$

$$C^{(1)}_{g_1^g}(z) = 4N_fT_f\left\{[2z-1]\ln\left(\frac{1-z}{z}\right)+3-4z\right\}$$

ALTARELLI, ELLIS, MARTINELLI, 1979; HUMPERT, VAN NEERVEN, 1981; BODWIN QUI, 1990.

## 2 Loop :

**polarized, unpolarized:**

Zijlstra, van Neerven 1992–1994;

Moments: Moch, Vermaseren, 1999

**unpolarized, heavy flavor:**

Laenen, Riemersma, Smith, van Neerven, 1993, 1994

Mellin Space: Alekhin, J.B., 2004

## 3 Loop :

**unpolarized:**

Moments : Larin, Nogueira, van Ritbergen, Vermaseren, 1994, 1997; Retey, Vermaseren, 2001; J.B., Vermaseren, 2004.

Complete : Moch, Vermaseren, Vogt, in preparation.

## Example : J.B., Vermaseren, 2004

$$
\begin{aligned}
C_2^{\mathrm{NS},16}(a_s) \quad = \quad & \frac{4047739719}{190590400}\, C_F\, a_s \\
+ \quad & \Bigg[\!\!\Bigg[ \left( \frac{4442667416304442887936697 0127}{3219318469217479564615680 00} \; \frac{24439538}{255255}\,\zeta_3 \right) {C_F}^2 \\
+ \quad & \left( \frac{1791830840849829422278308 7}{5942270587318281216000 0} - \frac{113298677}{1021020}\,\zeta_3 \right) C_F\, C_A \\
- \quad & \frac{14356837276190747211117 7}{27589113441120591360 00}\, C_F\, N_F \Bigg]\!\!\Bigg] {a_s}^2 \\
+ \quad & \Bigg[\!\!\Bigg[ \left( \frac{59290512768143}{3127445521200}\,\zeta_4 - \frac{27643576}{21879}\,\zeta_5 \right. \\
+ \quad & \frac{303681339759950972508467729384250597655916168 9}{803445801604077593342164786340334796800000 0} \\
+ \quad & \left. \frac{1494341926940450865387403}{5956740402060127680 00}\,\zeta_3 \right) {C_F}^3 \\
+ \quad & \left( \frac{59290512768143}{6254891042400}\,\zeta_4 + \frac{2628653778834757265588009355150331903 33}{56646805852503848671021043712000000} \right. \\
+ \quad & \left. \frac{47187263}{51051}\,\zeta_5 - \frac{153550504691714823 13}{4991403051835200}\,\zeta_3 \right) C_F\, {C_A}^2 \\
+ \quad & \left( \frac{722738493599967031231878988499 9}{76056398835262954714045440000} + \frac{64419601}{20675655}\,\zeta_3 \right) C_F\, {N_F}^2 \\
+ \quad & \left( \frac{77500266271187687528450917608900514652427 41}{1652500620329242273431025887166464000000} \right. \\
- \quad & \frac{28494820041389214915 31}{674116712167298400 0}\,\zeta_3 + \frac{983963}{21879}\,\zeta_5 \\
- \quad & \left. \frac{59290512768143}{2084963680800}\,\zeta_4 \right) {C_F}^2 C_A + \left( -\frac{552298563960959}{4021001384400}\,\zeta_3 \right. \\
- \quad & \left. \frac{4073207241348493196152222079933557529}{352977746994455372827884887040000 0} + \frac{64419601}{1531530}\,\zeta_4 \right) {C_F}^2 N_F \\
+ \quad & \left( \frac{598788865585667}{1850495446800}\,\zeta_3 - \frac{64419601}{1531530}\,\zeta_4 \right. \\
- \quad & \left. \frac{5828116349215429956471793586985365 47}{40462004180359891907872174080000 0} \right) C_F\, C_A\, N_F \Bigg]\!\!\Bigg] {a_s}^3
\end{aligned}
$$

**Agreement with :** an upcoming paper by Moch, Vermaseren, Vogt

# 4. New Mathematics in Perturbation Theory

Consider hard scattering processes in massless field theories:

QCD, QED, $m_i \to 0$

Factorization Theorem Leading Twist:

The cross section $\sigma$ factorizes as

$$\sigma = \sum_k \sigma_{k,W} \otimes f_k$$

$\sigma_W$ perturbative Wilson Coefficient

$f$ non-perturbative Parton Density

$\otimes$ Mellin convolution

$$[A \otimes B](x) = \int_0^1 dx_1 \int_0^1 dx_2 \delta(x - x_1 x_2) A(x_1) B(x_2)$$

$$\mathbf{M}[A \otimes B](N) = \mathbf{M}[A](N) \cdot \mathbf{M}[B](N)$$

with the Mellin transform :

$$\mathbf{M}[f(x)](N) = \int_0^1 dx x^{N-1} f(x), \quad Re[N] > c$$

**Observation :**

Feynman Amplitudes seem to obey the **Mellin Symmetry**

i.e. to significantly simplify **in Mellin Space**

Two Loop Coefficient functions:

van Neerven, Zijlstra 1992

.... several other pages for $c_2^{(+)}(x), c_2^G(x), c_L^{(q,G)}(x)$

$\Longrightarrow$ 77 Functions @ 2 Loops

$\Longrightarrow$ partly rather complicated arguments

$\Longrightarrow$ relations are not directly visible ...

The 77 functions do roughly correspond in number to the number of all possible harmonic sums up to weight w=4: 80.

Linear Representations of Mellin Transform by Harmonic Sums:

$$\mathbf{M}[F_w(x)](N) = S^w_{k_1,\ldots,k_m}(N) + P\left(S^{\tau'}_{k_1,\ldots,k_r}, \sigma^{\tau''}_{k_1,\ldots,k_p}\right)$$

$w = \sum_{i=1}^m |k_i|$ Weight $\tau', \tau'' < w$ $P$ is a polynomial.

| w | # | $\Sigma$ | |
|---|---|---|---|
| 1 | 2 | 2 | |
| 2 | 6 | 8 | |
| 3 | 18 | 26 | 2 Loop anom. Dimensions |
| 4 | 54 | 80 | 2 Loop Wilson Coefficients |
| 5 | 162 | 242 | 3 Loop anom. Dimensions |
| 6 | 486 | 728 | 3 Loop Wilson Coefficients |
| | $2 \cdot 3^{w-1}$ | $3^w - 1$ | |

# Algebraic Relations

First relation: L. Euler, 1775

$$S_{m,n} + S_{n,m} = S_m \cdot S_n + S_{m+n}, \quad m,n > 0$$

Generalized to alternating sums by

$$S_{m,n} + S_{n,m} = S_m \cdot S_n + S_{m \wedge n},\, m \wedge n = [|m| + |n|]\, \mathbf{sign}(m)\mathbf{sign}(n)$$

Ternary relations: Sita Ramachandra Rao, 1984; 4-ary relation: J.B., Kurth, 1998.

These & other relations hold widely independent of their **Value** and **Type**.

Determined by :
- Index Structure
- Multiplication Relation

Ramanujan: integer sums

Faa di Bruno: roots of multivar. algebraic equations

The Formalism applies as well to the Harmonic Polylogarithms. Remiddi, Vermaseren, 1999.

## Shuffle Products

Depth 2:

$$S_{a_1}(N) \sqcup\!\sqcup\, S_{a_2}(N) \quad = \quad S_{a_1,a_2}(N) + S_{a_2,a_1}(N)$$

Depth 3:

$$S_{a_1}(N) \sqcup\!\sqcup\, S_{a_2,a_3}(N) \quad = \quad S_{a_1,a_2,a_3}(N) + S_{a_2,a_1,a_3}(N) + S_{a_2,a_3,a_1}(N)$$

Depth 4: .....

## Algebraic Equations

Depth 2:

$$S_{a_1}(N) \sqcup\!\sqcup\, S_{a_2}(N) - S_{a_1}(N) S_{a_2}(N) - S_{a_1 \wedge a_2}(N) = 0$$

Depth 3:

$$S_{a_1}(N) \sqcup\!\sqcup\, S_{a_2,a_3}(N) \quad - \quad S_{a_1}(N) S_{a_2,a_3}(N) - S_{a_1 \wedge a_2,a_3}(N) - S_{a_2,a_1 \wedge a_3}(N) = 0$$

Depth 4: ....

# Basic Sums = # Permutations - # Independent Equations

# Theory of Words

Can we count the Basis in simpler way ? $\Longrightarrow$ YES.

**Free Algebras** and Elements of the Theory of Codes
$\Longrightarrow$ **Particle Physics**

**Only the multiplication relation and the Index structure matters**

$\mathfrak{A} = \{a, b, c, d, \ldots\}$ **Alphabet**

$a < b < c < d < \ldots$ **ordered**

$\mathfrak{A}^*(\mathfrak{A})$ **Set of all words W**

$W = a_1 \cdot a_2 \cdot a_{27} \ldots a_{532} \equiv$ **concatenation product (nc)**

$W = p \cdot x \cdot s$ **p** = **prefix; s** = **suffix**

Definition:

**A Lyndon** word is smaller than any of its suffixes.

Theorem:[Radford, 1979]

The shuffle algebra $K\langle\mathfrak{A}\rangle$ is freely generated by the Lyndon words.
I.e. the number of Lyndon words yields the number of basic elements.

Examples :

$\{a, a, \ldots, a, b\} = aaa \ldots ab$ 1 Lyndon word for these sets

$n \quad a's: \quad n_{basic}/n_{all} = 1/n \qquad n \equiv$ depth of the sums

Symmetries lead to a smaller fraction.

## Is there a general Counting Relation ?

E. Witt, 1937

$$l_n(n_1,\ldots,n_q) = \frac{1}{n}\sum_{d\|n_i}\mu(d)\frac{(n/d)!}{(n_1/d)!\ldots(n_q/d)!}, \qquad \sum_i n_i = n$$

$\mu(k)$ Möbius function

2nd Witt formula.

The Length of the Basis is a function mainly of the Depth.

**Observation:** Sums with index $-1$ do not occur.

$$N_{\neg-1}(w) = \frac{1}{2}\left[\left(1+\sqrt{2}\right)^w + \left(1-\sqrt{2}\right)^w\right]$$

$$N_{\neg-1}^{\rm basic}(w) = \frac{2}{w}\sum_{d|w}\mu\left(\frac{w}{d}\right)N_{\neg-1}(d)$$

J.B., 2004; Further Reduction: Structural Relations.

| Weight | Sums | a-basic | Sums $\neg-1$ | a-basic | str. Rel. | Fraction |
|---|---|---|---|---|---|---|
| 1 | 2 | 2 | 1 | 0 | 0 | 0.0 |
| 2 | 6 | 3 | 3 | 0 | 0 | 0.0 |
| 3 | 18 | 8 | 7 | 2 | 2 | 0.1111 |
| 4 | 54 | 18 | 17 | 5 | 3 | 0.0555 |
| 5 | 162 | 48 | 41 | 14 | 8 | 0.0494 |
| 6 | 486 | 116 | 99 | 28 | ? | <0.0576 |
| | 728 | 195 | 168 | 49 | <41 | <0.0563 |

# The Basic Functions :

## The final set of functions:

Trivial functions:

$$S_{\pm k}(N) \quad \longrightarrow \quad \psi^{(k-1)}(N+1)$$

For $w = 1, 2$ no non–trivial functions contribute to the anomalous dimensions and Wilson coefficients.

Non–trivial functions:

$N = 3$ : Two–Loop anomalous dimensions

$$\mathbf{M}\left[\frac{\mathrm{Li}_2(x)}{1+x}\right](N)$$

Yndurain et al., 1980

$N = 4$ : Two–Loop Wilson Coefficients

$$\mathbf{M}\left[\frac{\ln(1+x)}{1+x}\right](N), \quad \mathbf{M}\left[\frac{\mathrm{Li}_2(x)}{1-x}\right](N), \quad \mathbf{M}\left[\frac{\mathrm{S}_{1,2}(x)}{1\pm x}\right](N)$$

Structure Fct.: J.B., S. Moch, 2003,

Drell-Yan, Higgs-Prod., Fragmentation: J.B., V. Ravindran, 2004.

$N = 5$ : Three–Loop Anomalous Dimensions

$$\mathbf{M}\left[\frac{\mathrm{Li}_4(x)}{1\pm x}\right](N), \quad \mathbf{M}\left[\frac{S_{1,3}(x)}{1+x}\right](N), \quad \mathbf{M}\left[\frac{S_{2,2}(x)}{1\pm x}\right](N),$$

$$\mathbf{M}\left[\frac{S_{2,2}(-x)-\mathrm{Li}_2^2(-x)/2}{1\pm x}\right](N), \quad \mathbf{M}\left[\frac{\mathrm{Li}_2^2(x)}{1+x}\right](N)$$

J.B., S. Moch, 2004.

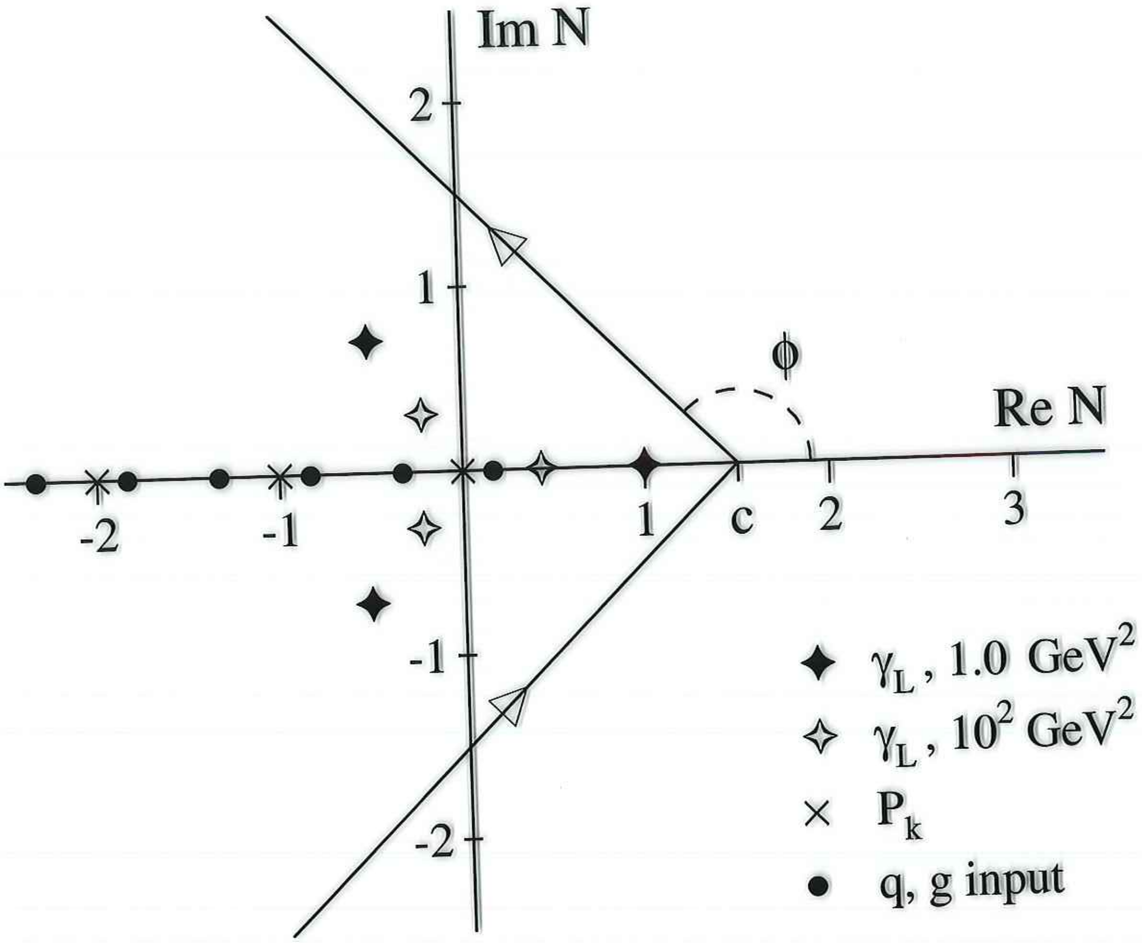
Essentially 14 Functions seem to rule the single scale processes of massless QCD.

| | Moments, $N \leq 40$ | Inversion |
|---|---|---|
| $[\mathrm{Li}_4(x) - \zeta_4]/(1-x)$ | 4.0D–12 | 5.1D–8 |
| $[S_{2,2}(x) - \zeta_4/4]/(1-x)$ | 8.2D–10 | 5.4D–8 |
| $[S_{2,2}(-x) - c_1]/(1-x)$ | 1.8D–10 | 4.5D–8 |
| $[\mathrm{Li}_2^2(-x) - \zeta_2^2/4]/(1-x)$ | 3.0D–10 | 4.3D–8 |
| $\mathrm{Li}_4(x)/(1+x)$ | 1.2D–12 | 8.0D–8 |
| $S_{1,3}(x)/(1+x)$ | 2.2D–12 | 3.3D–8 |
| $S_{2,2}(x)/(1+x)$ | 1.1D–12 | 1.8D–8 |
| $\mathrm{Li}_2^2(x)/(1+x)$ | 2.2D–12 | 3.0D–8 |
| $[S_{2,2}(-x) - \mathrm{Li}_2^2(-x)/2]/(1+x)$ | 7.2D–13 | 3.2D–8 |

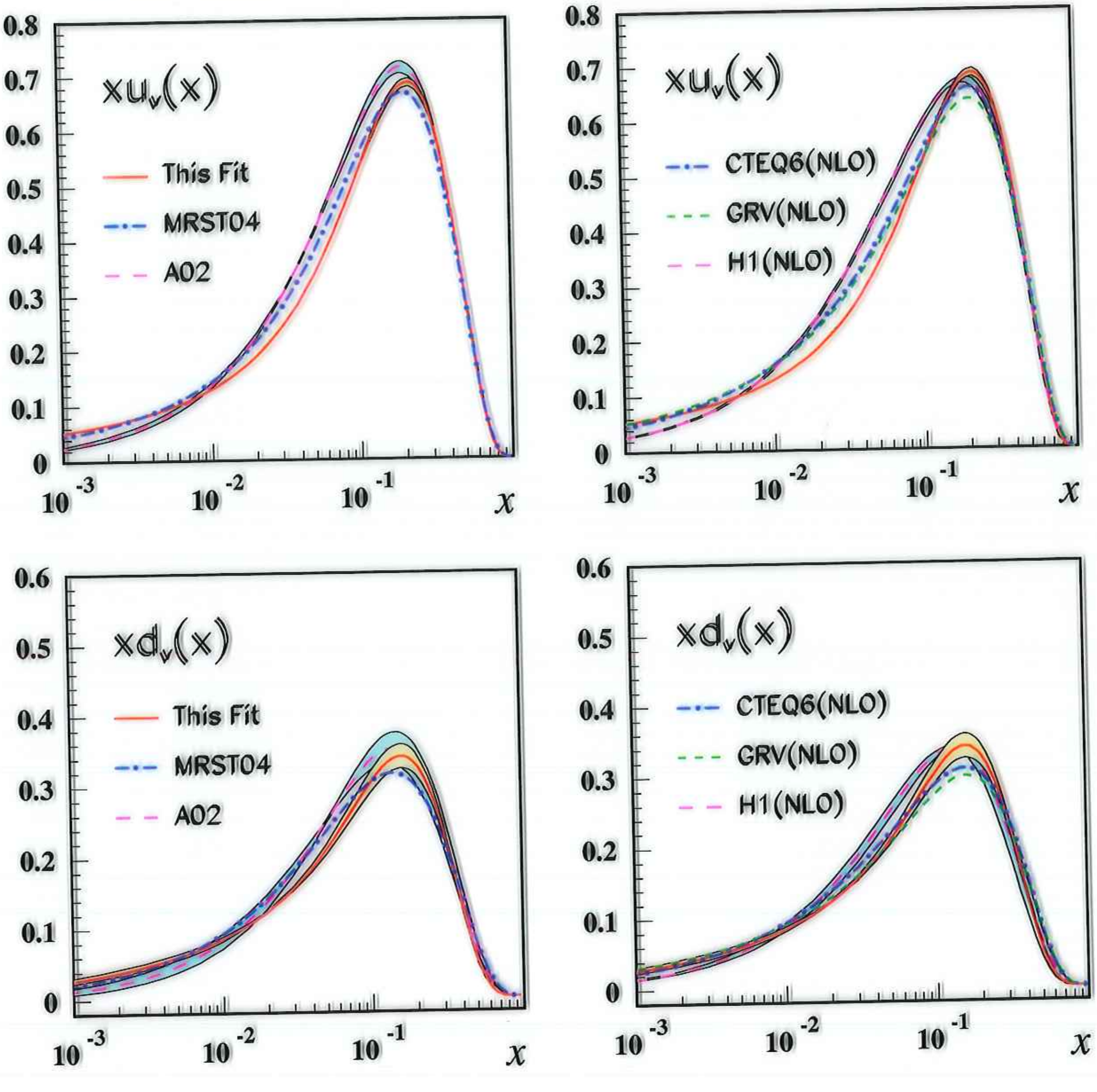
Relative accuracies of the representations.

J.B., S. Moch, DESY 05–007

Contour for the inverse Mellin transform:



# 5. QCD NS-Analysis to 3 Loops




$W^2 > 12.5\,\mathrm{GeV}^2, Q^2 > 4\,\mathrm{GeV}^2$

NNLO :

$$\alpha_s(M_Z^2) = 0.1139^{+0.0026}_{-0.0028}$$

J.B., H. Böttcher, A. Guffanti, 2004

# The World Data on $F_2$

| *Experiment* | $x$ | $Q^2, GeV^2$ | $F_2$ | *Norm* |
|---|---|---|---|---|
| BCDMS (100) | 0.35 – 0.75 | 11.75 – 75.00 | 51 | 1.018 |
| BCDMS (120) | 0.35 – 0.75 | 13.25 – 75.00 | 59 | 1.011 |
| BCDMS (200) | 0.35 – 0.75 | 32.50 – 137.50 | 50 | 1.017 |
| BCDMS (280) | 0.35 – 0.75 | 43.00 – 230.00 | 49 | 1.018 |
| NMC (comb) | 0.35 – 0.50 | 7.00 – 65.00 | 15 | 1.003 |
| SLAC (comb) | 0.30 – 0.62 | 7.30 – 21.39 | 57 | 1.003 |
| H1 (hQ2) | 0.40 – 0.65 | 200 – 30000 | 26 | 1.018 |
| ZEUS (hQ2) | 0.40 – 0.65 | 650 – 30000 | 15 | 1.001 |
| *proton* | | | 322 | |
| BCDMS (120) | 0.35 – 0.75 | 13.25 – 99.00 | 59 | 0.992 |
| BCDMS (200) | 0.35 – 0.75 | 32.50 – 137.50 | 50 | 0.993 |
| BCDMS (280) | 0.35 – 0.75 | 43.00 – 230.00 | 49 | 0.993 |
| NMC (comb) | 0.35 – 0.50 | 7.00 – 65.00 | 15 | 0.980 |
| SLAC (comb) | 0.30 – 0.62 | 10.00 – 21.40 | 59 | 0.980 |
| *deuteron* | | | 232 | |
| BCDMS (120) | 0.070 – 0.275 | 8.75 – 43.00 | 36 | 1.000 |
| BCDMS (200) | 0.070 – 0.275 | 17.00 – 75.00 | 29 | 1.000 |
| BCDMS (280) | 0.100 – 0.275 | 32.50 – 115.50 | 27 | 1.000 |
| NMC (comb) | 0.013 – 0.275 | 4.50 – 65.00 | 88 | 1.000 |
| SLAC (comb) | 0.153 – 0.293 | 4.18 – 5.50 | 28 | 1.000 |
| *non – singlet* | | | 208 | |
| *total* | | | 762 | |

- **CUTS**: $0.3 < x < 1.0$ for $F_2^p$ and $F_2^d$
  $0.0 < x < 0.3$ for $F_2^{ns} = 2(F_2^p - F_2^d)$
  $4.0 < Q^2 < 30000\, GeV^2$, $W^2 > 12.5\, GeV^2$

# Fully Correlated Error Calculation

- The fully correlated $1\sigma$ error for the parton density $f_q$ as given by Gaussian error propagation is

$$\sigma(f_q(x)^2 = \sum_{i,j=1}^{n_p} \left( \frac{\partial f_q}{\partial p_i} \frac{\partial f_q}{\partial p_j} \right) \mathrm{cov}(p_i, p_j) \ , \qquad (1)$$

where the $\partial f_q / \partial p_i$ are the derivatives of $f_q$ w.r.t. the parameters $p_i$ and the $\mathrm{cov}(p_i, p_j)$ are the elements of the covariance matrix as determined in the fit.

- The derivatives $\partial f_q / \partial p_i$ at the input scale $Q_0^2$ can be calculated analytically. Their values at $Q^2$ are given by evolution.

- The derivatives evolved in $\mathrm{MELLIN}$-$\mathrm{N}$ space are transformed back to $x$–space and can then be used according to the error propagation formula above.

$\Longrightarrow$ As an example the derivative of $f(x, a, b)$ w.r.t. parameter $a$ in $\mathrm{MELLIN}$–$\mathrm{N}$ space reads:

# Fit Results

- Parameter values and Covariance Matrix at the input scale $Q_0^2 = 4.0\,GeV^2$

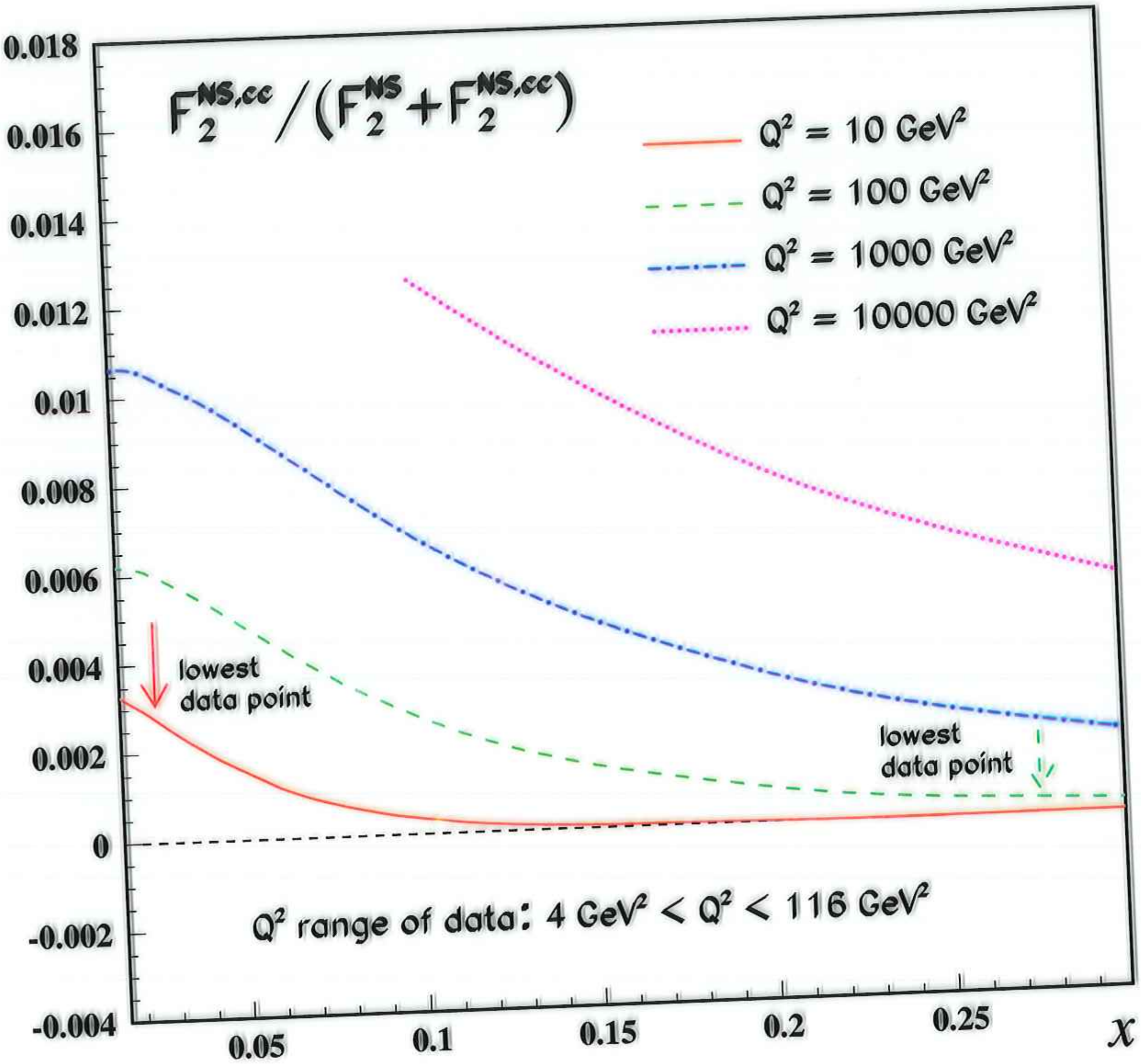
$$xq_i(x, Q_0^2) = A_i x^{a_i}(1-x)^{b_i}(1+\rho_i x^{\frac{1}{2}} + \gamma_i x)$$

| | | |
|---|---|---|
| $u_v$ | $a$ | 0.299 ± 0.007 |
| | $b$ | 4.157 ± 0.031 |
| | $\rho$ | 0.751 |
| | $\gamma$ | 28.833 |
| $d_v$ | $a$ | 0.488 ± 0.048 |
| | $b$ | 6.609 ± 0.332 |
| | $\rho$ | -1.690 |
| | $\gamma$ | 17.247 |
| $\Lambda_{QCD}^{(4)}$ | | 233 ± 34 $MeV$ |
| $\chi^2/ndf$ = 630/757 = 0.83 | | |

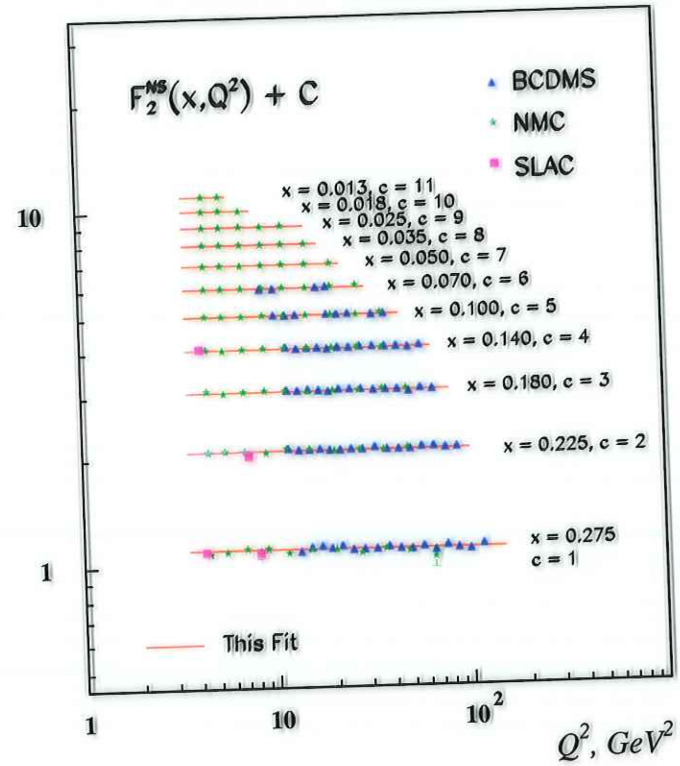
- Covariance Matrix at the input scale $Q_0^2 = 4.0\,GeV^2$

| | $\Lambda_{QCD}^{(4)}$ | $a_{u_v}$ | $b_{u_v}$ | $a_{d_v}$ | $b_{d_v}$ |
|---|---|---|---|---|---|
| $\Lambda_{QCD}^{(4)}$ | **1.15E-3** | | | | |
| $a_{u_v}$ | 1.03E-4 | **5.40E-5** | | | |
| $b_{u_v}$ | -8.45E-5 | 1.71E-4 | **9.59E-4** | | |
| $a_{d_v}$ | 4.17E-4 | 8.84E-6 | -4.35E-4 | **2.32E-3** | |
| $b_{d_v}$ | 2.32E-3 | 4.21E-4 | -2.28E-3 | 1.48E-2 | **1.10E-1** |

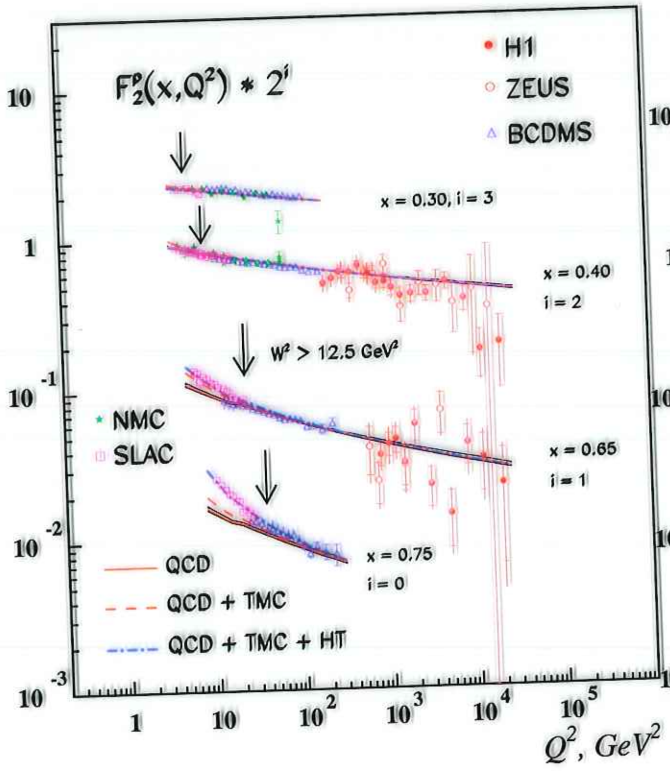
# Heavy Flavor NS-contributions



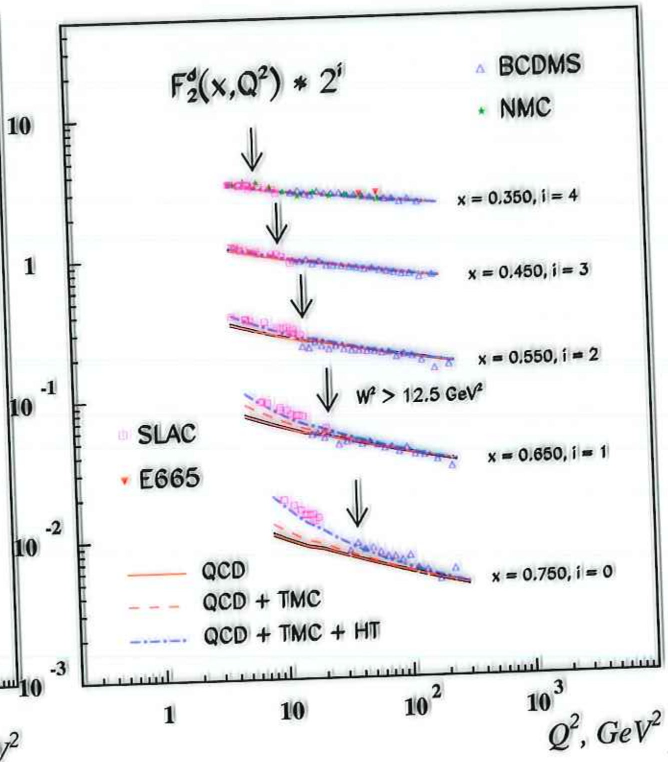
# Non-Singlet 3-Loop QCD Analysis




$x > 0.3$



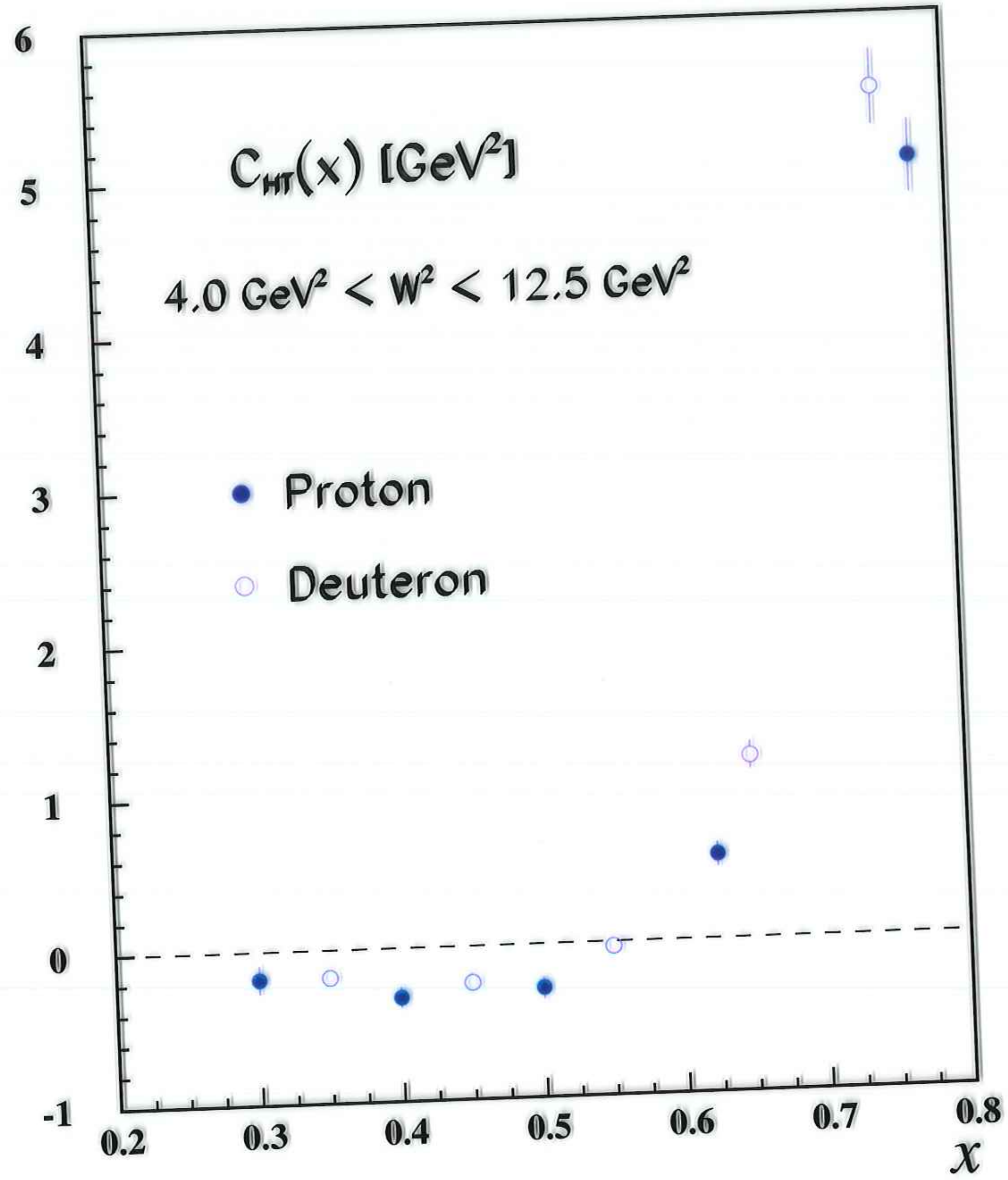

proton




deuteron

# Higher Twist Contributions:

$$4 < W^2 < 12.5\,\text{GeV}^2, Q^2 > 4\,\text{GeV}^2$$



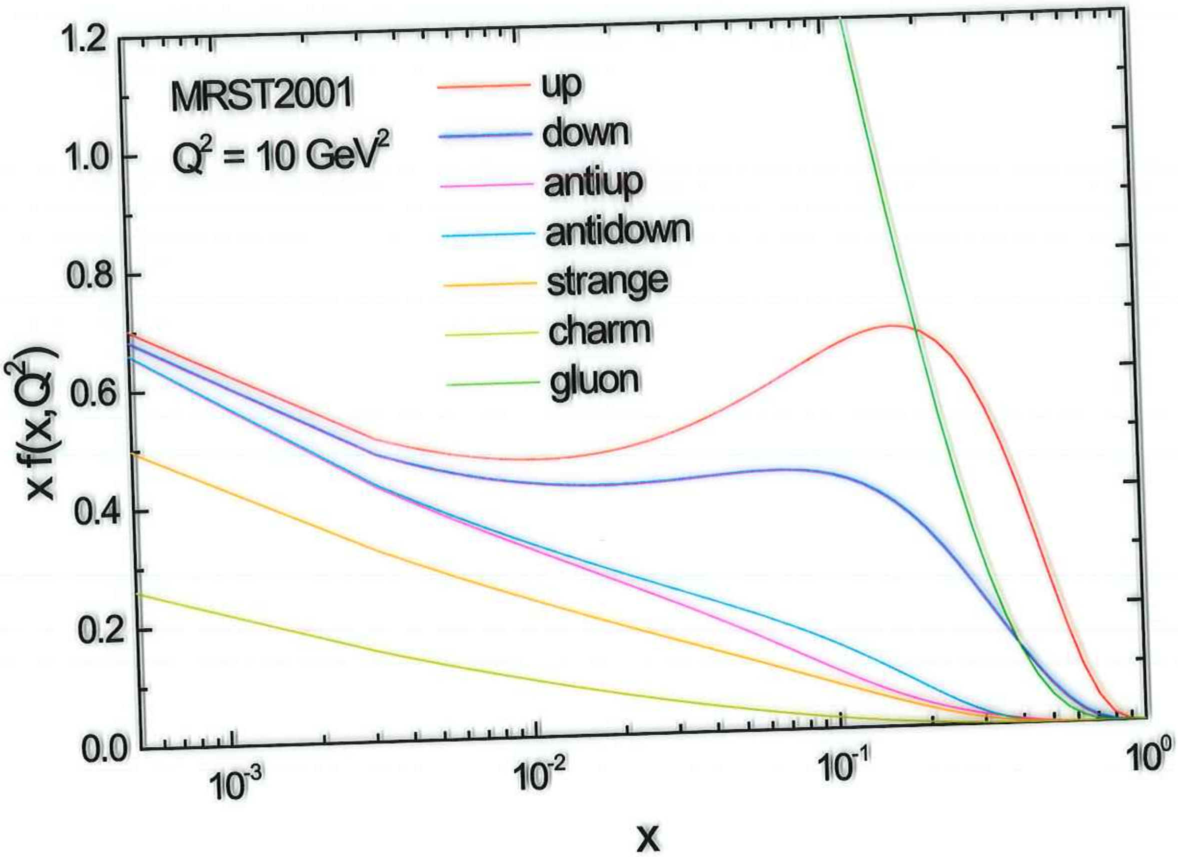
# MOMENTS AND LATTICE RESULTS

| $f$ | $n$ | This Fit | MRST04 | A02 |
|---|---|---|---|---|
| $u_v$ | 2 | $0.288 \pm 0.003$ | 0.285 | 0.304 |
| | 3 | $0.084 \pm 0.001$ | 0.082 | 0.087 |
| | 4 | $0.0319 \pm 0.0004$ | 0.032 | 0.033 |
| $d_v$ | 2 | $0.113 \pm 0.004$ | 0.115 | 0.120 |
| | 3 | $0.026 \pm 0.001$ | 0.028 | 0.028 |
| | 4 | $0.0078 \pm 0.0004$ | 0.009 | 0.010 |
| $u_v - d_v$ | 2 | $0.175 \pm 0.004$ | 0.171 | 0.184 |
| | 3 | $0.058 \pm 0.001$ | 0.055 | 0.059 |
| | 4 | $0.0241 \pm 0.0005$ | 0.022 | 0.024 |

Wait for upcoming results from LGT.

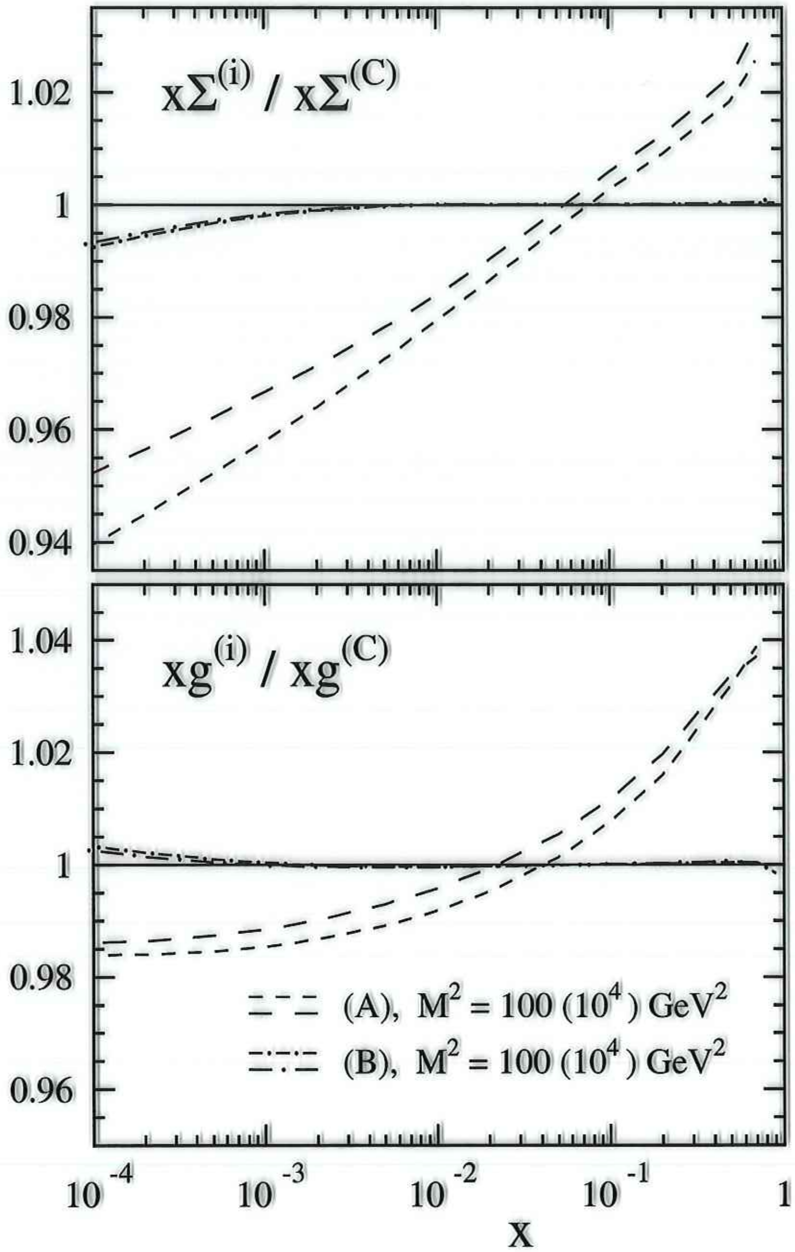
# 6. The Singlet Sector

## Parton Densities: Relative Size



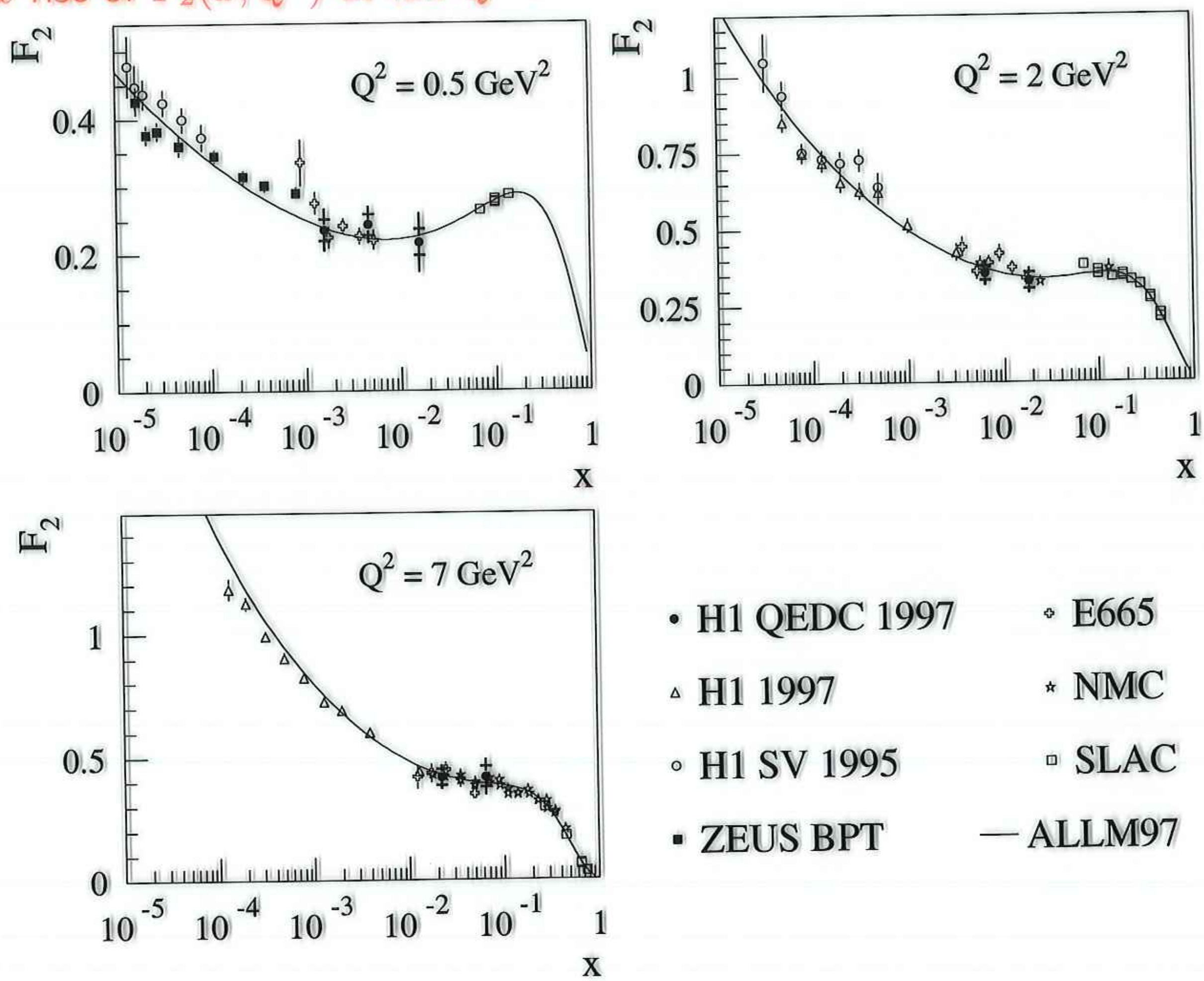
# PILE–UP EFFECTS:

## Iterative vs Exact Solution of Evolution Equations

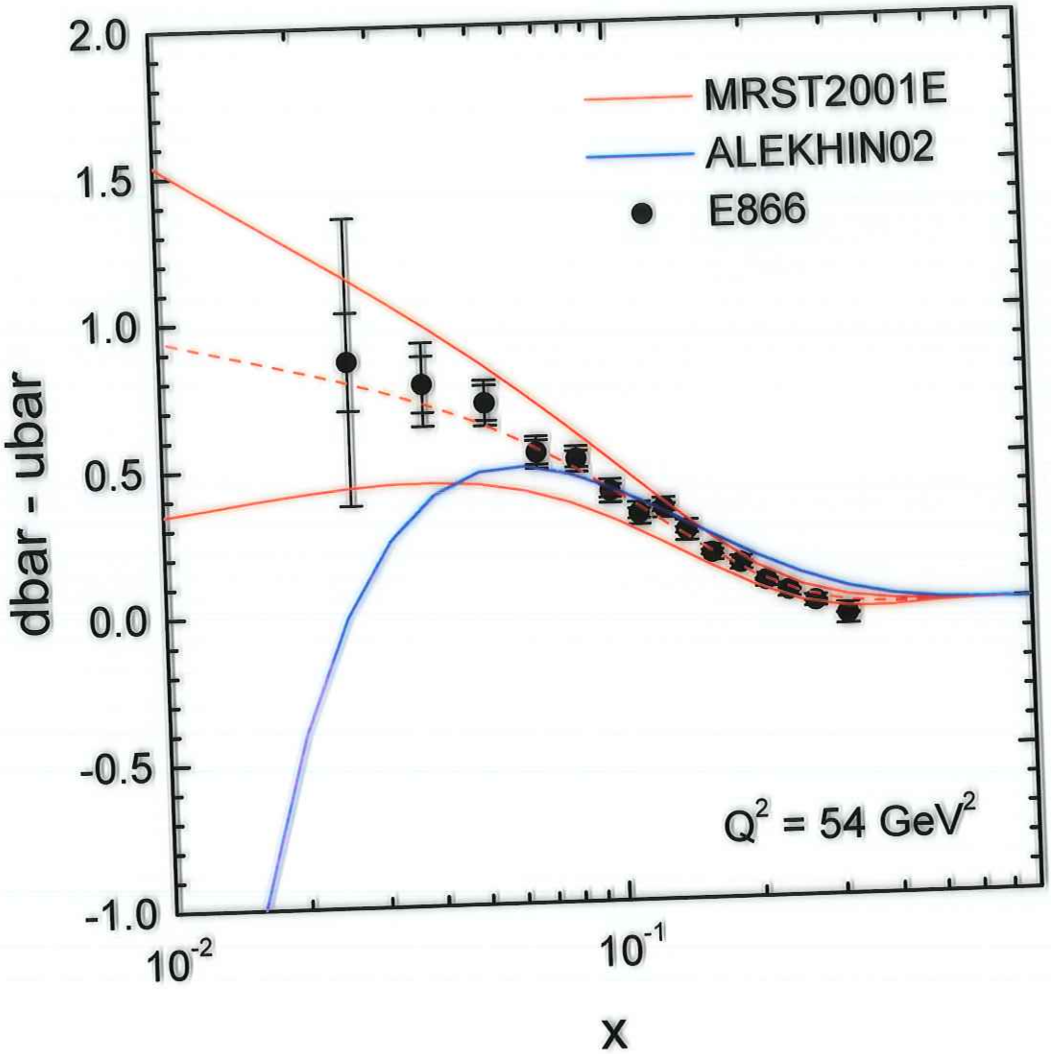



Blümlein, Riemersma, van Neerven, Vogt, 1996

$x$ rise of $F_2(x, Q^2)$ at low $Q^2$ :



# $\overline{d} - \overline{u}$

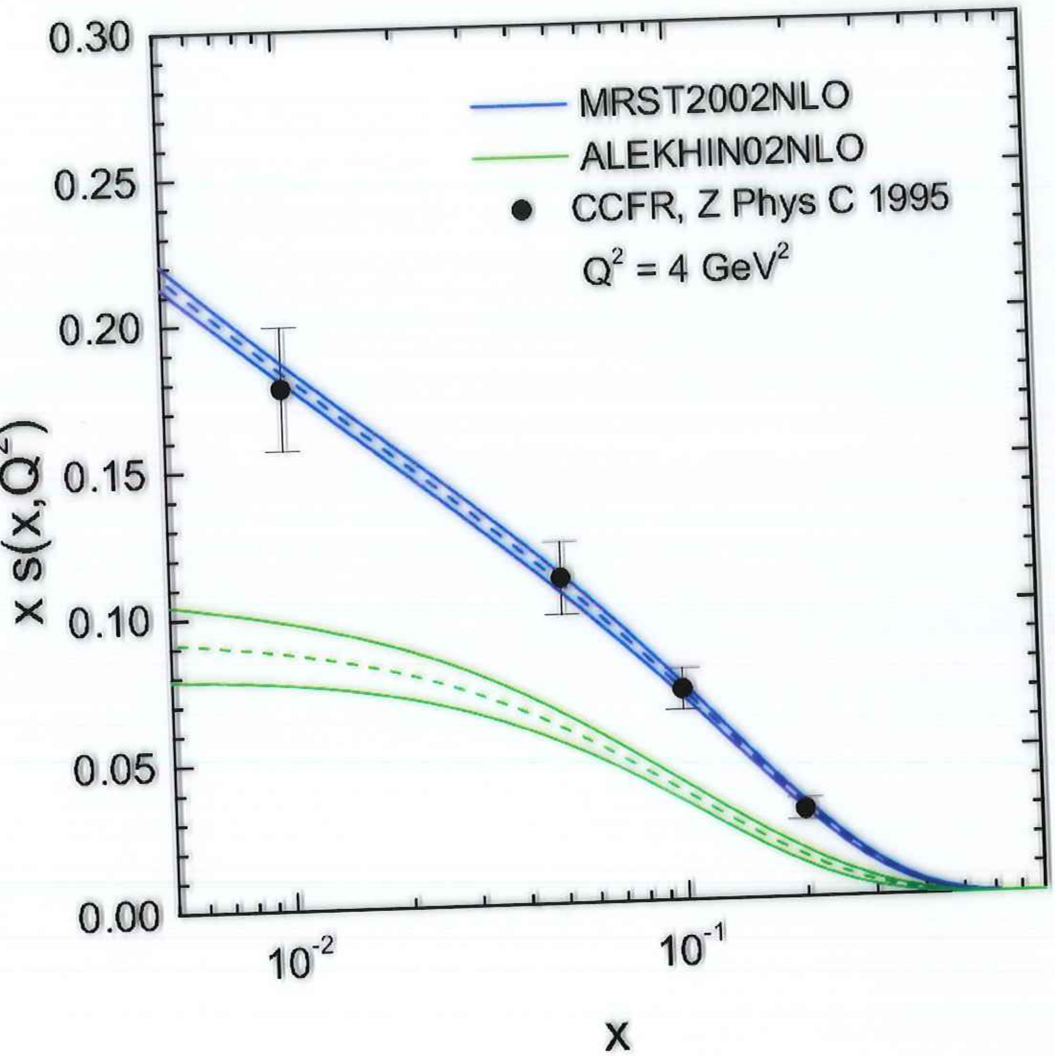



$$x(\overline{d}(x) - \overline{u}(x)) = 1.195x^{1.24}(1-x)^{9.10}(1 + 14.05x - 45.52x^2)$$

$Q^2 = 1\,\mathrm{GeV}^2$

$\Longrightarrow$ Drell-Yan to NNLO & Data Analysis

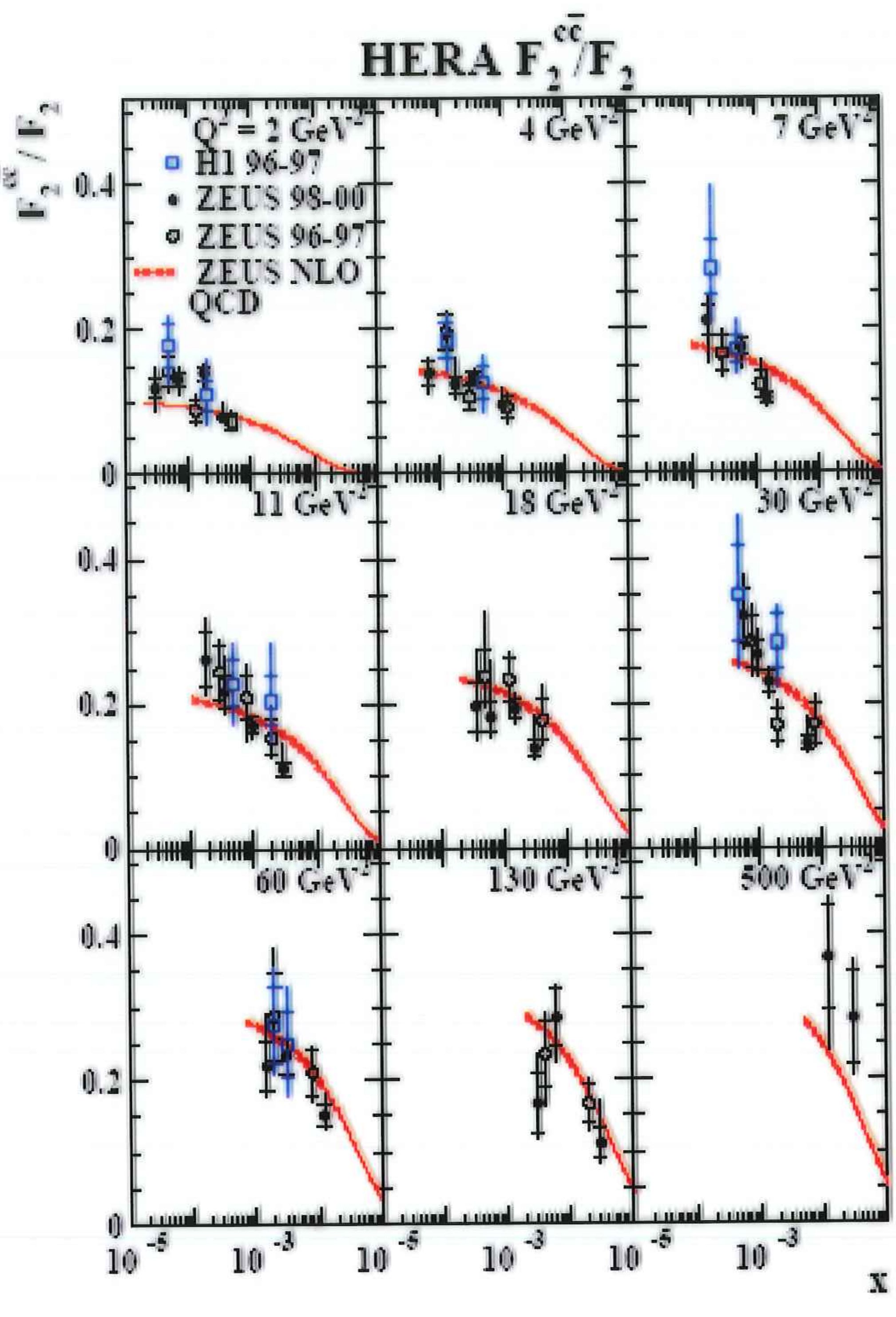
## Strange quark distribution



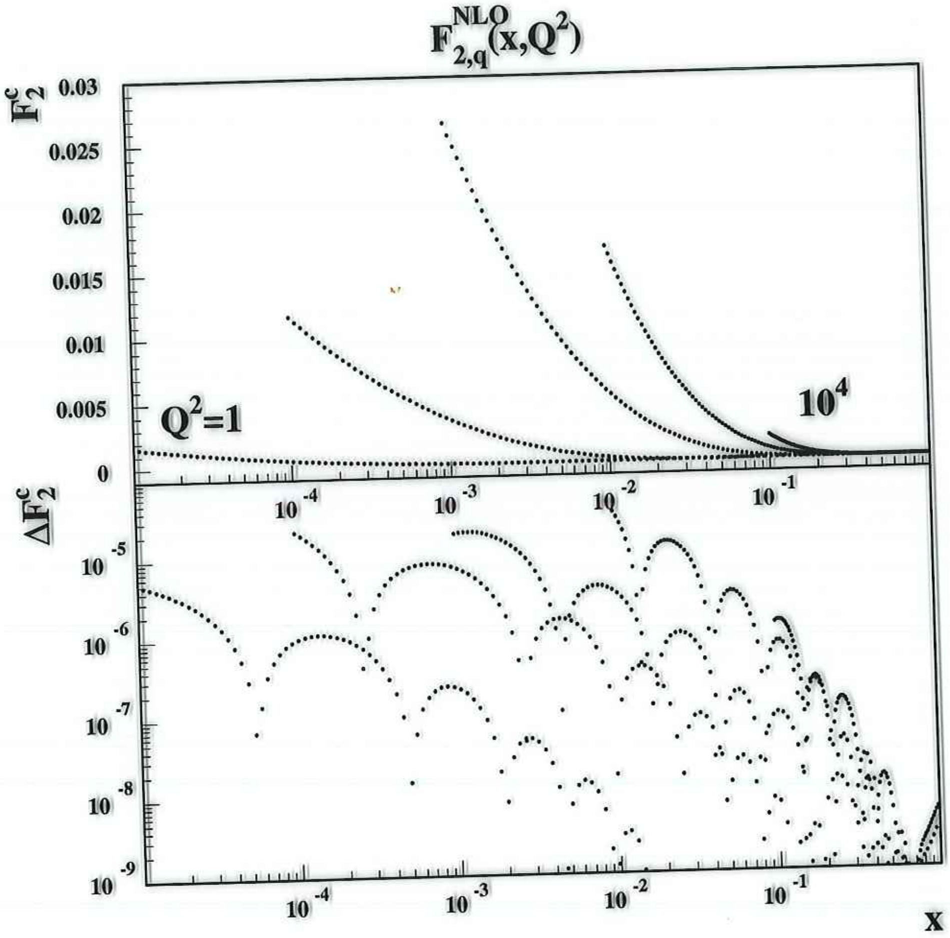

- CCFR : iron target, EMC effect. How large ?

**CAN HERMES MEASURE $s(x, Q^2)$ ?**

## $c\bar{c}$ Structure Function $F_2$



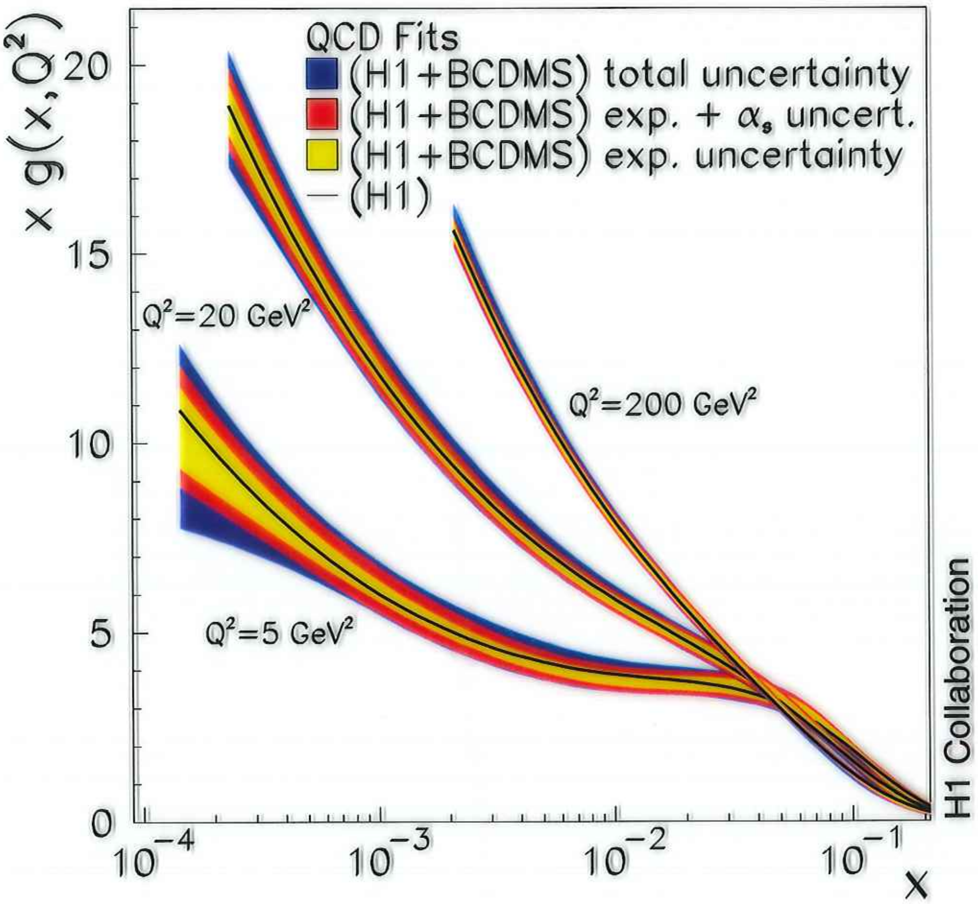
## Mellin-space representation :

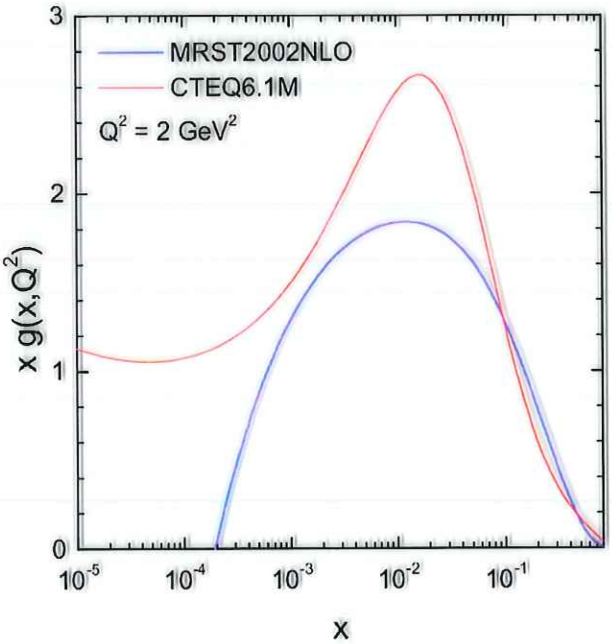



S. Alekhin and J.B., 2004

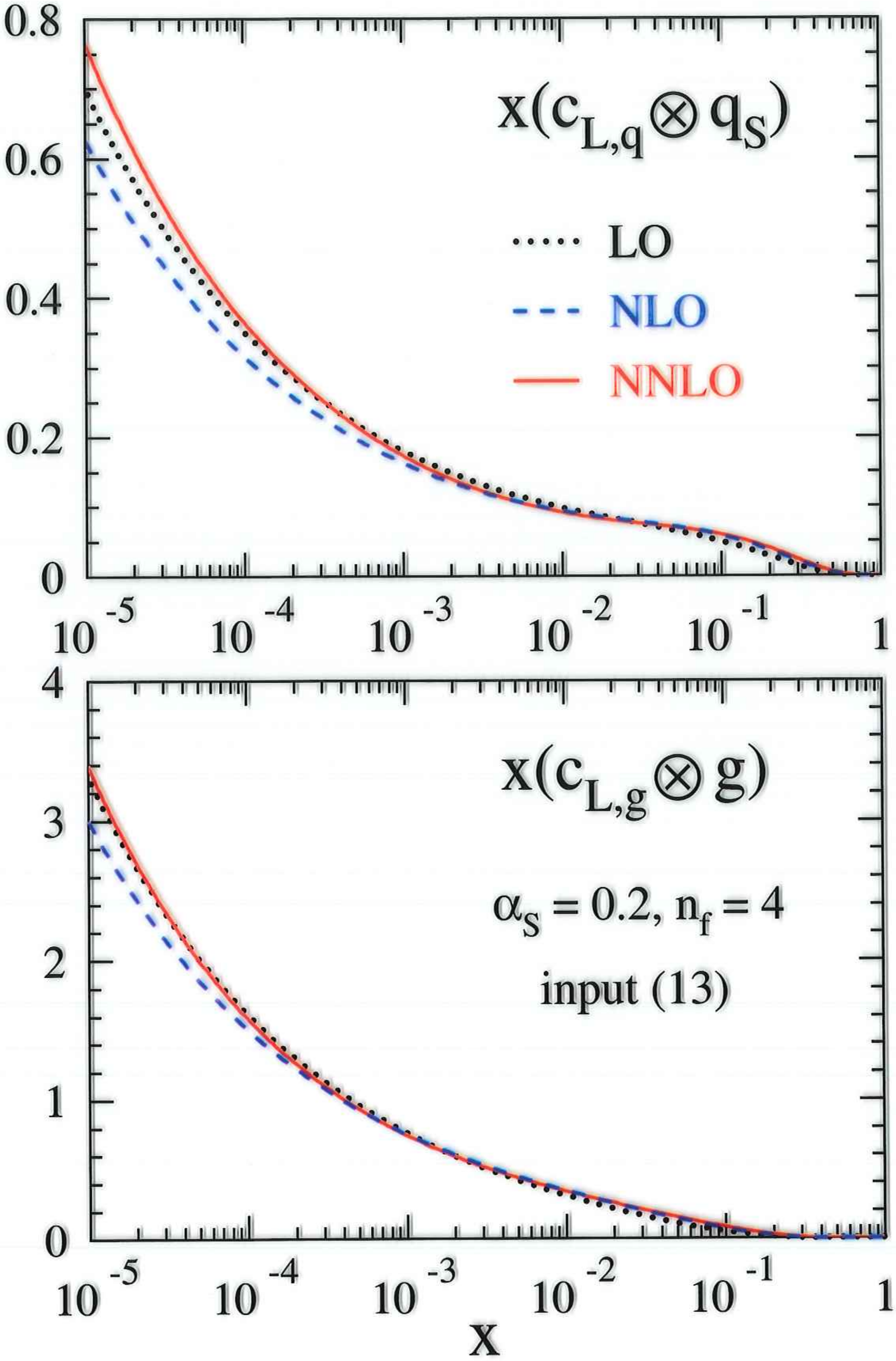
- necessary for scheme-invariant evolution.
- fast and accurate access to heavy flavor Wilson coefficients.

# Gluon Density



## $F_L(x, Q^2)$




Moch, Vermasern, Vogt, hep-ph/0411112

# 3. Small $x$ Resummations

STRUCTURE FUNCTIONS RISE AT SMALL x.
(SINGLET)

EVEN AT RATHER LOW $Q^2$! FIG

REGGE THEORY PREDICTS A FLAT BEHVIOUR.

→ WHAT IS DESCRIBED BY PERTURBATION THEORY?

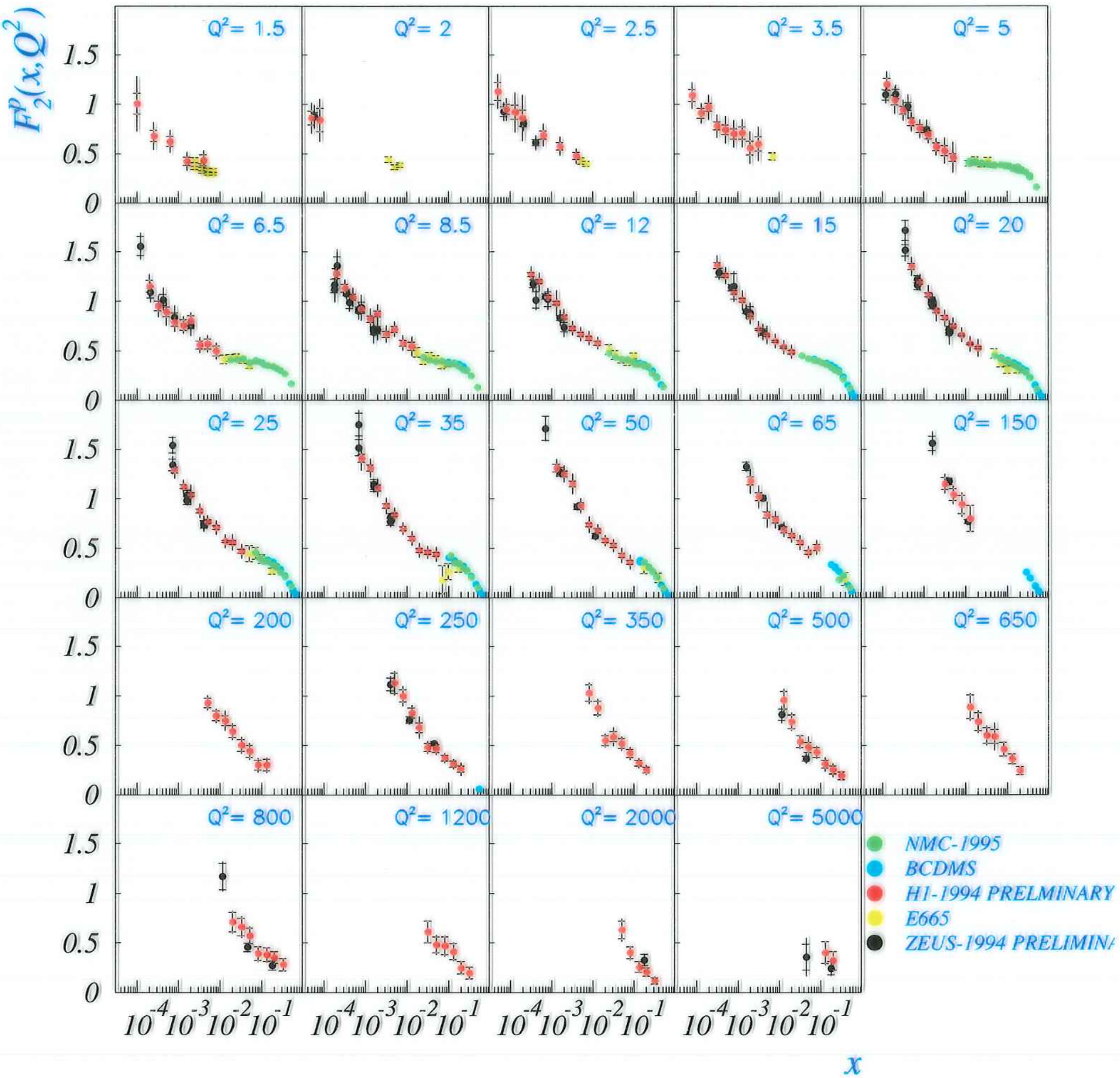
→ IS BFKL RESUMMATION ALLOWED?

- RISE AT LOW SCALE : NON-PERTURBATIVE
  RESUMMATIONS: $\sim \alpha_s \ll 1$

- RESUMMATIONS ARE ONLY USEFUL IF THEY ARE DOMINANT.

→ NOT THE CASE FOR BFKL.

→ QUESTION:
HOW MANY LESS SINGULAR TERMS DO WE NEED ELSE?




BLUMLEIN@PHOENIX, DEC02/15:20

# SINGULARITY STRUCTURE OF SMALL x EXPANSIONS

a) PERTURBATIVE: POLES.

SCALARS: $\Gamma \sim \sum_k c_k^{S}(N+1)\left[\frac{\alpha_s}{(N+1)^2}\right]^k$ $N=-1$

FERMIONS: $\Gamma \sim \sum_k c_k^{F} N \left(\frac{\alpha_s}{N^2}\right)^k$ $N=0$

VECTORS: $\Gamma \sim \sum_k c_k^{V} \left(\frac{\alpha_s}{N-1}\right)^k$ $N=1$

BALITZKI, KURAEV | 76-86
FADIN LIPATOV | 98

b) AFTER RESUMMATION:

! ALL POLES DISAPPEAR TO BRANCH POINTS

SCALARS: $\Gamma = (N+1)\left[\sqrt{1-\frac{4\alpha_s}{(N+1)^2}}-1\right]$ LOVELACE 75; JB, VAN NEERVEN 82

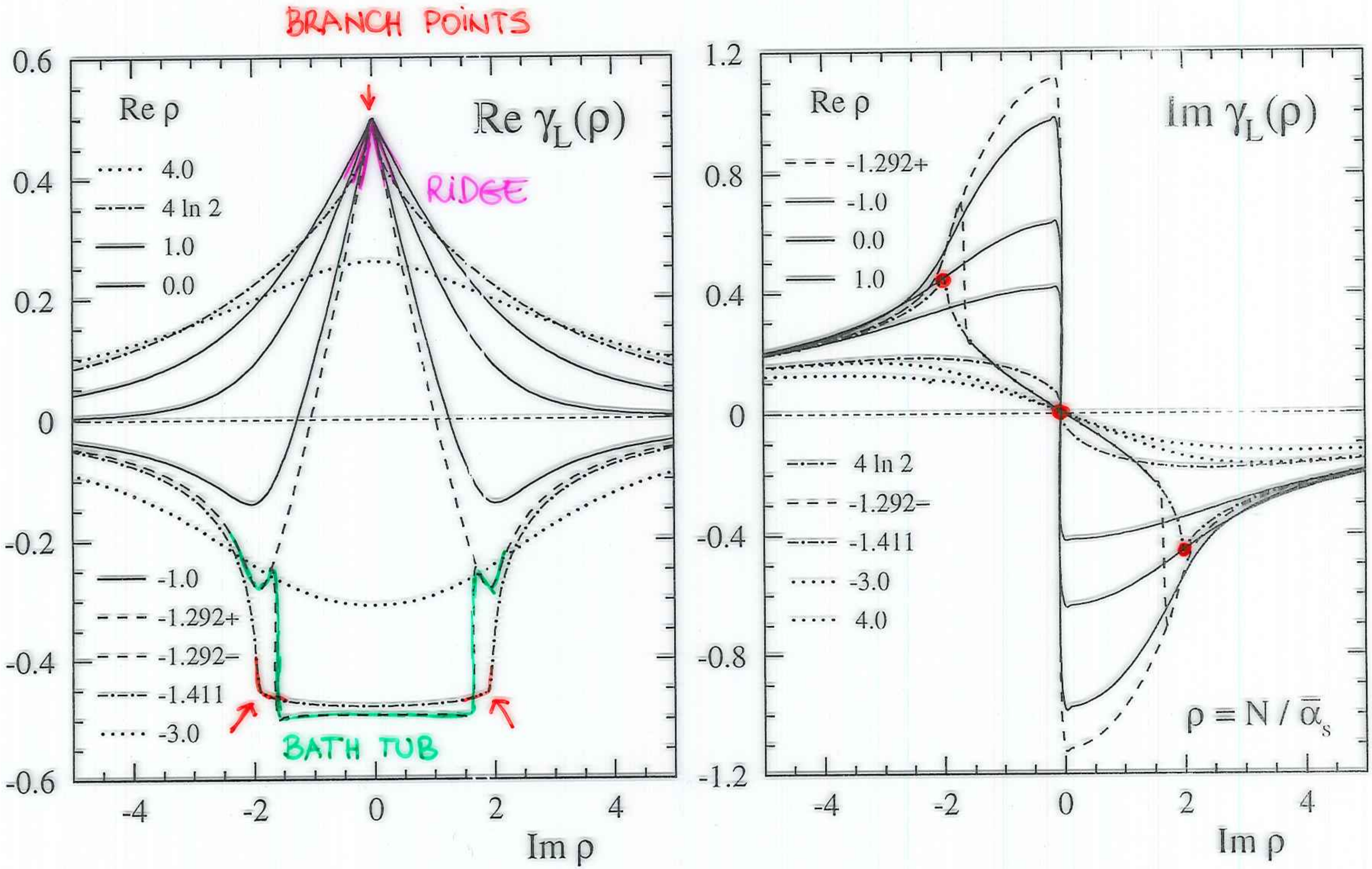
$N = -1 \pm \sqrt{4\alpha_s}$

FERMIONS: $\Gamma^{+} = N\left[\sqrt{1-\frac{8\alpha_s C_F}{N^2}}-1\right]$ $NS^{+}$ ('83)

KIRSCHNER, LIPATOV '83

$$\Gamma^{-} = -N\left\{1-\sqrt{1-\frac{8\alpha_s C_F}{N^2}\left[1-\frac{8\alpha_s N_c}{N}\frac{d}{dN}\phi(z)\right]}\right\}, \quad NS^{-}$$

$$\phi(z) = \log\left[e^{z^2/4} D_{-\frac{1}{2N_c^2}}(z)\right], \quad z = N/\sqrt{2N_c\alpha_s}.$$

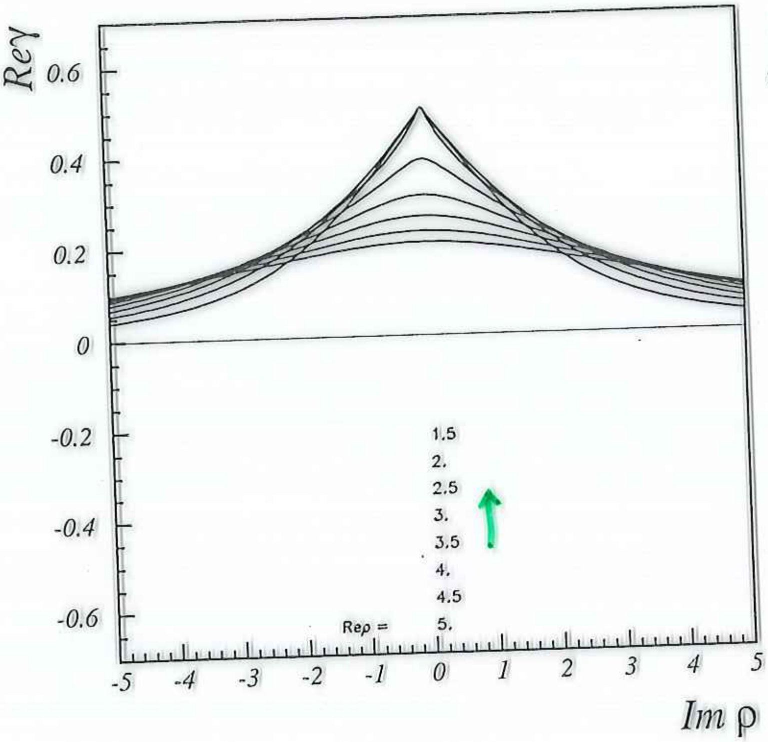

BRANCH POINTS
Re ρ
Re γL(ρ)
4.0
4 ln 2
1.0
0.0
RIDGE
-1.0
-1.292+
-1.292−
-1.411
-3.0
BATH TUB
Im ρ
Re ρ
Im γL(ρ)
-1.292+
-1.0
0.0
1.0
4 ln 2
-1.292−
-1.411
-3.0
4.0
ρ ≡ N / ᾱs
Im ρ

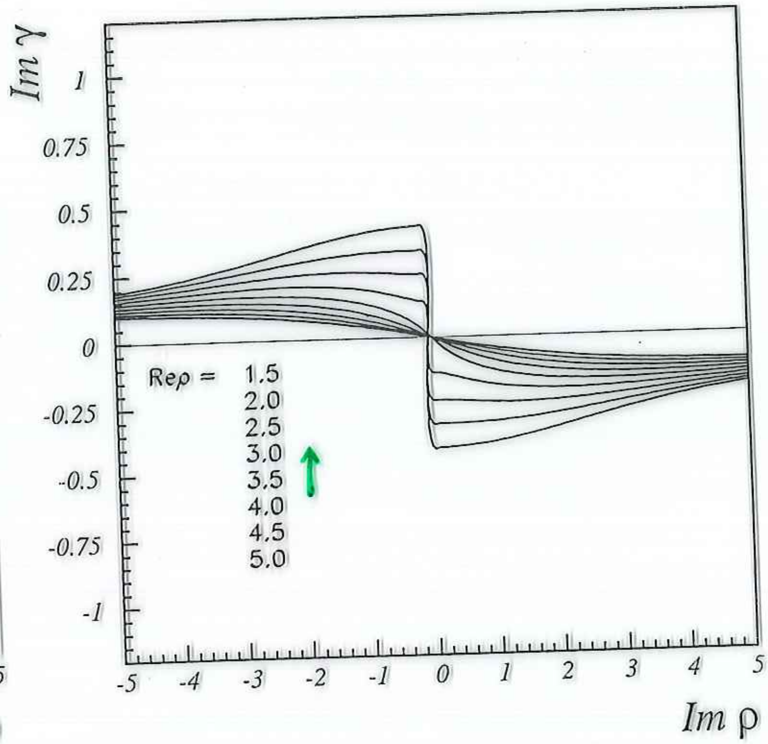
LO:

## The behaviour of $\gamma_c(\rho)$ for $\rho \in C$

J.B. ('94)

$Re\rho \geq 1.5$

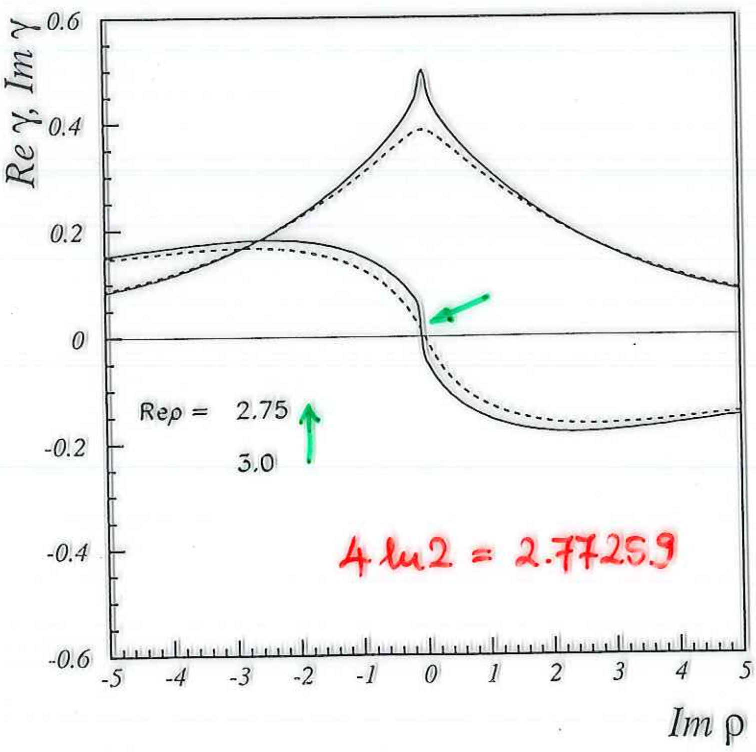



( USE: ADAPTIVE NEWTON ALGORITHM ).

$\rho = \frac{\bar{N}\pi}{\alpha_s N_c}$

$\bar{N} = N-1$



# LOCATION OF THE BRANCH POINTS

$$\rho = \frac{\ell - 1}{\bar{\alpha}_s} = 2\psi(1) - \psi(\gamma) - \psi(1-\gamma).$$

$$1 = \left[-\psi'(\gamma) + \psi'(1-\gamma)\right] \frac{\partial \gamma}{\partial \rho}$$

$$\frac{1}{\partial\gamma/\partial\rho} = \psi'(1-\gamma) - \psi'(\gamma) = 0$$

$$\psi'(z) = \frac{\pi^2}{2} \frac{1}{\sin^2 \pi z} = 0$$

$\gamma_1 = \frac{1}{2} + 0i$ $\qquad$ $\rho_1 = 4 \ln 2$

$\gamma_{2,3} = -0.425214 \pm i\, 0.473898$

$\rho_{2,3} = -1.4105 \pm i\, 1.9721.$

K. Ellis, Hautmann, Webber '95

JB '95

COMPLEX N-PLANE

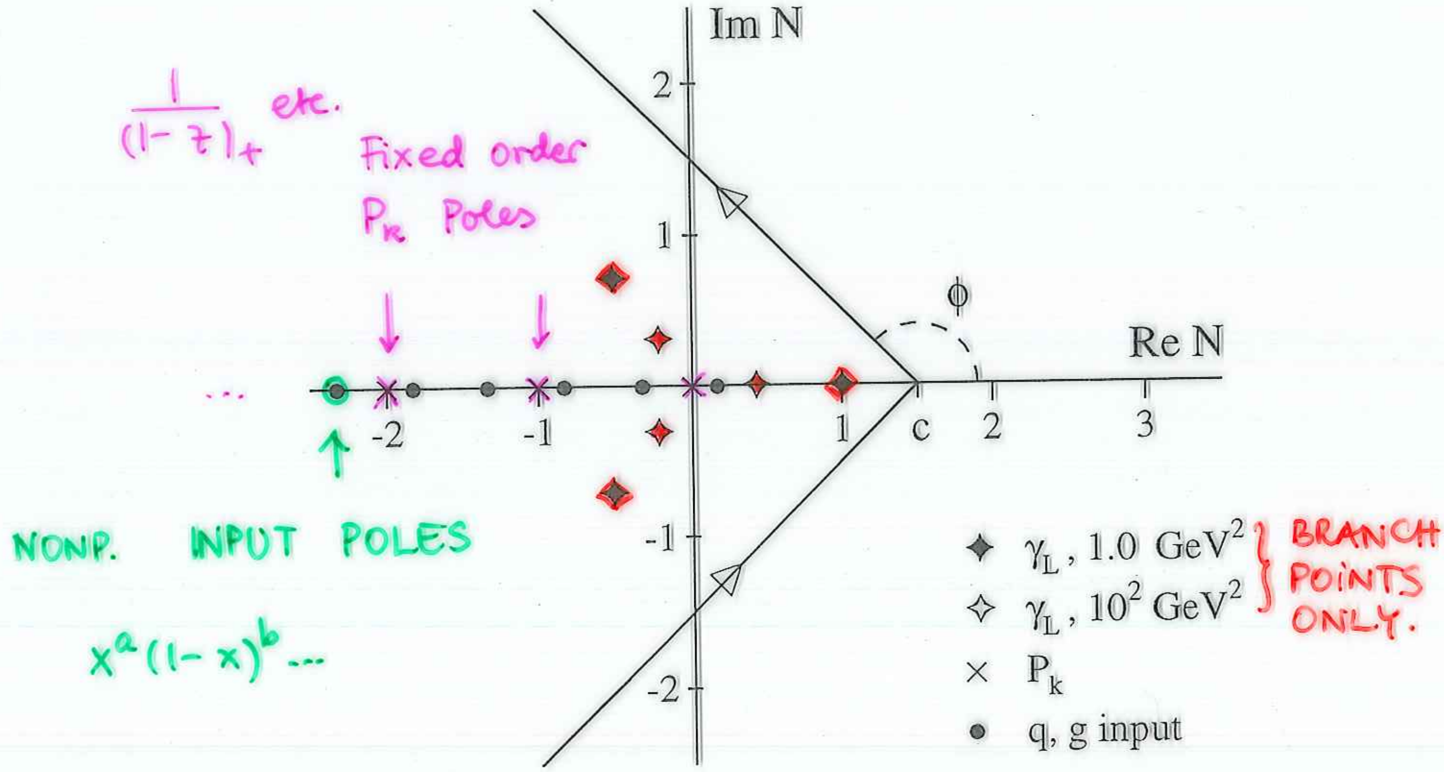
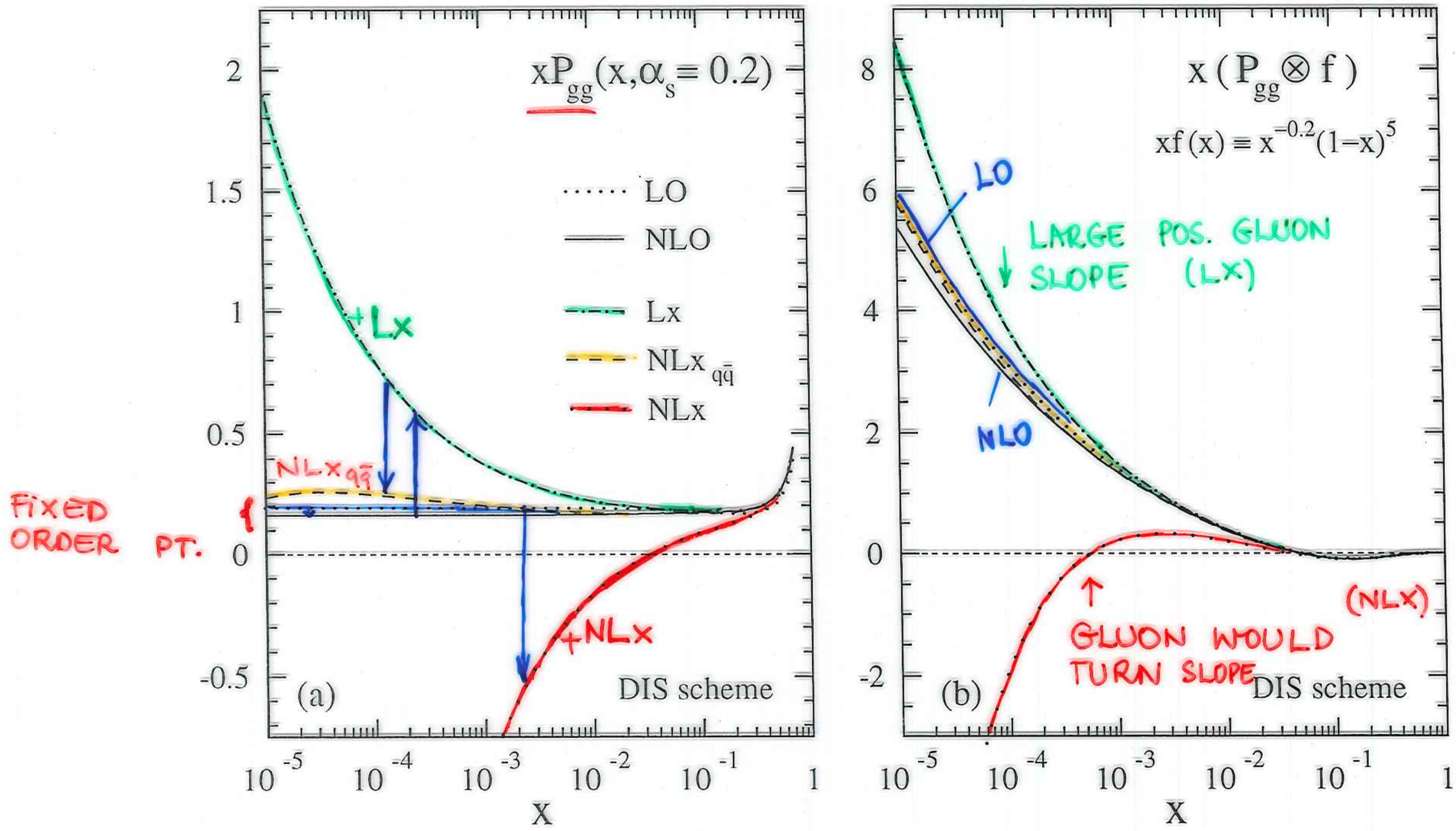



Fig. 4

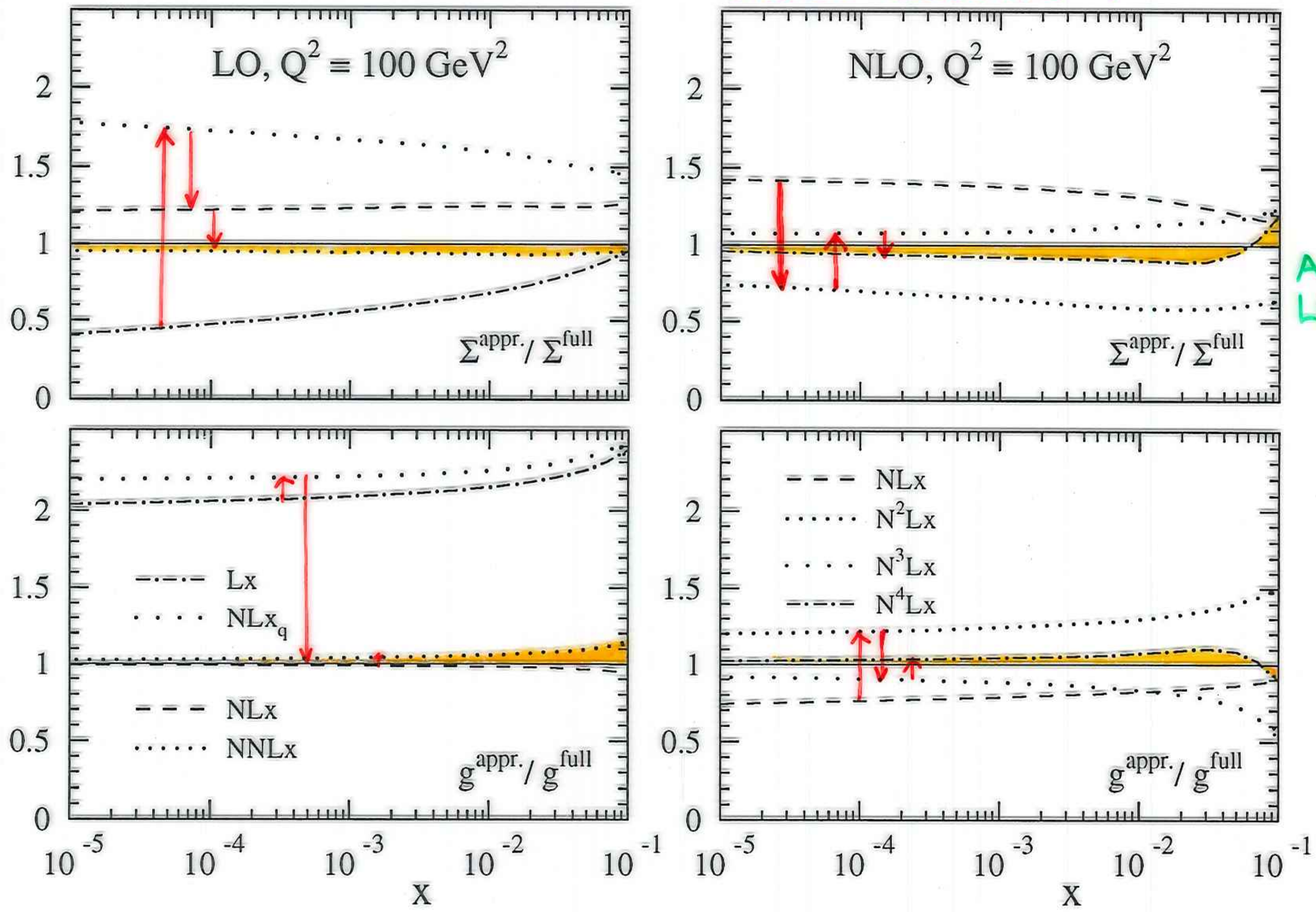
- SOLUTION OF THE RGE'S IN MELLIN SPACE
- EXACT ACCOUNT FOR ALL COMMUTATION RELATIONS $[P_{ij}^{\ell}, P_{ij}^{m}] \neq 0$ FOR $\ell \neq m$.




NO LESS SINGULAR TERMS CONSIDERED.

JB, VOGT '98

HOW MANY $1/(N-1)^k$ TERMS ARE NEEDED TO GET FIXED ORDER RESULTS ?




AT LEAST 4.

Fig. 6

JB, VOGT 1997

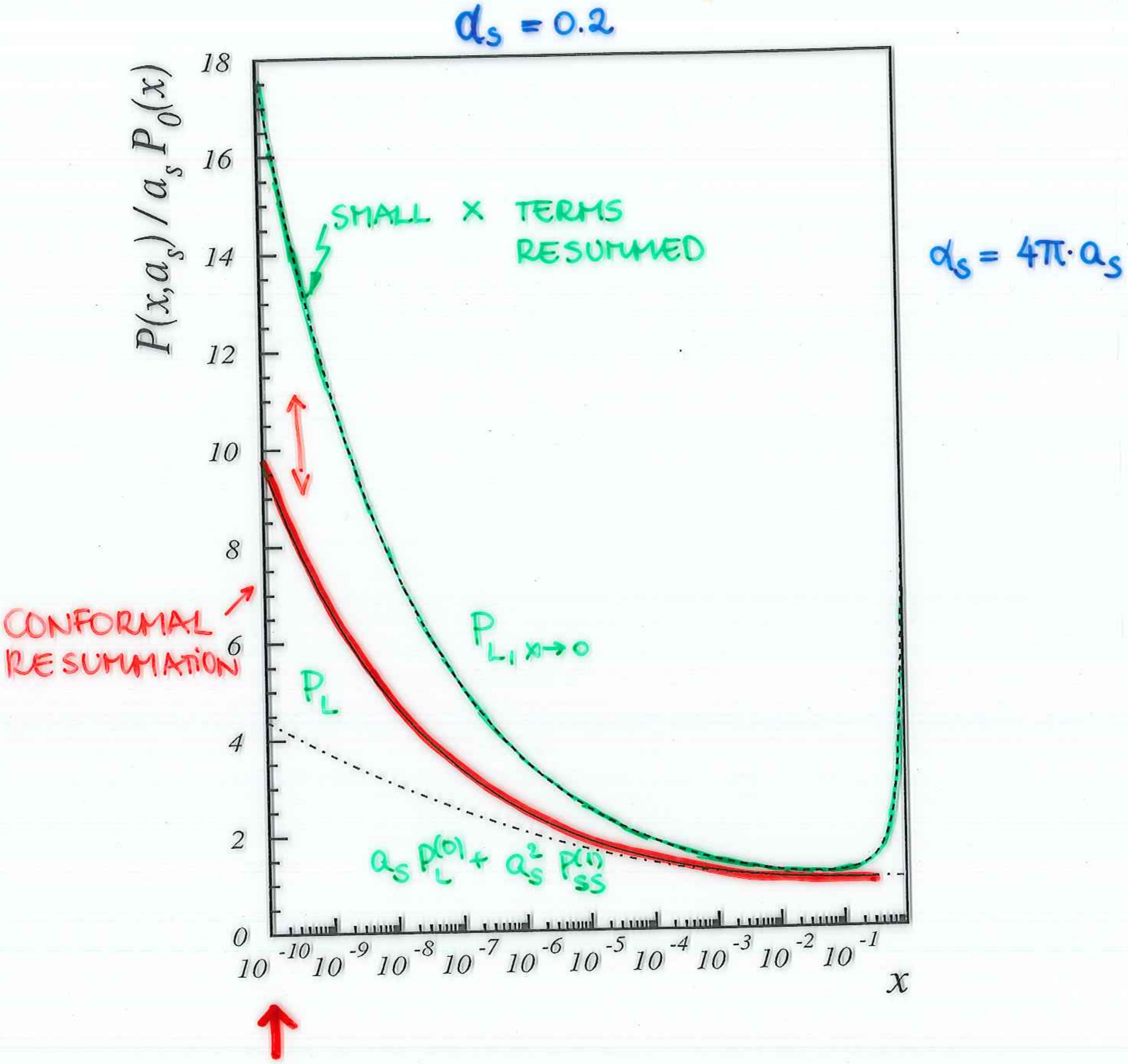



Figure 1: Fixed–order and resummed splitting functions $P(x, a_s)$ normalized to $a_s P_{SS}^{(0)}(x)$ for $\alpha_s = 0.2$. Dash-dotted line : $P = a_s P_L^{(0)} + a_s^2 P_{SS}^{(1)}$ Eqs. (7), (8). Solid line : $P = P_L$, Eq. (3). Dashed line : $P = P_{L,x\to 0}$, Eq. (11).

JB, VAN NEERVEN '98

$\phi^3_{D=6}$.

# 8. $\Lambda_{\rm QCD}$ and $\alpha_s(M_Z^2)$

| NLO | $\alpha_s(M_Z^2)$ | expt | theory | Ref. |
|---|---|---|---|---|
| CTEQ6 | 0.1165 | ±0.0065 | | [1] |
| MRST03 | 0.1165 | ±0.0020 | ±0.0030 | [2] |
| A02 | 0.1171 | ±0.0015 | ±0.0033 | [3] |
| ZEUS | 0.1166 | ±0.0049 | | [4] |
| H1 | 0.1150 | ±0.0017 | ±0.0050 | [5] |
| BCDMS | 0.110 | ±0.006 | | [6] |
| BB (pol) | 0.113 | ±0.004 | $^{+0.009}_{-0.006}$ | [7] |

| NNLO | $\alpha_s(M_Z^2)$ | expt | theory | Ref. |
|---|---|---|---|---|
| MRST03 | 0.1153 | ±0.0020 | ±0.0030 | [2] |
| A02 | 0.1143 | ±0.0014 | ±0.0009 | [3] |
| SY01(ep) | 0.1166 | ±0.0013 | | [8] |
| SY01($\nu$N) | 0.1153 | ±0.0063 | | [8] |
| BBG | 0.1139 | $+0.0026/-0.0028$ | | [9] |

BBG: $N_f = 4$: non-singlet data-analysis at $O(\alpha_s^3)$:
$\Lambda = 233 \pm 30$ MeV

Alpha Collab: $N_f = 2$ Lattice; non-pert. renormalization
$\Lambda = 245 \pm 16 \pm 16$ MeV

QCDSF Collab: $N_f = 2$ Lattice, pert. reno.
$\Lambda = 261 \pm +17 \pm 25$ MeV also other collab., (cf. PDG).

DIS: $\alpha_s(M_Z^2)$