The $O(\alpha^2)$ Initial State Radiation to e^+e^- Annihilation into a Neutral Vector Boson Revisited

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DESY, Zeuthen

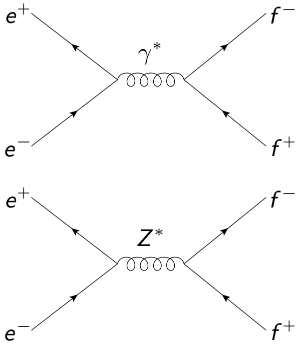
U. Freiburg/Br., May, 2019

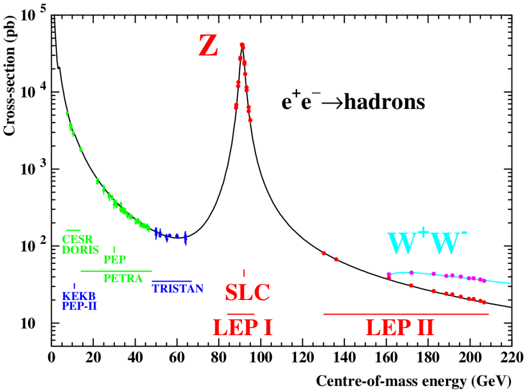
based on:

J. Blümlein, A. De Freitas, C. Raab and K. Schönwald, Phys.Lett. B791 (2019) 206-209, and in preparation.
J. Blümlein, A. De Freitas and W.L. van Neerven, Nucl. Phys. B855 (2012) 508-560.

Content

- Why is this calculation needed ?
- The Renormalization Group Method
- The Full Calculation
- Comparison of the Results and to Results in the Literature
- Conclusions

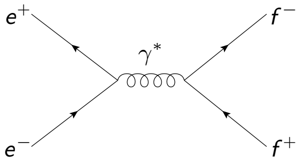


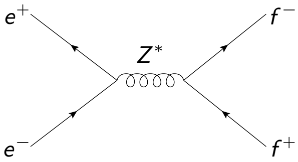


- We revisit the initial state corrections to $e^+ e^-$ annihilation to a neutral vector boson.
- These corrections are important for the prediction of the Z-boson peak at LEP, ILC and FCC-ee, and at Higgs factories through $e^+ e^- \to Z^* H^0$ and scanning the $t\bar{t}$-threshold.

We calculate the $O(\alpha^2)$ ISR corrections to the process:

$$e^- + e^+ \rightarrow \gamma^*/Z^* \rightarrow f^- + f^+$$





We calculate the $O(\alpha^2)$ ISR corrections to the process:

$$e^- + e^+ \to \gamma^*/Z^* \to f^- + f^+$$

with the invariants

$$(p_- + p_+)^2 = s, \qquad p_-^2 = p_+^2 = m_e^2, \qquad q^2 = s'$$

The initial state radiation (ISR) of n particles can be described according to the Drell-Yan mechanism

$$\frac{\mathrm{d}\sigma}{\mathrm{d}s'} = \frac{\sigma^0(s')}{4s} \int \mathrm{d}^4 q \delta^+(q^2 - s') \frac{1}{(2\pi)^{3n}} \prod_{i=1}^{n} \int \mathrm{d}^4 k_i \delta^+(k_i^2 - m_i^2) \delta^{(4)} (p_- + p_+ - q - K) \, |T^{(n)}|^2$$

where $\sigma^0(s')$ describes the leading order process and $T^{(n)}$ the matrix element of the ISR process.

Why do we have to revisit these matters ?

ISR corrections have been calculated up to $O(\alpha^2)$ in:

Nuclear Physics B297 (1988) 429–478
North-Holland, Amsterdam

HIGHER ORDER RADIATIVE CORRECTIONS AT LEP ENERGIES

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Received 21 August 1987

A complete two-loop $O(\alpha^2)$ initial state radiative correction to the Z-resonance shape is presented. The correction is compared with those expressions where only the soft-photon effects are resummed in all orders of perturbation theory. Our result shows that the soft-photon part constitutes the bulk of the radiative correction near the top of the Z-peak. The effect of non-photonic QED processes on the Z-resonance is found to be very small. The above results have been obtained by means of a standard Feynman diagram calculation. In addition we have also computed the cross sections by using the renormalization group method, where besides the leading logs $\ln(s/m^2)$, the next-to-leading ones also have been taken into account.

1. Introduction

Electron-positron colliding beam experiments which will be carried out in the near future at SLC and LEP will provide us with a wealth of information about the standard model of the electroweak interactions. A vast amount of literature [1] exists about the subjects one wants to investigate. We just mention topics such as the search for new particles like the Higgs boson, the top quark and maybe some supersymmetric partners of the particles in the standard model. Furthermore, one wants to make precise determinations of the electroweak parameters among which the most interesting are the mass and width of the Z boson. For the study of new physics effects, it is of the utmost importance to measure these quantities with a very high degree of accuracy. As has been extensively discussed in refs. [2, 3], the determination of the mass and width of the Z boson will be greatly affected by radiative corrections. In particular the pure QED part of the standard model leads to a distortion and a shift of the resonance peak. Up to now complete one-loop radiative corrections of the process $e^+e^- \to \mu^+\mu^-$ in which the Z is produced have been carried out [4]. The calculations reveal that the bulk of the corrections can be attributed to photonic contributions from the initial electron-positron state. Exponentiation of the lowest-order soft photon contribution shows [2, 3] that higher-order

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HADRONIC CONTRIBUTIONS TO $O(\alpha^2)$ RADIATIVE CORRECTIONS IN e^+e^- ANNIHILATION

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The complete hadronic contribution to $O(\alpha^2)$ real and virtual corrections from initial state radiation in e^+e^- annihilation is calculated using experimental information on the imaginary part of the hadronic vacuum polarization $\mathrm{Im}\,\Pi(q^2)$. For high energies both real and virtual corrections are expressed by four moments of $\mathrm{Im}\,\Pi(q^2)$ which characterize its behaviour in the low- and high-q^2 region. The formalism is applied to study the influence of hadronic radiation on the Z line shape and on the cross section in the continuum.

1. Introduction

Measurements of the total cross section for e^+e^- annihilation have reached a level of precision [1] where the influence of higher order radiative corrections is no longer negligible. Also the determination of the Z mass and width through the resonance line shape at future e^+e^- colliders will be influenced by radiative corrections and again the treatment to $O(\alpha)$ is insufficient [2].

In these reactions radiative corrections are dominantly due to initial-state radiation. To $O(\alpha^2)$ purely photonic contributions as well as those from real and virtual leptonic and hadronic states are relevant. For the leptonic and soft photonic case terms that are enhanced by powers of $\ln s/m^2$ have been calculated in ref. [3] and the remaining constant terms – typically of order 10^{-5} – are given in refs. [4,5].

Hadronic corrections are known to contribute approximately 50% to the large logarithms that appear in the vacuum polarization $\Pi(q^2)$ for large q^2. They are therefore expected to be as important in the high energy region for $O(\alpha^2)$ vertex corrections as the aforementioned large leptonic terms. Just as for $\Pi(q^2)$ these $O(\alpha^2)$ hadronic corrections are also determined by the quantity $R(s) \equiv \sigma_{had}/\sigma_{point}$ measured in lower energy e^+e^- collisions, assuming that $R(s)$ approaches a constant value for large s.

In the following we derive an expression for the hadronic contribution to the virtual $O(\alpha^2)$ corrections which is valid for arbitrary q^2. It becomes particularly simple in the large-q^2 region. The information contained in $R(s)$ can be condensed in its asymptotic behaviour together with three moments which fix the coefficients of the $\ln^n q^2/m^2$ terms. A similar approach will be developed for real soft and hard hadron radiation which in the high energy region depends on the same moments. The formalism can be easily applied to the special case of lepton radiation and reproduces earlier results for virtual radiation [3,4] and the logarithmically enhanced terms from real radiation [3]. However, in the latter case, we disagree with ref. [5] in subleading terms.

A special situation arises close to the Z peak. The Born cross section changes rapidly when the energy varies by $\Gamma_Z/2 \approx 1.3$ GeV. On the other hand, hadron production in e^+e^- annihilation has its threshold at $2m_\pi$, peaks at about 800 MeV and shows up drastic variations up to 4 GeV. The "soft" approximation that can be justified

[1] On leave of absence from Institute of Theoretical Physics, Warsaw University, PL-00681 Warsaw, Poland.
[2] Supported in part by Ministry of Science and Higher Education, Research Problem CPBP 01.03.

0370-2693/88/$ 03.50 © Elsevier Science Publishers B.V. 337
(North-Holland Physics Publishing Division)

Berends et al.: Complete $O(\alpha^2)$ ISR, (385 citations)
Kniehl et al.: so-called $O(\alpha^2)$ NS-corrections (process II of BBN). (96 citations)

Are these widely used results correct ?

Why do we have to revisit these matters ?

ISR corrections have been calculated up to $O(\alpha^2)$ in the asymptotic limit $m_e^2/s \ll 1$ with two different techniques:

1. Berends, Burgers, van Neerven (Nucl. Phys. B297 (1988)) (BBN)
   - full calculation with massive electrons in the limit $m_e^2 \ll s$ calculation in $d = 4$ with soft-hard separation, including soft and virtual photons, hard bremsstrahlung, as well as fermion pair production
   - Calculational Technique:

     o direct integration over the phase space in $d = 4$ with soft-hard photon separator

     o expansion in $m_e^2 \ll s$ on integrand level
2. Blümlein, De Freitas, van Neerven (Nucl. Phys. B855 (2012))
   - direct calculation of the asymptotic limit using operator matrix elements
   - technique based on asymptotic factorization
   - The logarithmically enhanced terms are correct in both papers
   - There are significant differences in results of these papers for the constant terms.

In the asymptotic region the cross section factorizes

$$\frac{\mathrm{d}\sigma_{ij}(s')}{\mathrm{d}s'} = \frac{\sigma^{(0)}(s')}{s} \sum_{l,k} \Gamma_{l,i}\left(z, \frac{\mu^2}{m_e^2}\right) \otimes \tilde{\sigma}_{lk}\left(z, \frac{s'}{\mu^2}\right) \otimes \Gamma_{k,j}\left(z, \frac{\mu^2}{m_e^2}\right)$$

into

- massless cross sections $\tilde{\sigma}_{ij}\left(z, \frac{s'}{\mu^2}\right)$
  Hamberg, van Neerven, Matsuura (Nucl. Phys. B 359 (1991))
  Harlander, Kilgore (Phys. Rev. Lett. 88 (2002))
- massive operator matrix elements $\Gamma_{ij}\left(z, \frac{\mu^2}{m_e^2}\right)$, which carry all mass dependence
  Blümlein, De Freitas, van Neerven (Nucl.Phys. B855 (2012))

$\sigma^{(0)}(s')$ is the Born cross section and the convolution $\otimes$ is given by

$$f(z) \otimes g(z) = \int_0^1 \mathrm{d}z_1 \int_0^1 \mathrm{d}z_2\, f(z_1) g(z_2) \delta(z - z_1 z_2).$$

Factorization in the Asymptotic Region

The comparison between both calculations shows:

- the one-loop, i.e. $O(\alpha)$, agrees between both calculations
- the logarithmically enhanced terms at two-loops ($O(\alpha^2)$) agree between both calculations
- the constant terms do not agree

$\Rightarrow$ breakdown of asymptotic factorization or errors?

Recalculation

The approach to the recalculation:

- **full integration over the phase space** in $d = 4$, i.e. no a-priori expansion in the mass
  - this is a four-fold integration, three integrations can be performed using standard techniques
  - integrand of the last integral contains rational, logarithmic and polylogarithmic expressions with involved argument structures
  - the last integration is performed in terms of iterated integrals after determining the minimal set of contributing letters
  - the final result is expressed as iterated integrals over involved square root valued letters of the kind

$$f_1 = \frac{y}{\sqrt{1-y}(w^2+4zy)w},$$
$$f_2 = \frac{y}{\sqrt{y}\sqrt{y(1-z)^2-16\rho^2}(w^2+4zw)w}$$

and related with

$$w = \sqrt{y^2(1-z^2) - 8\rho(1+z)y + 16\rho^2}, \qquad z = \frac{s'}{s}, \qquad \rho = \frac{m_e^2}{s}$$

$\Rightarrow$ the analytic results can be expanded in the electron mass

Strategy

- Calculate the cross section applying the renormalization group technique [our work 2011]
- Perform a full phase space calculation without any approximation and expand the result in the limit $m^2/s \ll 1$
- Compare the results and compare to Berends et al. and Kniehl et al. (1988).

The Born Cross Section

The Born Cross Section : $e^+e^- \to f,\bar{f} \quad f \neq e$

$$\frac{d\sigma^{(0)}(s)}{d\Omega} = \frac{\alpha^2}{4s} N_{C,f} \sqrt{1-\frac{4m_f}{s}} \Bigg[\left(1+\cos^2\theta + \frac{4m_f^2}{s}\sin^2\theta\right) G_1(s) - \frac{8m_f^2}{s} G_2(s)$$

$$+2\sqrt{1-\frac{4m_f^2}{s}} \cos\theta\, G_3(s) \Bigg]$$

$$\sigma^{(0)}(s) = \frac{4\pi\alpha^2}{3s} N_{C,f} \sqrt{1-\frac{4m_f}{s}} \left[\left(1+\frac{2m_f^2}{s}\right) G_1(s) - 6\frac{m_f^2}{s} G_2(s) \right]$$

$$G_1(s) = Q_e^2 Q_f^2 + 2Q_e Q_f v_e v_f \mathrm{Re}[\chi_Z(s)] + (v_e^2+a_e^2)(v_f^2+a_f^2)|\chi_Z(s)|^2$$

$$G_2(s) = (v_e^2+a_e^2)a_f^2|\chi_Z(s)|^2$$

$$G_3(s) = 2Q_e Q_f a_e a_f \mathrm{Re}[\chi_Z(s)] + 4v_e v_f a_e a_f |\chi_Z(s)|^2.$$

$$\chi_Z(s) = \frac{s}{s - M_Z^2 + iM_Z\Gamma_Z}$$

The Renormalization Group Technique

We represent the observable in Mellin space transforming $z = s'/s \; \epsilon[0,1]$:

The differential scattering cross section $\Sigma(z) = d\sigma_{ij}(z)/ds'$ is considered. This quantity reads in Mellin space

$$\mathbf{M}[\Sigma(z)](N) = \int_0^1 dz z^{N-1} \Sigma(z) \; .$$

In this representation the different Mellin convolutions to be performed in z–space simplify to ordinary products. The following representation is obtained

$$\frac{d\sigma_{ij}}{ds'}(N) = \frac{1}{s}\sigma^{(0)}(N) \sum_{l,k} \Gamma_{l,i}\left(N, \frac{\mu^2}{m^2}\right) \tilde{\sigma}_{lk}\left(N, \frac{s'}{\mu^2}\right) \Gamma_{k,j}\left(N, \frac{\mu^2}{m^2}\right) \; .$$

- Here Γ_{li} denote massive operator matrix elements and $\tilde{\sigma}_{lk}$ the massless Wilson coefficients, both being calculated in the $\overline{\rm MS}$ scheme.
- μ is the factorization mass, which cancels in the physical cross section.
- The initial state fermion mass dependence is solely encoded in Γ_{li}.

Γ_{li} and $\tilde{\sigma}_{lk}$ obey the following renormalization group equations:

$$\left[\left(\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g}\right)\delta_{al} + \gamma_{al}(N,g)\right]\Gamma_{li}\left(N, \frac{\mu^2}{m^2}, g(\mu^2)\right) = 0$$

$$\left[\left(\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g}\right)\delta_{la}\delta_{kb} - \gamma_{la}(N,g)\delta_{kb} - \gamma_{kb}(N,g)\delta_{la}\right]\tilde{\sigma}_{lk}\left(\frac{s'}{m^2}, g(\mu^2)\right) = 0$$

$$\left[\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g}\right]\sigma_{ij}\left(\frac{s'}{\mu^2}, g(\mu^2)\right) = 0$$

For the process under consideration we obtain to $O(a^2)$:

$$\left[\frac{\partial}{\partial\hat{L}} - \beta_0 a^2\frac{\partial}{\partial a} + \frac{1}{2}\gamma_{ee}(N,a)\right]\Gamma_{ee}\left(N, a, \frac{\mu^2}{m^2}\right) + \frac{1}{2}\gamma_{e\gamma}(N,a)\Gamma_{\gamma e}\left(N, a, \frac{\mu^2}{m^2}\right) = 0$$

$$\left[\frac{\partial}{\partial\hat{L}} - \beta_0 a^2\frac{\partial}{\partial a} - \gamma_{ee}(N,a)\right]\tilde{\sigma}_{ee}\left(N, a, \frac{s'}{\mu^2}\right) - \gamma_{\gamma e}(N,a)\tilde{\sigma}_{e\gamma}\left(N, a, \frac{s'}{\mu^2}\right) = 0$$

where $\partial/\partial\mu$ has been replaced by $2\partial/\partial\hat{L}$, with $\hat{L} = \ln(\mu^2/M^2)$.

The solutions of these equations are

$$\Gamma_{ee}\left(N, a, \frac{\mu^2}{m^2}\right) = 1 + a\left[-\frac{1}{2}\gamma_{ee}^{(0)}L + \Gamma_{ee}^{(0)}\right] + a^2\Bigg[\left\{\frac{1}{8}\gamma_{ee}^{(0)}\left(\gamma_{ee}^{(0)} - 2\beta_0\right) + \frac{1}{8}\gamma_{e\gamma}^{(0)}\gamma_{\gamma e}^{(0)}\right\}L^2$$
$$+\frac{1}{2}\left\{-\gamma_{ee}^{(1)} + 2\beta_0\Gamma_{ee}^{(0)} - \gamma_{ee}^{(0)}\Gamma_{ee}^{(0)} - \gamma_{e\gamma}^{(0)}\Gamma_{\gamma e}^{(0)}\right\}L + \Gamma_{ee}^{(1)}\Bigg] + O(a^3),$$

$$\tilde{\sigma}_{ee}\left(N, a, \frac{s'}{\mu^2}\right) = 1 + a\left[-\frac{1}{2}\gamma_{ee}^{(0)}\lambda + \tilde{\sigma}_{ee}^{(0)}\right] + a^2\Bigg[\left\{\frac{1}{2}\gamma_{ee}^{(0)}\left(\gamma_{ee}^{(0)} + \beta_0\right) + \frac{1}{4}\gamma_{e\gamma}^{(0)}\gamma_{\gamma e}^{(0)}\right\}\lambda^2$$
$$+1 + \left\{-\gamma_{ee}^{(1)} - \beta_0\tilde{\sigma}_{ee}^{(0)} - \gamma_{ee}^{(0)}\tilde{\sigma}_{ee}^{(0)} - \gamma_{\gamma e}^{(0)}\tilde{\sigma}_{e\gamma}^{(0)}\right\}\lambda + \tilde{\sigma}_{ee}^{(1)}\Bigg] + O(a^3),$$

$$\Gamma_{\gamma e}\left(N, a, \frac{\mu^2}{m^2}\right) = a\left[-\frac{1}{2}\gamma_{\gamma e}^{(0)}L + \Gamma_{\gamma e}^{(0)}\right] + O(a^2)$$

$$\tilde{\sigma}_{e\gamma}\left(N, a, \frac{\mu^2}{m^2}\right) = a\left[-\frac{1}{2}\gamma_{e\gamma}^{(0)}\lambda + \tilde{\sigma}_{e\gamma}^{(0)}\right] + O(a^2)\ ,$$

with the logarithms $L = \ln\left(\frac{\mu^2}{m^2}\right)$ and $\lambda = \ln\left(\frac{s'}{\mu^2}\right)$

Introducing the splitting functions in N-space

$$P_{ij}^{(l)}(N) = \int_0^1 dz z^{N-1} P_{ij}^{(l)}(z) = -\gamma_{ij}^{(l)}(N)$$

one obtains

$$\frac{d\sigma_{e^+e^-}}{ds'} = \frac{1}{s}\sigma^{(0)}(s)\Bigg\{1 + a_0\left[P_{ee}^{(0)}\mathbf{L} + \left(\tilde{\sigma}_{ee}^{(0)} + 2\Gamma_{ee}^{(0)}\right)\right]$$
$$+a_0^2\Bigg\{\left[\frac{1}{2}P_{ee}^{(0)} \otimes P_{ee}^{(0)} - \frac{\beta_0}{2}P_{ee}^{(0)} + \frac{1}{4}P_{e\gamma}^{(0)} \otimes P_{\gamma e}^{(0)}\right]\mathbf{L}^2$$
$$+\left[P_{ee}^{(1)} + P_{ee}^{(0)} \otimes \left(\tilde{\sigma}_{ee}^{(0)} + 2\Gamma_{ee}^{(0)}\right) - \beta_0\tilde{\sigma}_{ee}^{(0)} + P_{\gamma e}^{(0)} \otimes \tilde{\sigma}_{e\gamma}^{(0)} + \Gamma_{\gamma e}^{(0)} \otimes P_{e\gamma}^{(0)}\right]\mathbf{L}$$
$$+\left(2\Gamma_{ee}^{(1)} + \tilde{\sigma}_{ee}^{(1)}\right) + 2\Gamma_{ee}^{0)} \otimes \tilde{\sigma}_{ee}^{(0)} + 2\tilde{\sigma}_{e\gamma}^{(0)} \otimes \Gamma_{\gamma e}^{(0)} + \Gamma_{ee}^{(0)} \otimes \Gamma_{ee}^{(0)}\Bigg\}\Bigg\}$$

with

$$\mathbf{L} = \ln\left(\frac{s'}{m^2}\right) = \ln\left(\frac{s}{m^2}\right) + \ln(z); \quad \hat{\mathbf{L}} \equiv \ln(s/m^2) \; .$$

It is convenient to represent the differential scattering cross section in terms of three contributions, the flavor non-singlet terms with a single fermion line (I), those with a closed fermion line (II), and the pure-singlet terms (III). These contributions are :

$$
\begin{aligned}
\frac{d\sigma^{\mathrm{I}}_{e^+e^-}}{ds'} &= \frac{1}{s}\sigma^{(0)}(s)\Bigg\{1 + a_0\left[P_{ee}^{(0)}\mathbf{L} + \left(\tilde{\sigma}_{ee}^{(0)} + 2\Gamma_{ee}^{(0)}\right)\right] \\
&\quad + a_0^2\Bigg\{\frac{1}{2}P_{ee}^{(0)} \otimes P_{ee}^{(0)}\mathbf{L}^2 + \left[P_{ee}^{(1),\mathrm{I}} + P_{ee}^{(0)} \otimes \left(\tilde{\sigma}_{ee}^{(0)} + 2\Gamma_{ee}^{(0)}\right)\right]\mathbf{L} \\
&\quad + \left(2\Gamma_{ee}^{(1),\mathrm{I}} + \tilde{\sigma}_{ee}^{(1),\mathrm{I}}\right) + 2\Gamma_{ee}^{0)} \otimes \tilde{\sigma}_{ee}^{(0)} + \Gamma_{ee}^{(0)} \otimes \Gamma_{ee}^{(0)}\Bigg\}\Bigg\} \\
\frac{d\sigma^{\mathrm{II}}_{e^+e^-}}{ds'} &= \frac{1}{s}\sigma^{(0)}(s)a_0^2\left\{-\frac{\beta_0}{2}P_{ee}^{(0)}\mathbf{L}^2 + \left[P_{ee}^{(1),\mathrm{II}} - \beta_0\tilde{\sigma}_{ee}^{(0)}\right]\mathbf{L} + \left(2\Gamma_{ee}^{(1),\mathrm{II}} + \tilde{\sigma}_{ee}^{(1),\mathrm{II}}\right)\right\} \\
\frac{d\sigma^{\mathrm{III}}_{e^+e^-}}{ds'} &= \frac{1}{s}\sigma^{(0)}(s)a_0^2\Bigg\{\frac{1}{4}P_{e\gamma}^{(0)} \otimes P_{\gamma e}^{(0)}\mathbf{L}^2 + \left[P_{ee}^{(1),\mathrm{III}} + P_{\gamma e}^{(0)} \otimes \tilde{\sigma}_{e\gamma}^{(0)} + \Gamma_{\gamma e}^{(0)} \otimes P_{e\gamma}^{(0)}\right]\mathbf{L} \\
&\quad + \left(2\Gamma_{ee}^{(1),\mathrm{III}} + \tilde{\sigma}_{ee}^{(1),\mathrm{III}}\right) + 2\tilde{\sigma}_{e\gamma}^{(0)} \otimes \Gamma_{\gamma e}^{(0)}\Bigg\}
\end{aligned}
$$

- $\tilde{\sigma}_{ij}^{(k)}$ denotes the respective contribution of the massless Drell-Yan (DY) cross section.

Different ingredients to the calculation :

- Splitting functions P_{ij} to $O(\alpha^2)$

E.G. Floratos, D.A. Ross and C.T. Sachrajda, Nucl. Phys. B **129** (1977) 66 [Erratum-ibid. B **139** (1978) 545];
Nucl. Phys. B **152** (1979) 493;
A. Gonzalez-Arroyo, C. Lopez and F.J. Yndurain, Nucl. Phys. B **153** (1979) 161;
A. Gonzalez-Arroyo and C. Lopez, Nucl. Phys. B **166** (1980) 429;
E.G. Floratos, C. Kounnas and R. Lacaze, Nucl. Phys. B **192** (1981) 417;
G. Curci, W. Furmanski and R. Petronzio, Nucl. Phys. B **175** (1980) 27;
W. Furmanski and R. Petronzio, Phys. Lett. B **97** (1980) 437;
R. Hamberg and W.L. van Neerven, Nucl. Phys. B **379** (1992) 143;
R.K. Ellis and W. Vogelsang, arXiv:hep-ph/9602356;
S. Moch and J.A.M. Vermaseren, Nucl. Phys. B **573** (2000) 853;
J. Ablinger *et al.*, Nucl. Phys. B **882** (2014) 263; Nucl. Phys. B **886** (2014) 733; Nucl. Phys. B **890** (2014) 48;
Nucl. Phys. B **922** (2017) 1.

- massless Drell-Yan Cross Section $\tilde{\sigma}_{ij}$ to $O(\alpha^2)$

R. Hamberg, W.L. van Neerven and T. Matsuura, Nucl. Phys. B **359** (1991) 343 [E: B **644** (2002) 403];
R.V. Harlander and W.B. Kilgore, Phys. Rev. Lett. **88** (2002) 201801.

- massive OMEs Γ_{ij} to $O(\alpha^2)$ $\Longrightarrow$ this calculation.

(Some errors at $O(\alpha)$ in earlier work corrected.)

Renormalization concerns the wave function, the charge renormalization, and the ultraviolet singularities of the local operators.

i) Wave function renormalization

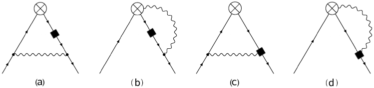
The bare wave function is renormalized by

$$\psi_0 = \sqrt{Z_2(\varepsilon)}\psi$$

$$\begin{aligned} Z_2 &= 1 + \hat{a} S_\varepsilon \left(\frac{m^2}{\mu^2}\right)^{\varepsilon/2} \left[\frac{6}{\varepsilon} - 4 + \left(4 + \frac{3}{4}\zeta_2\right)\varepsilon\right] \\ &+ \hat{a}^2 S_\varepsilon^2 \left(\frac{m^2}{\mu^2}\right)^{\varepsilon} \left\{\left[18\frac{1}{\varepsilon^2} - \frac{51}{2}\frac{1}{\varepsilon} + \left(\frac{433}{8} - \frac{147}{2}\zeta_2 + 96\zeta_2 \ln(2) - 24\zeta_3\right)\right]_{\rm I} \right. \\ &\left. + \left[16\frac{1}{\varepsilon^2} - \frac{38}{3}\frac{1}{\varepsilon} + \left(\frac{1139}{18} - 28\zeta_2\right)\right]_{\rm II}\right\} . \end{aligned}$$

Broadhurst, et. al. Z. Phys. C 48 (1990). Melnikov and Ritbergen, Nucl. Phys. B 591 (2000)

Counterterms:



$$Z_{CT} = \int_0^1 dx\, x^N \Bigg\{ -\frac{72}{\varepsilon^2}\delta(1-x) - \frac{24}{\varepsilon}\mathcal{D}_0(x) - 24\mathcal{D}_1(x) + 16\mathcal{D}_0(x) - (64+18\zeta_2)\,\delta(1-x)\Bigg\} - 12 + N\left[\left(\frac{24}{\varepsilon}+8\right)\delta(1-x) + 24\mathcal{D}_0(x)\right]\Bigg\}$$

$$\mathcal{D}_k(x) = \left[\frac{\ln^{k-1}(1-x)}{1-x}\right]$$

ii) Charge Renormalization

Massive fermions $\rightarrow$ coupling constant first obtained in MOM-scheme, where

$$Z_g^{\mathrm{MOM}^2} = 1 + a^{\mathrm{MOM}}(\mu^2)\beta_{0,H}\left(\frac{m_e^2}{\mu^2}\right)^{\varepsilon/2}\exp\left(\sum_{k=2}^{\infty}\frac{\zeta_k}{k}\left(\frac{\varepsilon}{2}\right)^k\right) + O\left(a^{\mathrm{MOM}}(\mu^2)^2\right)$$

We then transform to the $\overline{\mathrm{MS}}$ scheme using

$$Z_g^{\mathrm{MOM}^2}a^{\mathrm{MOM}}(\mu^2) = Z_g^{\overline{\mathrm{MS}}^2}a^{\overline{\mathrm{MS}}}(\mu^2), \qquad a \equiv \frac{\alpha}{4\pi}$$

which implies

$$a^{\mathrm{MOM}} = a^{\overline{\mathrm{MS}}} - \beta_{0,H}\ln\left(\frac{m_e^2}{\mu^2}\right)a^{\overline{\mathrm{MS}}^2} + O\left(a^{\overline{\mathrm{MS}}^3}\right)$$

with

$$\beta_{0,H} = -\frac{4}{3}$$

iii) Renormalization of the composite operators:
The inverse Z-factors are given in the MOM-scheme by

$$Z_{ij}^{-1}(a^{\text{MOM}}, n_f + 1, \mu) = \delta_{ij} - a^{\text{MOM}} \frac{\gamma_{ij}^{(0)}}{\varepsilon} + a^{\text{MOM}^2} \Bigg[\frac{1}{\varepsilon}\left(-\frac{1}{2}\gamma_{ij}^{(1)} - \delta a_1^{\text{MOM}} \gamma_{ij}^{(0)} \right) + \frac{1}{\varepsilon^2}\left(\frac{1}{2}\gamma_{il}^{(0)}\gamma_{lj}^{(0)} + \beta_0 \gamma_{ij}^{(0)} \right) \Bigg] + O(a^{\text{MOM}^3})$$

with

$$\delta a_1^{\text{MOM}} = S_\varepsilon \frac{2\beta_{0,H}}{\varepsilon} \left(\frac{m_e^2}{\mu^2} \right)^{\varepsilon/2} \exp\left[\sum_{i=2}^{\infty} \frac{\zeta_i}{i} \left(\frac{\varepsilon}{2} \right)^i \right]$$

The renormalized OMEs are given by

$$A_{ij}^{\text{MOM}} = \delta_{ij} + a^{\text{MOM}} \left[\hat{A}_{ij}^{(1)} + Z_{i,j}^{-1,(1)} \right] + a^{\text{MOM}^2} \left[\hat{A}_{ij}^{(2)} + Z_{i,j}^{-1,(2)} + Z_{i,j}^{-1,(1)} \hat{A}_{ij}^{(1)} \right] + \ldots$$

The corresponding Z-factors are given by

$$
\begin{aligned}
\left[Z_{ee}^{\mathrm{I}}(\varepsilon, N)\right]^{-1} &= 1 + a^{\mathsf{MOM}} S_\varepsilon \frac{1}{\varepsilon} P_{ee}^{(0)}(N) + a^{\mathsf{MOM}\,2} S_\varepsilon^2 \left\{ \frac{1}{2\varepsilon^2} P_{ee}^{(0)\,2}(N) + \frac{1}{2\varepsilon} P_{ee}^{(1),\mathrm{NS}}(N) \right\} \\
&\quad + O(a^{\mathsf{MOM}\,3}) \\
\left[Z_{ee}^{\mathrm{II}}(\varepsilon, N)\right]^{-1} &= a^{\mathsf{MOM}\,2} \left\{ -\frac{1}{2\varepsilon^2} \beta_0 P_{ee}^{(0)} + \frac{2}{\varepsilon^2} \beta_{0,H} \left(\frac{m_e^2}{\mu_2} \right)^{\varepsilon/2} \exp\left[\sum_{i=2}^{\infty} \frac{\zeta_i}{i} \left(\frac{\varepsilon}{2} \right)^i \right] + \frac{1}{2\varepsilon} P_{ee}^{(1),\mathrm{II}} \right\} \\
&\quad + O(a^{\mathsf{MOM}\,3}) \\
\left[Z_{ee}^{\mathrm{III}}(\varepsilon, N)\right]^{-1} &= a^{\mathsf{MOM}\,2} S_\varepsilon^2 \left\{ \frac{1}{2\varepsilon^2} P_{e\gamma}^{(0)}(N) P_{\gamma e}^{(0)}(N) + \frac{1}{2\varepsilon} \left[P_{ee}^{(1),\mathrm{III}}(N) \right] \right\} + O(a^{\mathsf{MOM}\,3}) \\
\left[Z_{e\gamma}(\varepsilon, N)\right]^{-1} &= a^{\mathsf{MOM}} S_\varepsilon \frac{1}{\varepsilon} P_{e\gamma}^{(0)}(N) + O(a^{\mathsf{MOM}\,2}) \\
\left[Z_{\gamma e}^{\mathrm{NS}}(\varepsilon, N)\right]^{-1} &= a^{\mathsf{MOM}} S_\varepsilon \frac{1}{\varepsilon} P_{\gamma e}^{(0)}(N) + O(a^{\mathsf{MOM}\,2})
\end{aligned}
$$

UV Un-renormalized Operator-Matrix Elements : after wave function and charge renormalization

$$\hat{A}_{ee}^{(1)} = a^{\rm MOM} S_\varepsilon \left(\frac{m^2}{\mu^2}\right)^{\varepsilon/2} \left\{\frac{1}{\varepsilon} P_{ee}^{(0)} + \Gamma_{ee}^{(0)} + \varepsilon \overline{\Gamma}_{ee}^{(0)}\right\}$$

$$\hat{A}_{ee}^{(2),\rm I} = a^{\rm MOM\,2} S_\varepsilon^2 \left(\frac{m^2}{\mu^2}\right)^{\varepsilon} \left\{\frac{1}{2\varepsilon^2} P_{ee}^{(0)} \otimes P_{ee}^{(0)} - \frac{1}{2\varepsilon}\left[P_{ee}^{(1),\rm I} + 2\Gamma_{ee}^{(0)} \otimes P_{ee}^{(0)}\right] + \Gamma_{ee}^{(1),\rm I}\right\}$$

$$\hat{A}_{ee}^{(2),\rm II} = a^{\rm MOM\,2} S_\varepsilon^2 \left(\frac{m^2}{\mu^2}\right)^{\varepsilon} \left\{\frac{1}{\varepsilon^2} \beta_0 P_{ee}^{(0)} - \frac{1}{\varepsilon}\left[\frac{1}{2} P_{ee}^{(1),\rm II} + 2\beta_0 \Gamma_{ee}^{(0)}\right] + \Gamma_{ee}^{(1),\rm II}\right\}$$

$$\hat{A}_{ee}^{(2),\rm III} = a^{\rm MOM\,2} S_\varepsilon^2 \left(\frac{m^2}{\mu^2}\right)^{\varepsilon} \left\{\frac{1}{2\varepsilon^2} P_{e\gamma}^{(0)} \otimes P_{\gamma e}^{(0)} - \frac{1}{\varepsilon}\left\{\frac{1}{2} P_{ee}^{(1),\rm III} + \Gamma_{\gamma e}^{(0)} \otimes P_{e\gamma}^{(0)}\right\} + \Gamma_{ee}^{(1),\rm III}\right\}$$

- cf. also : M. Buza et al., Nucl. Phys. B **472** (1996) 611;

I. Bierenbaum, J. Blümlein, S. Klein, Nucl. Phys. B **780** (2007) 40.

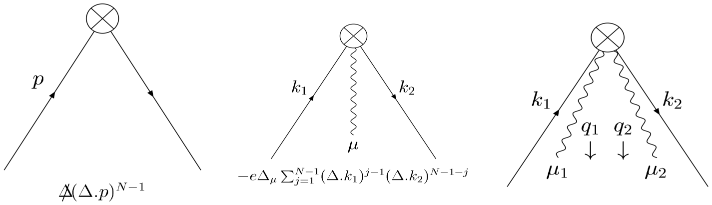
The renormalized OMEs in the $\overline{\rm MS}$-scheme are finally given by

$$A_{ee}^{\overline{\rm MS},\rm I}(N) = a^{\overline{\rm MS}}\left[-\frac{1}{2}P_{ee}^{(0)}\ln\left(\frac{m_e^2}{\mu^2}\right)+\Gamma_{ee}^{(0)}\right]+a^{\overline{\rm MS}^2}\left[\frac{1}{8}P_{ee}^{(0)^2}\ln^2\left(\frac{m_e^2}{\mu^2}\right)\right.$$
$$\left.-\frac{1}{2}\left[P_{ee}^{(1),\rm I}+P_{ee}^{(0)}\Gamma_{ee}^{(0)}\right]\ln\left(\frac{m_e^2}{\mu^2}\right)+\hat{\Gamma}_{ee}^{(1),\rm I}+P_{ee}^{(0)}\overline{\Gamma}_{ee}^{(0)}\right]+O(a^{\overline{\rm MS}})$$

$$A_{ee}^{\overline{\rm MS},\rm II}(N) = a^{\overline{\rm MS}^2}\left[\frac{\beta_{0,H}}{4}P_{ee}^{(0)}\ln^2\left(\frac{m_e^2}{\mu^2}\right)-\left[\frac{1}{2}P_{ee}^{(1),\rm II}+\beta_{0,H}\Gamma_{ee}^{0}\right]\ln\left(\frac{m_e^2}{\mu^2}\right)\right.$$
$$\left.+\hat{\Gamma}_{ee}^{(1),\rm II}+2\beta_{0,H}\overline{\Gamma}_{ee}^{(0)}\right]+O(a^{\overline{\rm MS}^3})$$

$$A_{ee}^{\overline{\rm MS},\rm III}(N) = a^{\overline{\rm MS}^2}\left[\frac{1}{8}P_{e\gamma}^{(0)}P_{\gamma e}^{(0)}\ln^2\left(\frac{m_e^2}{\mu^2}\right)-\frac{1}{2}\left[P_{ee}^{(1),\rm III}\right.\right.$$
$$\left.\left.+P_{e\gamma}^{(0)}\Gamma_{\gamma e}^{(0)}(N)\right]\ln\left(\frac{m_e^2}{\mu^2}\right)+\hat{\Gamma}_{ee}^{(1),\rm III}+P_{e\gamma}^{(0)}\overline{\Gamma}_{\gamma e}^{(0)}\right]+O(a^{\overline{\rm MS}^3})$$

Here $\Gamma_{ij}^{(0)}$ and $\overline{\Gamma}_{ij}^{(0)}$ denote the constant and linear term in ε of the unrenormalized one-loop massive OMEs, and $\hat{\Gamma}_{ij}^{(1)}$ the corresponding constant part of the two-loop OMEs.



$$\Gamma_{e^+e^+} = \Gamma_{e^-e^-} = \langle e| O_F^{\mathrm{NS,S}} |e\rangle ,$$
$$\Gamma_{e^+\gamma} = \Gamma_{e^-\gamma} = \langle \gamma| O_F^{\mathrm{S}} |\gamma\rangle ,$$
$$\Gamma_{\gamma e^+} = \Gamma_{\gamma e^-} = \langle e| O_V^{\mathrm{S}} |e\rangle ,$$

$$O^{\mathrm{NS,S}}_{F;\mu_1,\ldots,\mu_N} = i^{N-1}\mathrm{S}\left[\bar\psi\gamma_{\mu_1}D_{\mu_2}\ldots D_{\mu_N}\psi\right] - \text{traces},$$
$$O^{\mathrm{S}}_{V;\mu_1,\ldots,\mu_N} = 2i^{N-2}\mathrm{S}\left[F_{\mu_1\alpha}D_{\mu_2}\ldots D_{\mu_{N-1}}F^{\alpha}_{\mu_N}\right] - \text{traces}$$

- technique has been used to derive deep-inelastic scattering (DIS) structure functions in the asymptotic limit $Q^2 \gg m^2$ up to $O(\alpha_s^3)$
- in the context of DIS proven to work at α_s^2 in the
  - non-singlet process
    Buza, Matiounine, Smith, van Neerven (Nucl.Phys. B485 (1997))
    Blümlein, Falcioni, De Freitas (Nucl.Phys. B910 (2016))
  - pure-singlet process
    Blümlein, De Freitas, Raab, Schönwald (Nucl. Phys B (2019), in print, arXiv:1903.06155)

  through analytic calculations

The one–loop splitting functions are factorization–scheme invariant and are given in x-space by

$$
\begin{aligned}
P_{ee}^{(0)}(x) &= 8\mathcal{D}_0(x) - 4(1+x) + 6\delta(1-x) = 4\left[\frac{1+x^2}{1-x}\right]_+ , \\
P_{e\gamma}^{(0)}(x) &= 4\left[x^2 + (1-x)^2\right] , \\
P_{\gamma e}^{(0)}(x) &= 4\left[\frac{1+(1-x)^2}{x}\right] ,
\end{aligned}
$$

The $+$-prescription is defined by

$$
\int_0^1 dx\,[f(x)]_+\, g(x) = \int_0^1 dx f(x)\,[g(x) - g(1)]\,,
$$

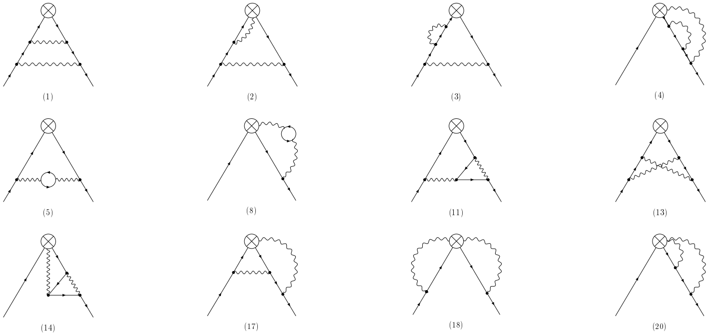
with $g(x) \,\epsilon\, \mathcal{D}[0,1]$ denoting a test function.

The $O(\varepsilon^0)$ terms are

$$\begin{aligned}
\Gamma_{ee}^{(0)}(x) &= -8\mathcal{D}_1(x) - 4\mathcal{D}_0(x) + 4\delta(1-x) + 2(1+x)\left[2\ln(1-x)+1\right] \\
&= -4\left[\frac{1+x^2}{1-x}\left\{\ln(1-x)+\frac{1}{2}\right\}\right]_+ \\
\Gamma_{e\gamma}^{(0)}(x) &= 0 \\
\Gamma_{\gamma e}^{(0)}(x) &= -2\frac{1+(1-x)^2}{x}\left[2\ln(x)+1\right],
\end{aligned}$$

The linear term in $\varepsilon\, \overline{\Gamma}_{ee}^{(0)}(x)$ reads

$$\begin{aligned}
\overline{\Gamma}_{ee}^{(0)}(x) &= -4\mathcal{D}_2(x) - 4\mathcal{D}_1(x) - \zeta_2\mathcal{D}_0(x) - \left(4+\frac{3}{4}\zeta_2\right)\delta(1-x) \\
&\quad +2(1+x)\left[\ln^2(1-x)+\ln(1-x)+\frac{1}{4}\zeta_2\right] \\
&= -2\left[\frac{1+x^2}{1-x}\left\{\ln^2(1-x)+\ln(1-x)+\frac{1}{4}\zeta_2\right\}\right]_+ .
\end{aligned}$$



Two-loop diagrams contributing to the massive operator matrix element $A_{ee}(N, \alpha)$.
The antisymmetric diagrams count twice.

All the diagrams can be written in terms of integrals of these type:

$$A^{a,b}_{\nu_1,\nu_2,\nu_3,\nu_4,\nu_5} = \int \frac{d^D k_1}{(4\pi)^D} \frac{d^D k_2}{(4\pi)^D} \frac{(\Delta \cdot k_1)^a (\Delta \cdot k_2)^b}{D_1^{\nu_1} D_2^{\nu_2} D_3^{\nu_3} D_4^{\nu_4} D_5^{\nu_5}}$$

$$B^{a,b}_{\nu_1,\nu_2,\nu_3,\nu_4,\nu_5} = \int \frac{d^D k_1}{(4\pi)^D} \frac{d^D k_2}{(4\pi)^D} \frac{k_2 \cdot p (\Delta \cdot k_1)^a (\Delta \cdot k_2)^b}{D_1^{\nu_1} D_2^{\nu_2} D_3^{\nu_3} D_4^{\nu_4} D_5^{\nu_5}}$$

$$E^{a,b}_{\nu_1,\nu_2,\nu_3,\nu_4,\nu_5} = \int \frac{d^D k_1}{(4\pi)^D} \frac{d^D k_2}{(4\pi)^D} \frac{(\Delta \cdot k_1)^a (\Delta \cdot k_2)^b}{D_1^{\nu_1} D_2^{\nu_2} D_3^{\nu_3} D_4^{\nu_4} D_5^{\nu_5}} \sum_{j=0}^{n-1} (\Delta \cdot k_1)^j (\Delta \cdot k_2)^{n-1-j}$$

$$F^{a,b}_{\nu_1,\nu_2,\nu_3,\nu_4,\nu_5} = \int \frac{d^D k_1}{(4\pi)^D} \frac{d^D k_2}{(4\pi)^D} \frac{(\Delta \cdot k_1)^a (\Delta \cdot k_2)^b}{D_1^{\nu_1} D_2^{\nu_2} D_3^{\nu_3} D_4^{\nu_4} D_5^{\nu_5}} \sum_{j=0}^{n-1} (\Delta \cdot p)^j (\Delta \cdot k_1)^{n-1-j}$$

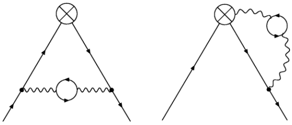
where,

$$D_1 = k_1^2 - m^2 \; ; \quad D_2 = k_2^2 - m^2 \; ; \quad D_3 = (k_1 - p)^2 \; ;$$
$$D_4 = (k_1 - k_2)^2 \; ; \quad D_5 = (k_2 - k_1 + p)^2 - m^2 \; ; \quad D_6 = (k_2 - p)^2$$

Result: Processes I + IV

The result for the the matrix element $\hat{\Gamma}_{ee}^{(1),\mathrm{I}}$ is

$$
\frac{1+3x^2}{1-x}\left[6\zeta_2 \ln(x) - 8\ln(x)\mathrm{Li}_2(1-x) - 4\ln^2(x)\ln(1-x)\right] + \left(\frac{122}{3}x + 22 + \frac{32}{1-x}\right)\zeta_2 + (8-112\zeta_2)\,\mathcal{D}_1(x)
$$
$$
+16\frac{1+x^2}{1-x}\left[2\mathrm{Li}_3(-x) - \ln(x)\mathrm{Li}_2(-x)\right] + \frac{80}{3(1-x)} + 56(1+x)\zeta_2\ln(1-x) + (16 - 52\zeta_2 + 128\zeta_3)\,\mathcal{D}_0(x)
$$
$$
+\left(\frac{22}{3}x + 32 + \frac{64}{3(1-x)^2} - \frac{51}{1-x} - \frac{16}{3(1-x)^3}\right)\ln^2(x) - (92+20x)\ln^2(1-x) + 14(x-2)\ln(1-x) + 120\mathcal{D}_2(x)
$$
$$
+\left(\frac{178}{3} - 36x + \frac{64}{3(1-x)^2} - \frac{140}{3(1-x)} - \frac{48}{1+x}\right)\ln(x) - \frac{1}{3}(1+x)\ln^3(x) + 4\frac{x^2-8x-6}{1-x}\ln(x)\ln(1-x)
$$
$$
-2\frac{1+17x^2}{1-x}\ln(x)\ln^2(1-x) - \frac{112}{3}(1+x)\ln^3(1-x) + 32\frac{1+x}{1-x}\left[\ln(x)\ln(1+x) + \mathrm{Li}_2(-x)\right] - 22x - \frac{62}{3}
$$
$$
-4\frac{13x^2+9}{1-x}\mathrm{S}_{1,2}(1-x) + 4\frac{5-11x^2}{1-x}\left[\ln(1-x)\mathrm{Li}_2(1-x) - \mathrm{Li}_3(1-x) - 2\zeta_3\right] + \frac{4(16x^2-10x-27)}{3(1-x)}\mathrm{Li}_2(1-x)
$$
$$
+\frac{224}{3}\mathcal{D}_3(x) + \left[\frac{433}{8} - \frac{67}{45}\pi^4 + \left(\frac{37}{2} - 48\ln(2)\right)\zeta_2 + 58\zeta_3\right]\delta(1-x) + (-1)^n\left\{\frac{2(1-x)(45x^2+74x+45)}{3(1+x)^2}\right.
$$
$$
+\frac{2(9+12x+30x^2-20x^3-15x^4)}{3(1+x)^3}\ln(x) + \frac{4(x^2+10x-3)}{3(1+x)}\left(\zeta_2 + 2\mathrm{Li}_2(-x) + 2\ln(x)\ln(1+x)\right)
$$
$$
+\frac{1+x^2}{1+x}\left[36\zeta_3 - 24\zeta_2\ln(1+x) + 8\zeta_2\ln(x) - \frac{2}{3}\ln^3(x) + 40\mathrm{Li}_3(-x) - 4\ln^2(x)\ln(1+x) - 24\ln(x)\ln^2(1+x)\right.
$$
$$
\left. -24\ln(x)\mathrm{Li}_2(-x) - 48\ln(1+x)\mathrm{Li}_2(-x) - 8\ln(x)\mathrm{Li}_2(1-x) - 16\mathrm{S}_{1,2}(1-x) - 48\mathrm{S}_{1,2}(-x)\right]
$$
$$
\left. -\frac{16(x^4+12x^3+12x^2+8x+3)}{3(1+x)^3}\mathrm{Li}_2(1-x) + 4x\frac{1-x-5x^2+x^3}{(1+x)^3}\ln^2(x)\right\}
$$

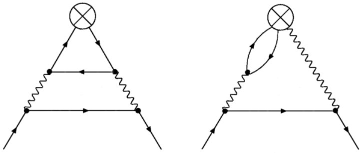


The result for $\hat{\Gamma}_{ee}^{(1),\mathrm{II}}$ is

$$\begin{aligned}\hat{\Gamma}_{ee}^{(1),\mathrm{II}} = & \frac{76}{27}x - \frac{572}{27} - \left(12x + \frac{4}{3} + \frac{8}{1-x}\right)\ln(x) + \frac{128}{9(1-x)^2} + \frac{80}{27(1-x)} - \frac{64}{9(1-x)^3} \\ & - \frac{32}{9}\left(\frac{1}{(1-x)^2} - \frac{5}{(1-x)^3} + \frac{2}{(1-x)^4}\right)\ln(x) + \frac{16}{3}(1+x)\left(\ln(1-x) + \ln^2(1-x)\right) \\ & - \frac{2(1+x^2)}{3(1-x)}\ln^2(x) + \left(\frac{224}{27} - \frac{8}{3}\zeta_2\right)\mathcal{D}_0(x) + \frac{4}{3}(1+x)\zeta_2 - \frac{32}{3}\left(\mathcal{D}_1(x) + \mathcal{D}_2(x)\right) \\ & + \left(\frac{8}{3}\zeta_3 + 10\zeta_2 - \frac{1411}{162}\right)\delta(1-x)\end{aligned}$$

Result: Process III

The result for $\hat{\Gamma}_{ee}^{(1),\mathrm{III}}$ is



$$
\begin{aligned}
\hat{\Gamma}_{ee}^{(1),\mathrm{III}} \quad = \quad & \frac{2}{x}(1-x)(4x^2+13x+4)\zeta_2 + \frac{1}{3x}(8x^3+135x^2+75x+32)\ln^2(x) \\
& + \left[\frac{304}{9x} - \frac{80}{9}x^2 - \frac{32}{3}x + 108 - \frac{32}{1+x} - \frac{64(1+2x)}{3(1+x)^3}\right]\ln(x) - \frac{224}{27}x^2 \\
& + 16\frac{1-x}{3x}(x^2+4x+1)\left[2\ln(x)\ln(1+x) - \mathrm{Li}_2(1-x) + 2\mathrm{Li}_2(-x)\right] \\
& + (1+x)\Bigg[4\zeta_2\ln(x) + \frac{14}{3}\ln^3(x) - 32\ln(x)\mathrm{Li}_2(-x) - 16\ln(x)\mathrm{Li}_2(x) + 64\mathrm{Li}_3(-x) \\
& + 32\mathrm{Li}_3(x) + 16\zeta_3\Bigg] - \frac{182}{3}x + 50 - \frac{32}{1+x} + \frac{800}{27x} + \frac{64}{3(1+x)^2}
\end{aligned}
$$

The first moment vanishes for all three contributions $\hat{\Gamma}_{ee}^{(1),\mathrm{I}}$, $\hat{\Gamma}_{ee}^{(1),\mathrm{II}}$ and $\hat{\Gamma}_{ee}^{(1),\mathrm{II}}$.

→ Fermion number conservation is satisfied.

The 2–loop corrections to the process $e^+e^- \to Z^0$ can be organized in the following form :

$$\frac{d\sigma_{e^+e^-}}{ds'} = \frac{1}{s}\sigma^{(0)}(s)\left\{1 + a_0\left[T_{11}\hat{\mathbf{L}} + T_{10}\right] + a_0^2\left[T_{22}\hat{\mathbf{L}}^2 + T_{21}\hat{\mathbf{L}} + T_{20}\right]\right\}$$

- Universal Corrections : $T_{ii}(z)$ $\Longrightarrow$ depend on LO splitting functions and β_0

$$T_{11} = 8\mathcal{D}_0(z) - 4(1+z) + 6\delta(1-z) = 4\left[\frac{1+z^2}{1-z}\right]_+ ,$$

$$\begin{aligned} T_{22} = & \left\{64\mathcal{D}_1(z) + 48D_0(z) + (18 - 32\zeta_2)\delta(1-z)\right. \\ & \left. -32\frac{\ln(z)}{1-z} - 32(1+z)\ln(1-z) + 24(1+z)\ln(z) - 8(5+z)\right\}_{\mathrm{I}} \\ & +\frac{2}{3}\left\{8\mathcal{D}_0(z) - 4(1+z) + 6\delta(1-z)\right\}_{\mathrm{II}} \\ & +16\left\{\frac{1}{2}(1-z)\ln(z) + \frac{1}{4}(1-z) + \frac{1}{3}\frac{1}{3z}(1-z^3)\right\}_{\mathrm{III}} . \end{aligned}$$

The Cross Section at $O(\alpha)$ and the Logarithmic 2-Loop Contributions

- $O(\alpha)$ Term : $T_{10}(z)$ $\Longrightarrow$ depend on LO OME + LO DY

$$T_{10} = -4\left[\frac{1+z^2}{1-z}\right]_+ + 2(4\zeta_2 - 1)\delta(1-z)$$

$$T_{11}\hat{\mathbf{L}} + T_{10} = P_{ee}^{(0)}(z)\left[\hat{\mathbf{L}} - 1\right] + 2(4\zeta_2 - 1)\delta(1-z)\ .$$

Complete 1-Loop Result.

- $O(\alpha^2\hat{\mathbf{L}})$ Terms : $T_{21}(z)$ $\Longrightarrow$ depend on LO,NLO splitting fcts., LO OME + LO DY

Contributions to the three main processes I-III :

$$T_{21}^{\mathrm{I}} = 16\left\{-8\mathcal{D}_1(z) - (7-4\zeta_2)\mathcal{D}_0(z) + \left(-\frac{45}{16} + \frac{11}{2}\zeta_2 + 3\zeta_3\right)\delta(1-z)\right.$$
$$+\left(\frac{1+z^2}{1-z}\right)\left[\ln(z)\ln(1-z) - \ln^2(z) + \frac{11}{4}\ln(z)\right]$$
$$\left.+(1+z)\left[4\ln(1-z) + \frac{1}{4}\ln^2(z) - \frac{7}{4}\ln(z) - 2\zeta_2\right] - \ln(z) + 3 + 4z\right\}$$

$$
\begin{aligned}
T_{21}^{\mathrm{II}} &= 16\Biggl\{\frac{4}{3}\mathcal{D}_1(z) - \frac{10}{9}\mathcal{D}_0(z) - \frac{17}{12}\delta(1-z) \\
&\quad -\frac{2}{3}\frac{\ln(z)}{1-z} - \frac{1}{3}(1+z)\left[2\ln(1-z) - \ln(z)\right] - \frac{1}{9} + \frac{11}{9}z\Biggr\} \\
T_{21}^{\mathrm{III}} &= 16\Biggl\{(1+z)\left[2\mathrm{Li}_2(1-z) - \ln^2(z) + 2\ln(z)\ln(1-z)\right] \\
&\quad + \left(\frac{4}{3}\frac{1}{z} + 1 - z - \frac{4}{3}z^2\right)\ln(1-z) - \left(\frac{2}{3}\frac{1}{z} + 1 - \frac{1}{2}z - \frac{4}{3}z^2\right)\ln(z) \\
&\quad -\frac{8}{9}\frac{1}{z} - \frac{8}{3} + \frac{8}{3}z + \frac{8}{9}z^2\Biggr\}
\end{aligned}
$$

Up to this point, we find agreement with Berends et al. (1988).

The Full Calculation

- To clarify the disagreement, only one rigorous method exists:
- **The recalculation of the process without any approximation.**
- Thus one also obtains a numerical etalon to which the subsequent expansion in m^2/s can be thoroughly compared.
- There is a formal problem to solve too:
  Does the massive Drell-Yan process factorize ?

$O(\alpha^2)$ process I

2 photon emission:

- $T_2^{\mathrm{S}_2}$: both emitted photons are soft $\surd$
- $T_2^{\mathrm{V}_2}$: both photons are virtual $\surd$
- $T_2^{\mathrm{S}_1\mathrm{V}_1}$: one photon is soft, one is virtual $\surd$
- $T_2^{\mathrm{S}_1\mathrm{H}_1}$: one photon is soft, one is hard $\surd$
- $T_2^{\mathrm{V}_1\mathrm{H}_1}$: one photon is virtual, one is hard (coming soon)
- $T_2^{\mathrm{H}_2}$: both emitted photons are hard, (coming soon)

All contributions which have been calculated for this process so far agree with
Berends, Burgers, van Neerven (Nucl. Phys. B297 (1988))

Here and in the following we only report the vector-case.
There are differences in the axial-vector case (not clear from Berends et al.)
Since we can work in 4-dimensions (only Abelian couplings) we can treat γ_5 without a further finite renormalization.

$O(\alpha^2)$: New Letters

$$
\begin{aligned}
f_{d_1} &= \frac{1}{\sqrt{(1-t)(16r^2-8r(1+z)t+(1-z)^2t^2)}}\\
f_{d_2} &= \frac{t}{\sqrt{(1-t)(16r^2-8r(1+z)t+(1-z)^2t^2)}}\\
f_{d_3} &= \frac{1}{t\sqrt{(1-t)(16r^2-8r(1+z)t+(1-z)^2t^2)}}\\
f_{d_4} &= \frac{1}{(16r^2+(4z-8r(1+z))t+(1-z)^2t^2)\sqrt{(1-t)(16r^2-8r(1+z)t+(1-z)^2t^2)}}\\
f_{d_5} &= \frac{t}{(16r^2+(4z-8r(1+z))t+(1-z)^2t^2)\sqrt{(1-t)(16r^2-8r(1+z)t+(1-z)^2t^2)}}\\
f_{d_6} &= \frac{1}{(16r^2+(4z-8r(1+z))t+(1-z)^2t^2)\sqrt{16r^2-8r(1+z)t+(1-z)^2t^2}}\\
f_{d_7} &= \frac{t}{(16r^2+(4z-8r(1+z))t+(1-z)^2t^2)\sqrt{16r^2-8r(1+z)t+(1-z)^2t^2}}\\
f_{d_8} &= \frac{1-z}{(4r-(1-z)t)\sqrt{1-t}}\\
f_{w_3} &= \frac{1}{t\sqrt{1-t}}\,.
\end{aligned}
$$

+ HPL letters

Iterative integrals:

$$
\mathrm{H}^*_{a,\vec{b}}(t) = \int_t^1 dy f_a(y)\mathrm{H}^*_{\vec{b}}(y); \qquad r = \rho = m^2/s, \quad z = s'/s
$$

also used:

$$
\mathrm{H}_{a,\vec{b}}(t) = \int_0^t dy f_a(y)\mathrm{H}_{\vec{b}}(y)
$$

$O(\alpha^2)$ Process II (full form)

$$
\begin{aligned}
\frac{d\sigma^{(2),\mathrm{II}}(z,\rho)}{ds'} = \frac{\sigma^{(0)}(s')}{s}\, a^2 \Bigg\{ & \frac{64}{3} z(1-z)(1+z-4\rho)\mathrm{H}^*_{w_3,d_7} + \frac{256}{3} z\rho(1+z-4\rho)\mathrm{H}^*_{w_3,d_6} \\
& + \frac{128z(1-4\rho^2)(1-z+2\rho)(1-z-4\rho)}{3(1-z)^2}\mathrm{H}^*_{d_8,d_7} \\
& + \frac{512z\rho(1-4\rho^2)(1-z+2\rho)(1-z-4\rho)}{3(1-z)^3}\mathrm{H}^*_{d_8,d_6} \\
& + \frac{16}{9(1-z)^2}\Big[(1+z)^2\big(4-9z+4z^2\big) + 2\big(9-16z+13z^2-2z^3\big)\rho + 32\rho^2\Big]\mathrm{H}^*_{d_2} \\
& + \frac{512z\rho}{9(1-z)^4}\Big[3(1-z)^4 z - (1-z)^3\big(4+z^2\big)\rho - 2\big(9-29z+38z^2-17z^3+3z^4\big)\rho^2 \\
& - 4(2-z)\big(3+6z-5z^2\big)\rho^3 + 16\big(7-8z+9z^2\big)\rho^4 + 128(3-z)\rho^5\Big]\mathrm{H}^*_{d_4} \\
& - \frac{16}{9(1-z)^4}\Big[3-34z+129z^2-212z^3+129z^4-34z^5+3z^6+8\big(2-16z+9z^2 \\
& + 4z^3-5z^4+2z^5\big)\rho + 16z\big(12-13z+18z^2-z^3\big)\rho^2 + 32\big(1+22z-7z^2\big)\rho^3\Big]\mathrm{H}^*_{d_1} \\
& - \frac{128z}{9(1-z)^4}\Big[1+7z-47z^2+86z^3-47z^4+7z^5+z^6-2\big(7-55z+54z^2 \\
& + 16z^3-17z^4+3z^5\big)\rho - 4\big(39-16z+16z^2+4z^3+5z^4\big)\rho^2 \\
& + 16\big(8-23z+22z^2+9z^3\big)\rho^3 + 128\big(7+2z-z^2\big)\rho^4\Big]\mathrm{H}^*_{d_5} - \frac{64}{3}(2z+(1-z)\rho)\mathrm{H}^*_{d_3} \\
& + \left[\frac{16}{3\sqrt{1-4\rho}}(1+z-4\rho)\mathrm{H}^*_{w_3} + \frac{32(1-4\rho^2)(1-z+2\rho)(1-z-4\rho)}{3(1-z)^3\sqrt{1-4\rho}}\mathrm{H}^*_{d_8}\right] \\
& \times \ln\left(\frac{1-z-4\rho-\sqrt{1-4\rho}\sqrt{(1-z)^2-8(1+z)\rho+16\rho^2}}{1-z-4\rho+\sqrt{1-4\rho}\sqrt{(1-z)^2-8(1+z)\rho+16\rho^2}}\right)\Bigg\}
\end{aligned}
$$

DESY.

$O(\alpha^2)$ Process II ($s \gg m^2$)

$$\frac{d\sigma^{(2),\mathrm{II}}(z,\rho)}{ds'} = \frac{\sigma^{(0)}(s')}{s}\left(\frac{\alpha}{4\pi}\right)^2 \Biggl\{\frac{8}{3}\frac{1+z^2}{1-z}L^2 - \Biggl[\frac{16}{9}\frac{11-12z+11z^2}{1-z} + \frac{16}{3}\frac{1+z^2}{1-z}\mathrm{H}_0$$
$$+\frac{32}{3}\frac{1+z^2}{1-z}\mathrm{H}_1\Biggr]L + \frac{32}{9(1-z)^3}\left(7-13z+8z^2-13z^3+7z^4\right) - \frac{16z}{9(1-z)^4}\bigl(3-36z$$
$$+94z^2-72z^3+19z^4\bigr)\mathrm{H}_0 - \frac{8z^2}{3(1-z)}\mathrm{H}_0^2 + \left(\frac{32}{9}\frac{11-12z+11z^2}{1-z} + \frac{16}{3}\frac{2+z^2}{1-z}\mathrm{H}_0\right)\mathrm{H}_1$$
$$+\frac{32}{3}\frac{1+z^2}{1-z}\mathrm{H}_1^2 + \frac{16z^2}{3(1-z)}\mathrm{H}_{0,1} - \frac{16\left(2+3z^2\right)}{3(1-x)}\zeta_2\Biggr\} + \mathcal{O}\left(\frac{m^2}{s}L^2\right).$$

equal mass case: differs from Berends et al. and from Kniehl et al.
$\mu^+\mu^-$-production: agrees with Berends et al. and Kniehl et al.: Here the electron mass can be set to zero and one is in the massless-quark situation, for which this calculation has originally been perfromed.

Result: Process II

- as an example we find the difference term to BBN for process II:

$$\delta_{II} = \frac{8}{3}\int_0^1 \frac{dy}{y}\sqrt{1-y}(2+y)\left[\frac{(1-z)(1-(4-z)z)y}{4z+(1-z^2)y} - \frac{1+z^2}{1-z}\ln\left(1+\frac{(1-z)^2y}{4z}\right)\right]$$

$$= -\frac{128}{9}\left[3+\frac{1}{(1-z)^3}-\frac{2}{(1-z)^2}-2z\right] - 16\left[1+\frac{5z}{3}+\frac{8}{9}\frac{1}{(1-z)^4}-\frac{20}{9}\frac{1}{(1-z)^3}\right.$$

$$\left.+\frac{4}{9}\frac{1}{(1-z)^2}\right]\ln(z)+\frac{8}{3}\frac{1+z^2}{1-z}\left[\frac{10}{9}-\frac{14}{3}\ln(z)-\ln^2(z)\right],$$

- in this case the difference can be attributed to the neglection of initial state electron masses
- in the pure-singlet process a calculation done for massless partons was reused
  Schellekens, van Neerven (Phys.Rev. D21 (1980))

$\Rightarrow$ our results agree with the ones obtained using the RGE technique

$O(\alpha^2)$ Process III ($s \gg m^2$)

$$
\begin{aligned}
\frac{d\sigma^{(2),\mathrm{III}}_{\mathrm{dir}}}{ds'} = \frac{\sigma^{(0)}(s')}{s}\left(\frac{\alpha}{4\pi}\right)^2 \Biggl\{ \Biggl[& \frac{4(1-z)\bigl(4+7z+4z^2\bigr)}{3z} + 8(1+z)\mathrm{H}_0\Biggr] L^2 - \Biggl[\frac{128(1-z)\bigl(1+4z+z^2\bigr)}{9z} \\
& + \frac{8\bigl(4+6z-3z^2-8z^3\bigr)}{3z}\mathrm{H}_0 + 16(1+z)\mathrm{H}_0^2 + \frac{16(1-z)\bigl(4+7z+4z^2\bigr)}{3z}\mathrm{H}_1 + 32(1+z)\mathrm{H}_{0,1} \\
& - 32(1+z)\zeta_2\Biggr] L - \frac{2(1-z)}{27z(1+z)^2}\bigl(80-2463z-5041z^2-2949z^3-163z^4\bigr) \\
& - \Biggl[\frac{4}{9z(1+z)^3}\bigl(40+3z-345z^2-445z^3+213z^4+318z^5+64z^6\bigr) \\
& - \frac{64(1-z)\bigl(1+4z+z^2\bigr)}{3z}\mathrm{H}_{-1}\Biggr]\mathrm{H}_0 - \frac{4\bigl(12+21z-27z^2-4z^3\bigr)}{3z}\mathrm{H}_0^2 - 8(1+z)\mathrm{H}_0^3 \\
& + \Biggl[\frac{256(1-z)\bigl(1+4z+z^2\bigr)}{9z} + \frac{8(1-z)\bigl(4+7z+4z^2\bigr)}{3z}\mathrm{H}_0\Biggr]\mathrm{H}_1 + \frac{16(1-z)\bigl(4+7z+4z^2\bigr)}{3z}\mathrm{H}_1^2 \\
& + \Biggl[\frac{8\bigl(4+9z-3z^2-12z^3\bigr)}{3z} + 16(1+z)\mathrm{H}_0\Biggr]\mathrm{H}_{0,1} - \Biggl[\frac{64(1-z)\bigl(1+4z+z^2\bigr)}{3z} \\
& - 64(1+z)\mathrm{H}_0\Biggr]\mathrm{H}_{0,-1} + 32(1+z)\mathrm{H}_{0,0,1} - 128(1+z)\mathrm{H}_{0,0,-1} + 64(1+z)\mathrm{H}_{0,1,1} \\
& - \Biggl[\frac{8\bigl(8+3z+3z^2-16z^3\bigr)}{3z} + 48(1+z)\mathrm{H}_0\Biggr]\zeta_2\Biggr\} + \mathcal{O}\left(\frac{m^2}{s}L^2\right).
\end{aligned}
$$

differs from Berends et al.

$O(\alpha^2)$ Process III ($s \gg m^2$)

$$\begin{aligned}
\frac{d\sigma^{(2),\mathrm{III}}_{\mathrm{interf}}}{ds'} = {} & \frac{\sigma^{(0)}(s')}{s}\left(\frac{\alpha}{4\pi}\right)^2 \Bigg\{ -160(1-z) - \left[16(5+4z) - 80(1+z)\mathrm{H}_{-1} + \frac{48\left(2+2z+z^2\right)}{z}\mathrm{H}_{-1}^2\right]\mathrm{H}_0 \\
& - \left[52z - \frac{40\left(2+2z+z^2\right)}{z}\mathrm{H}_{-1}\right]\mathrm{H}_0^2 - \frac{16}{3}z\mathrm{H}_0^3 + \left[8(5-4z)\mathrm{H}_0 - \frac{8\left(4-6z+3z^2\right)}{z}\mathrm{H}_0^2\right]\mathrm{H}_1 \\
& - \frac{4\left(4-6z+3z^2\right)}{z}\mathrm{H}_0\mathrm{H}_1^2 - \left[8(5-4z) - \frac{8\left(8-2z+5z^2\right)}{z}\mathrm{H}_0 - \frac{8\left(4-6z+3z^2\right)}{z}\mathrm{H}_1\right]\mathrm{H}_{0,1} \\
& - \left[80(1+z) + \frac{32\left(5+2z^2\right)}{z}\mathrm{H}_0 - \frac{96\left(2+2z+z^2\right)}{z}\mathrm{H}_{-1}\right]\mathrm{H}_{0,-1} - \frac{32\left(2+2z+z^2\right)}{z}\mathrm{H}_{0,0,1} \\
& + \frac{16\left(10-10z+3z^2\right)}{z}\mathrm{H}_{0,0,-1} - \frac{8\left(4-6z+3z^2\right)}{z}\mathrm{H}_{0,1,1} - \frac{96\left(2+2z+z^2\right)}{z}\mathrm{H}_{0,-1,-1} \\
& + \left[8(10+z) + 160\mathrm{H}_0 - \frac{8\left(4-6z+3z^2\right)}{z}\mathrm{H}_1 - \frac{48\left(2+2z+z^2\right)}{z}\mathrm{H}_{-1}\right]\zeta_2 + 32(5+z)\zeta_3 \Bigg\} \\
& + \mathcal{O}\left(\frac{m^2}{s}\right)
\end{aligned}$$

also calculated by A.N. Schellekens (Thesis, Nijmegen, 1981)

$O(\alpha^2)$ Process IV ($s \gg m^2$)

$$
\begin{aligned}
\frac{d\sigma^{(2),\mathrm{IV}}}{ds'} = \frac{\sigma^{(0)}(s')}{s}\left(\frac{\alpha}{4\pi}\right)^2 \Bigg\{ & -\Bigg[8(8-7z) + \frac{8\left(5-2z^2\right)}{1-z}\mathrm{H}_0 + \frac{8\left(1+z^2\right)}{1-z}\left(\mathrm{H}_0^2 + 2\mathrm{H}_0\mathrm{H}_1\right.\\
& \left. - 2\mathrm{H}_{0,1} + 2\zeta_2\right)\Bigg] L + \frac{8\left(27-42z+23z^2\right)}{1-z} + \Bigg[\frac{8}{(1-z)^2(1+z)}\left(3+10z-11z^2+22z^3-8z^4\right)\\
& + \frac{64(1+z)}{1-z}\mathrm{H}_{-1}\Bigg]\mathrm{H}_0 - \frac{8(1+z)^2}{1-z}\mathrm{H}_0^2 - \frac{8\left(1+2z^2\right)}{3(1-z)}\mathrm{H}_0^3 + \Bigg[16(8-7z) - \frac{8\left(3-2z-2z^2\right)}{1-z}\mathrm{H}_0\\
& + \frac{16\left(2+z^2\right)}{1-z}\mathrm{H}_0^2\Bigg]\mathrm{H}_1 + \frac{16}{1-z}\mathrm{H}_0\mathrm{H}_1^2 + \Bigg[\frac{8\left(13-2z-6z^2\right)}{1-z} - \frac{16\left(5+4z^2\right)}{1-z}\mathrm{H}_0 + \frac{32z^2}{1-z}\mathrm{H}_1\Bigg]\mathrm{H}_{0,1}\\
& - \Bigg[\frac{64(1+z)}{1-z} - \frac{32\left(1+z^2\right)}{1-z}\mathrm{H}_0\Bigg]\mathrm{H}_{0,-1} + \frac{128\left(1+z^2\right)}{1-z}\mathrm{H}_{0,0,1} - \frac{64\left(1+z^2\right)}{1-z}\mathrm{H}_{0,0,-1}\\
& - \frac{32\left(1+2z^2\right)}{1-z}\mathrm{H}_{0,1,1} - \Bigg[\frac{24\left(3-2z-2z^2\right)}{1-z} + \frac{16\left(2+3z^2\right)}{1-z}\mathrm{H}_0 + \frac{32z^2}{1-z}\mathrm{H}_1\Bigg]\zeta_2\\
& - \frac{16\left(3+z^2\right)}{1-z}\zeta_3\Bigg\} + \mathcal{O}\left(\frac{m^2}{s}L\right).
\end{aligned}
$$

equal mass case: differs from Berends et al.

The higher power terms of this contributions are exactly those found in the I+IV term of the BDN paper form 2011.

$O(\alpha^2)$ Drell-Yan Contributions

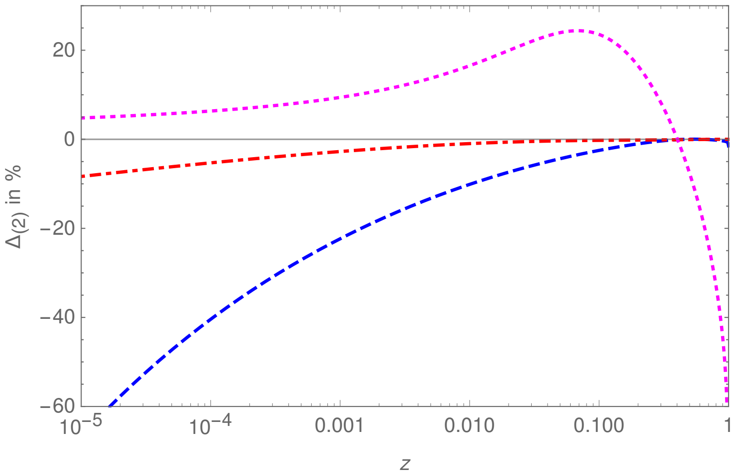
$$\frac{d\sigma^{(2),\mathrm{BB}}}{ds'} = \frac{\sigma^{(0)}(s')}{s}\left(\frac{\alpha}{4\pi}\right)^2 \Bigg\{\frac{40}{3}(1-z^2) + \bigg[\frac{8}{3}\big(3+4z+3z^2\big) - \frac{32}{3}(1+z)^2\mathrm{H}_{-1}\bigg]\mathrm{H}_0 + \frac{8}{3}(1+z)^2\mathrm{H}_0^2 + \frac{32}{3}(1+z)^2\mathrm{H}_{0,-1} - \frac{16}{3}(1+z)^2\zeta_2\Bigg\} + \mathcal{O}\left(\frac{m^2}{s}\right)$$

$$\frac{d\sigma^{(2),\mathrm{BC}}}{ds'} = \frac{\sigma^{(0)}(s')}{s}\left(\frac{\alpha}{4\pi}\right)^2 \Bigg\{2(1-z)(27+13z) + \bigg[4(9+11z) + 24(1+z)^2\mathrm{H}_{-1} - 24(1+z)^2\mathrm{H}_{-1}^2\bigg]\mathrm{H}_0 + \bigg[2\big(6-8z-15z^2\big) + 20(1+z)^2\mathrm{H}_{-1}\bigg]\mathrm{H}_0^2 + \frac{4}{3}\big(1+4z+z^2\big)\mathrm{H}_0^3 + 36(1-z^2)\mathrm{H}_0\mathrm{H}_1 - \bigg[36(1-z^2) - 16\big(1+3z+z^2\big)\mathrm{H}_0\bigg]\mathrm{H}_{0,1} - \bigg[24(1+z)^2 + 24(1+z)^2\mathrm{H}_0 - 48(1+z)^2\mathrm{H}_{-1}\bigg]\mathrm{H}_{0,-1} - 32\big(1+3z+z^2\big)\mathrm{H}_{0,0,1} + 8(1+z)^2\mathrm{H}_{0,0,-1} - 48(1+z)^2\mathrm{H}_{0,-1,-1} + \bigg[24(2-z)(1+z) + 8\big(3+8z+3z^2\big)\mathrm{H}_0 - 24(1+z)^2\mathrm{H}_{-1}\bigg]\zeta_2 + 32\big(1+3z+z^2\big)\zeta_3\Bigg\} + \mathcal{O}\left(\frac{m^2}{s}\right)$$

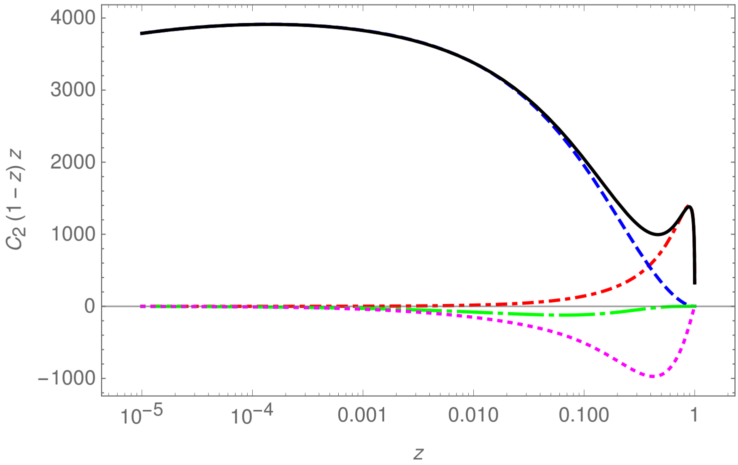
agreement with Hamberg, van Neerven, Matsuura (Nucl. Phys. B 359 (1991))

not contained in Berends, Burgers, van Neerven (Nucl. Phys. B297 (1988))

Recalculation – Numerical Illustration



- Relative deviation from BBN of process II (red), process III (blue) and process IV (magenta) contribution in %.



- Illustration of the Wilson coefficients for process II (red), process III (blue), process IV $\times 10$ (magenta) and remaining DY $\times 100$ (green) multiplied with the factor $z(1-z)$. The black line represents the whole contribution to initial state fermion-pair radiation.

Conclusions

- We calculated the $O(\alpha^2)$ massive operator matrix elements in QED, which contribute to the 2-loop initial state corrections for $e^+e^- \to Z^*/\gamma^*$ in the limit $m_f^2/s \to 0$ using the renormalization group method for the electron-contributions.
- We have obtained all logarithmic contributions $O((\alpha L)^2), O(\alpha^2 L), O(\alpha L)$ and the constant contributions $O(\alpha)$ correctly.
- The literal application of the $s \gg m_f^2$ expansion, as proposed by BBN seemed for nearly one decade not to deliver the result obtained by conventional integration. However, we found by a full calculation that the previous $O(\alpha^2)$ results are incorrect.
- On the other hand, we obtained our results for the 2-loop matrix elements by two independent methods, which agree on the results. Furthermore, the complete OMEs obey Fermion number conservation, and renormalize as expected. The 2-loop anomalous dimensions are correctly obtained.
- In case of massless external lines, massive OMEs can be calculated without any problem and the results agree in all cases investigated with that obtained in the limit $m^2/\mu^2 \to 0$. This now also applies for massive external states.

Conclusions

- There are contributions to the $O(\alpha^2)$ corrections with vanishing OME, which also appear in the massless Drell–Yan process. They have to be included (missing at BBN).
- Due to the axial-vector couplings of the Z-boson the corrections in the vector- and axial-vector case are not the same (as already known from the Drell-Yan process).
- The differences at $O(\alpha^2)$ are large, reaching the order of the logarithmic terms in part of the kinematic region.
- Very soon the calculation for the 2-photon case will be finished, allowing then to perform detailed phenomenological studies.
- The newly obtained corrections are of relevance at high luminosity e^+e^- facilities.