Precision Quantum Field Theory and Symbolic Integration: DESY & RISC

Johannes Blümlein, DESY | May 24, 2022

Outline

1. DESY-RISC collaboration since 2007

2. Results in Precision QFT
   - Heavy Quark Corrections to Deep-Inelastic Scattering
   - Massive 3-Loop Formfactors
   - Precision QED Corrections to $e^+e^- \to \gamma^*/Z^*$
   - 3–loop anomalous dimensions and massless Wilson coefficients

3. Further Plans: Physics

DESY-RISC collaboration since 2007

In June 2005 Bruno Buchberger gave a plenary talk at ACAT 2005 at DESY on the Theorema Project and first contacts with RISC were established.

I visited RISC firstly in September 2005 giving a talk on results on harmonic sums, structural relations and special numbers. Carsten Schneider and I met only later, since he was traveling.

DESY-RISC collaboration since 2007

Contract signing

2007

2012

2017

A survey on the activities

(during the period since 2017 (total))

- 53 publications (100)
- 31 journal publications
- 20 proceedings contributions
- ed. 2 books with topical reviews (3)
- More than 2800 citations
- Joined 3 EU networks together: LHCPhenonet, Higgstools, SAGEX (each for 4 years)
- Run a large computer cluster: 16 Tbyte RAM and 230 Tbyte fast disc $\Longrightarrow$ qftquad-cluster
- Various long-term internships at RISC and DESY, including full post-doc positions and longer research visits of PhD students from the other node
- Physics part of the research: precision predictions and analysis of collider Data from HERA (DESY), LHC (CERN), and preparing for EIC (Brookhaven) and FCC_ee (CERN)
- Research Topics:
    - Calculation of quantum field-theoretic quantities in collider phenomenology
    - Research in mathematics and algorithmics using computer algebra: talk by C. Schneider

A survey on the activities

Participating scientists over the years:

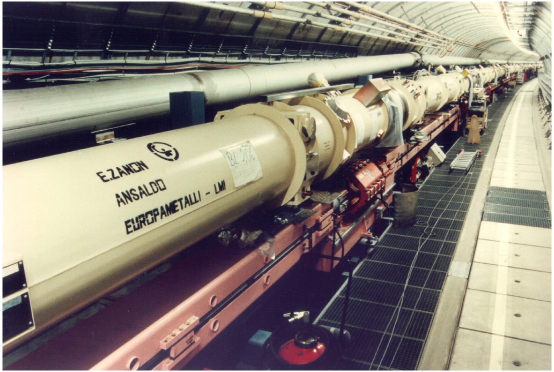
- DESY:
  J. Blümlein, A. Behring, I. Bierenbaum, A. De Freitas, A. Hasselhuhn, S. Klein, A. Maier, P. Marquard, N. Rana, T. Riemann, M. Saragnese, K. Schönwald, F. Wißbrock.
- RISC:
  C. Schneider, P. Paule, J. Ablinger, N. Faddeev, M. Kauers, C. Raab, S. Radu, M. Round, F. Stan.
- Other institutions involved:
  RWTH Aachen, KIT, U. Mainz, East Lansing, MI, FSU Tallahassee,FL, RICAM, CERN, IHES Bures sur Yvette, U. Jena.

Results in Precision QFT

- Heavy Quark Corrections to Deep-Inelastic Scattering
- Massive Formfactors
- Precision QED Corrections to $e^+e^- \rightarrow \gamma^*/Z^*$
- 3–loop anomalous dimensions and massless Wilson coefficients
- Some applications to classical Einstein gravity and black hole physics

Examples for Colliders

LHC

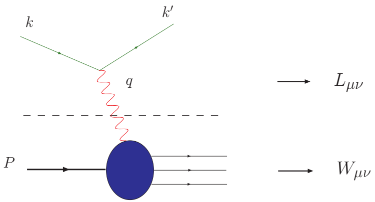


HERA ring

Heavy Quark Corrections to Deep-Inelastic Scattering

- The scaling violations of the heavy quark corrections are quite different from those of massless quarks.
- Work in the region $Q^2 \gg m_Q^2$, also to avoid higher twist corrections.
- Under these conditions the heavy flavor corrections are given by the massive operator matrix elements (OMEs) A_{ij} and the massless process-dependent Wilson coefficients.
- Analytic calculations are possible to the 3–loop level for single and two mass corrections.
- The corrections are needed in the unpolarized and the polarized case.
- The massive OMEs also form the transition matrix element in the Variable Flavor Number Scheme describing the behaviour of massive and massless partons in the high energy range at colliders.
- Measurement goals:
  - $\alpha_s(M_Z^2) = 0.1147 \pm 0.0008$
  - $m_c(m_c) = 1.252 \pm 0.018(exp) \begin{array}{c} +0.03 \\ -0.02 \end{array} (scale), \begin{array}{c} +0.00 \\ -0.07 \end{array} (thy)\mathrm{GeV}$ $(\overline{\mathrm{MS}}$-scheme)

Deep–inelastic scattering



- Kinematic invariants:

$$Q^2 = -q^2, \qquad x = \frac{Q^2}{2P.q}$$

- The cross section factorizes into leptonic and hadronic tensor:

$$\frac{d^2\sigma}{dQ^2 dx} \sim L_{\mu\nu} W^{\mu\nu}$$

- The hadronic tensor can be expressed through structure functions:

$$W_{\mu\nu} = \frac{1}{4\pi} \int d^4\xi \exp(iq\xi) \langle P, | \left[J_\mu^{\rm em}(\xi), J_\nu^{\rm em}(\xi) \right] | P \rangle$$

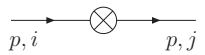
$$= \frac{1}{2x} \left(g_{\mu\nu} + \frac{q_\mu q_\nu}{Q^2} \right) F_L(x, Q^2) + \frac{2x}{Q^2} \left(P_\mu P_\nu + \frac{q_\mu P_\nu + q_\nu P_\mu}{2x} - \frac{Q^2}{4x^2} g_{\mu\nu} \right) F_2(x, Q^2)$$

$$+ i\epsilon_{\mu\nu\rho\sigma} \frac{q^\rho S^\sigma}{q \cdot P} g_1(x, Q^2) + i\epsilon_{\mu\nu\rho\sigma} \frac{q^\rho \left(q \cdot P S^\sigma - q \cdot S P^\sigma\right)}{(q \cdot P)^2} g_2(x, Q^2)$$

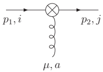
- F_L, F_2, g_1 and g_2 contain contributions from both, charm and bottom quarks.

Calculation of the 3-loop operator matrix elements

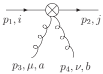
The OMEs are calculated using the QCD Feynman rules together with the following operator insertion Feynman rules:



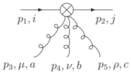
$$\delta^{ij} \not\Delta \gamma_\pm (\Delta \cdot p)^{N-1}\,, \quad N \geq 1$$



$$g t^a_{ji} \Delta^\mu \not\Delta \gamma_\pm \sum_{j=0}^{N-2} (\Delta \cdot p_1)^j (\Delta \cdot p_2)^{N-j-2}\,, \quad N \geq 2$$



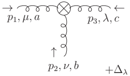
$$g^2 \Delta^\mu \Delta^\nu \not\Delta \gamma_\pm \sum_{j=0}^{N-3} \sum_{l=j+1}^{N-2} (\Delta p_2)^j (\Delta p_1)^{N-l-2} \Big[(t^a t^b)_{ji} (\Delta p_1 + \Delta p_4)^{l-j-1} + (t^b t^a)_{ji} (\Delta p_1 + \Delta p_3)^{l-j-1} \Big]\,, \quad N \geq 3$$



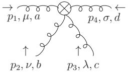
$$\begin{aligned}
& g^3 \Delta_\mu \Delta_\nu \Delta_\rho \not\Delta \gamma_\pm \sum_{j=0}^{N-4} \sum_{l=j+1}^{N-3} \sum_{m=l+1}^{N-2} (\Delta.p_2)^j (\Delta.p_1)^{N-m-2} \\
& \Big[\ (t^a t^b t^c)_{ji} (\Delta.p_4 + \Delta.p_5 + \Delta.p_1)^{l-j-1} (\Delta.p_5 + \Delta.p_1)^{m-l-1} \\
& + (t^a t^c t^b)_{ji} (\Delta.p_4 + \Delta.p_5 + \Delta.p_1)^{l-j-1} (\Delta.p_4 + \Delta.p_1)^{m-l-1} \\
& + (t^b t^a t^c)_{ji} (\Delta.p_3 + \Delta.p_5 + \Delta.p_1)^{l-j-1} (\Delta.p_5 + \Delta.p_1)^{m-l-1} \\
& + (t^b t^c t^a)_{ji} (\Delta.p_3 + \Delta.p_5 + \Delta.p_1)^{l-j-1} (\Delta.p_3 + \Delta.p_1)^{m-l-1} \\
& + (t^c t^a t^b)_{ji} (\Delta.p_3 + \Delta.p_4 + \Delta.p_1)^{l-j-1} (\Delta.p_4 + \Delta.p_1)^{m-l-1} \\
& + (t^c t^b t^a)_{ji} (\Delta.p_3 + \Delta.p_4 + \Delta.p_1)^{l-j-1} (\Delta.p_3 + \Delta.p_1)^{m-l-1} \Big]\,, \\
& N \geq 4
\end{aligned}$$

$$\gamma_+ = 1\,, \quad \gamma_- = \gamma_5\,.$$

$$\frac{1+(-1)^N}{2} \delta^{ab} (\Delta \cdot p)^{N-2} \Big[g_{\mu\nu} (\Delta \cdot p)^2 - (\Delta_\mu p_\nu + \Delta_\nu p_\mu) \Delta \cdot p + p^2 \Delta_\mu \Delta_\nu \Big]\,, \quad N \geq 2$$



$$\begin{aligned}
& -ig \frac{1+(-1)^N}{2} f^{abc} \Bigg(\\
& \Big[(\Delta_\nu g_{\lambda\mu} - \Delta_\lambda g_{\mu\nu}) \Delta \cdot p_1 + \Delta_\mu (p_{1,\nu} \Delta_\lambda - p_{1,\lambda} \Delta_\nu) \Big] (\Delta \cdot p_1)^{N-2} \\
& + \Delta_\lambda \Big[\Delta \cdot p_1 p_{2,\mu} \Delta_\nu + \Delta \cdot p_2 p_{1,\nu} \Delta_\mu - \Delta \cdot p_1 \Delta \cdot p_2 g_{\mu\nu} - p_1 \cdot p_2 \Delta_\mu \Delta_\nu \Big] \\
& \times \sum_{j=0}^{N-3} (-\Delta \cdot p_1)^j (\Delta \cdot p_2)^{N-3-j} \\
& + \left\{ \begin{smallmatrix} p_1 \to p_2 \to p_3 \to p_1 \\ \mu \to \nu \to \lambda \to \mu \end{smallmatrix} \right\} + \left\{ \begin{smallmatrix} p_1 \to p_3 \to p_2 \to p_1 \\ \mu \to \lambda \to \nu \to \mu \end{smallmatrix} \right\} \Bigg)\,, \quad N \geq 2
\end{aligned}$$



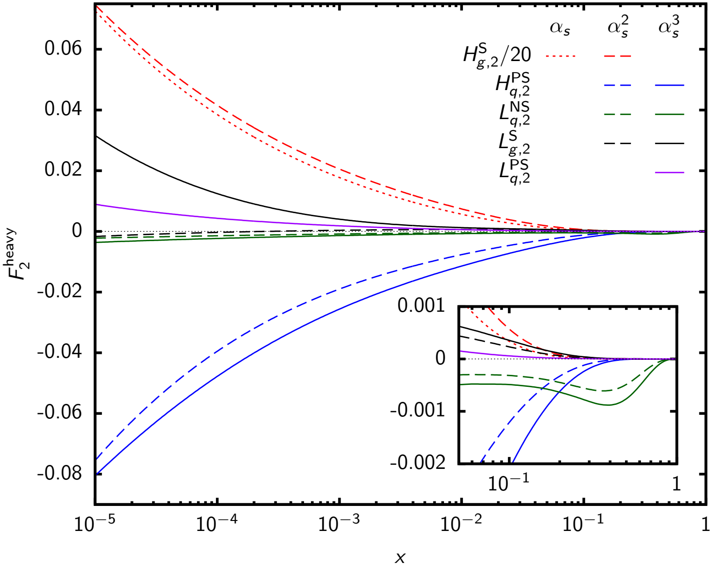
$$\begin{aligned}
& g^2 \frac{1+(-1)^N}{2} \Big(f^{abe} f^{cde} O_{\mu\nu\lambda\sigma}(p_1, p_2, p_3, p_4) \\
& + f^{ace} f^{bde} O_{\mu\lambda\nu\sigma}(p_1, p_3, p_2, p_4) + f^{ade} f^{bce} O_{\mu\sigma\nu\lambda}(p_1, p_4, p_2, p_3) \Big)\,, \\
& O_{\mu\nu\lambda\sigma}(p_1, p_2, p_3, p_4) = \Delta_\nu \Delta_\lambda \Big\{ -g_{\mu\sigma} (\Delta \cdot p_3 + \Delta \cdot p_4)^{N-2} \\
& + [p_{4,\mu} \Delta_\sigma - \Delta \cdot p_4 g_{\mu\sigma}] \sum_{i=0}^{N-3} (\Delta \cdot p_3 + \Delta \cdot p_4)^i (\Delta \cdot p_4)^{N-3-i} \\
& - [p_{1,\sigma} \Delta_\mu - \Delta \cdot p_1 g_{\mu\sigma}] \sum_{i=0}^{N-3} (-\Delta \cdot p_1)^i (\Delta \cdot p_3 + \Delta \cdot p_4)^{N-3-i} \\
& + [\Delta \cdot p_1 \Delta \cdot p_4 g_{\mu\sigma} + p_1 \cdot p_4 \Delta_\mu \Delta_\sigma - \Delta \cdot p_4 p_{1,\sigma} \Delta_\mu - \Delta \cdot p_1 p_{4,\mu} \Delta_\sigma] \\
& \times \sum_{i=0}^{N-4} \sum_{j=0}^{i} (-\Delta \cdot p_1)^{N-4-i} (\Delta \cdot p_3 + \Delta \cdot p_4)^{i-j} (\Delta \cdot p_4)^j \Big\} \\
& - \left\{ \begin{smallmatrix} p_1 \leftrightarrow p_2 \\ \mu \leftrightarrow \nu \end{smallmatrix} \right\} - \left\{ \begin{smallmatrix} p_3 \leftrightarrow p_4 \\ \lambda \leftrightarrow \sigma \end{smallmatrix} \right\} + \left\{ \begin{smallmatrix} p_1 \leftrightarrow p_2,\ p_3 \leftrightarrow p_4 \\ \mu \leftrightarrow \nu,\ \lambda \leftrightarrow \sigma \end{smallmatrix} \right\}\,, \quad N \geq 2
\end{aligned}$$

The Wilson Coefficients at large Q^2

$$
\begin{aligned}
L_{q,(2,L)}^{\rm NS}(N_F+1) &= a_s^2\left[A_{qq,Q}^{(2),\rm NS}(N_F+1)\delta_2 + \hat{C}_{q,(2,L)}^{(2),\rm NS}(N_F)\right] + a_s^3\left[A_{qq,Q}^{(3),\rm NS}(N_F+1)\delta_2 + A_{qq,Q}^{(2),\rm NS}(N_F+1)C_{q,(2,L)}^{(1),\rm NS}(N_F+1) + \hat{C}_{q,(2,L)}^{(3),\rm NS}(N_F)\right] \\
L_{q,(2,L)}^{\rm PS}(N_F+1) &= a_s^3\left[A_{qq,Q}^{(3),\rm PS}(N_F+1)\delta_2 + N_F A_{gq,Q}^{(2),\rm NS}(N_F)\tilde{C}_{g,(2,L)}^{(1),\rm NS}(N_F+1) + N_F \hat{\tilde{C}}_{q,(2,L)}^{(3),\rm PS}(N_F)\right] \\
L_{g,(2,L)}^{\rm S}(N_F+1) &= a_s^2 A_{gg,Q}^{(1)}(N_F+1)N_F\tilde{C}_{g,(2,L)}^{(2)}(N_F+1) + a_s^3\left[A_{qg,Q}^{(3)}(N_F+1)\delta_2 + A_{gg,Q}^{(1)}(N_F+1)N_F\tilde{C}_{g,(2,L)}^{(2)}(N_F+1)\right. \\
&\quad \left. + A_{gg,Q}^{(2)}(N_F+1)N_F\tilde{C}_{g,(2,L)}^{(1)}(N_F+1) + A_{Qg}^{(1)}(N_F+1)N_F\tilde{C}_{q,(2,L)}^{(2),\rm PS}(N_F+1) + N_F\hat{\tilde{C}}_{g,(2,L)}^{(3)}(N_F)\right] \\
H_{q,(2,L)}^{\rm PS}(N_F+1) &= a_s^2\left[A_{Qq}^{(2),\rm PS}(N_F+1)\delta_2 + \tilde{C}_{q,(2,L)}^{(2),\rm PS}(N_F+1)\right] \\
&\quad + a_s^3\left[A_{Qq}^{(3),\rm PS}(N_F+1)\delta_2 + A_{gq,Q}^{(2)}(N_F+1)\tilde{C}_{g,(1,L)}^{(2)}(N_F+1) + A_{Qq}^{(2),\rm PS}(N_F+1)\tilde{C}_{q,(2,L)}^{(1),\rm NS}(N_F+1) + \tilde{C}_{q,(2,L)}^{(3),\rm PS}(N_F+1)\right] \\
H_{g,(2,L)}^{\rm S}(N_F+1) &= a_s\left[A_{Qg}^{(1)}(N_F+1)\delta_2 + \tilde{C}_{g,(2,L)}^{(1)}(N_F+1)\right] \\
&\quad + a_s^2\left[A_{Qg}^{(2)}(N_F+1)\delta_2 + A_{Qg}^{(1)}(N_F+1)\tilde{C}_{q,(2,L)}^{(1)}(N_F+1) + A_{gg,Q}^{(1)}(N_F+1)\tilde{C}_{g,(2,L)}^{(1)}(N_F+1) + \tilde{C}_{g,(2,L)}^{(2)}(N_F+1)\right] \\
&\quad + a_s^3\left[A_{Qg}^{(3)}(N_F+1)\delta_2 + A_{Qg}^{(2)}(N_F+1)\tilde{C}_{q,(2,L)}^{(1)}(N_F+1) + A_{gg,Q}^{(2)}(N_F+1)\tilde{C}_{g,(2,L)}^{(1)}(N_F+1)\right. \\
&\quad \left. + A_{Qg}^{(1)}(N_F+1)\tilde{C}_{q,(2,L)}^{(2),\rm S}(N_F+1) + A_{gg,Q}^{(1)}(N_F+1)\tilde{C}_{g,(2,L)}^{(1)}(N_F+1) + \tilde{C}_{g,(2,L)}^{(3)}(N_F+1)\right]
\end{aligned}
$$

- All first order factorizable contributions and $O(1000)$ fixed moments of $A_{Qg}^{(3)}$ are known.
- The case for two different masses obeys an analogous representation.

Heavy Flavor contribution to F_2



The variable flavor number scheme

- Matching conditions for parton distribution functions:

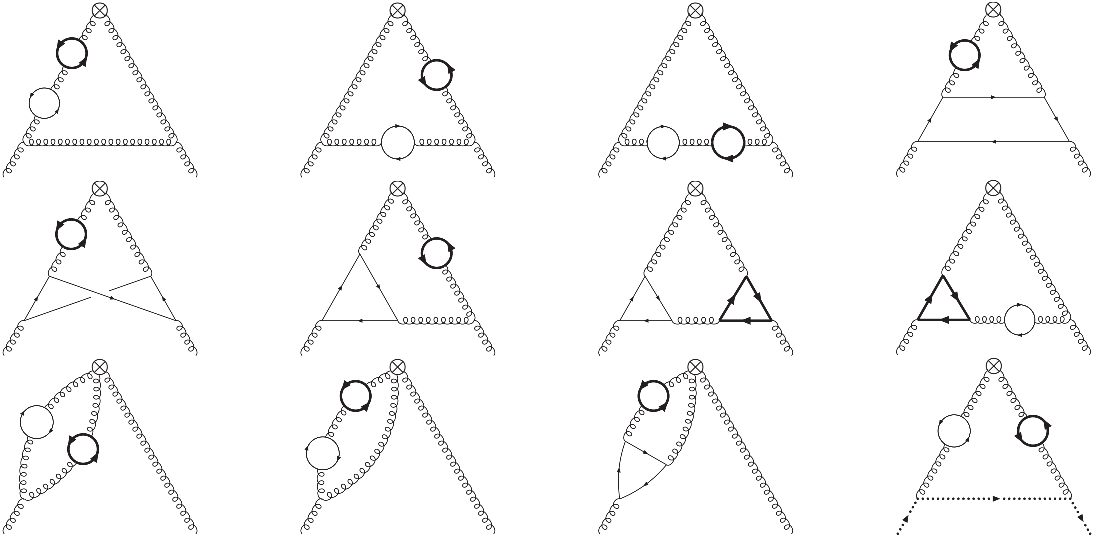
$$f_k(N_F+2) + f_{\bar{k}}(N_F+2) = A_{qq,Q}^{\rm NS}\left(N_F+2, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2}\right) \cdot \left[f_k(N_F) + f_{\bar{k}}(N_F)\right] + \frac{1}{N_F} A_{qq,Q}^{\rm PS}\left(N_F+2, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2}\right) \cdot \Sigma(N_F)$$
$$+ \frac{1}{N_F} A_{qg,Q}\left(N_F+2, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2}\right) \cdot G(N_F)\,,$$

$$f_Q(N_F+2) + f_{\bar{Q}}(N_F+2) = A_{Qq}^{\rm PS}\left(N_F+2, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2},\right) \cdot \Sigma(N_F) + A_{Qg}\left(N_F+2, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2}\right) \cdot G(N_F)\,,$$

$$\Sigma(N_F+2) = \left[A_{qq,Q}^{\rm NS}\left(N_F+2, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2}\right) + A_{qq,Q}^{\rm PS}\left(N_F+2, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2}\right) + A_{Qq}^{\rm PS}\left(N_F+2, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2}\right)\right] \cdot \Sigma(N_F)$$
$$+ \left[A_{qg,Q}\left(N_F+2, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2}\right) + A_{Qg}\left(N_F+2, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2}\right)\right] \cdot G(N_F)\,,$$

$$G(N_F+2) = A_{gq,Q}\left(N_F+2, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2}\right) \cdot \Sigma(N_F) + A_{gg,Q}\left(N_F+2, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2}\right) \cdot G(N_F)\,.$$

2-mass contributions



2-mass contributions

$A_{qq,Q}^{(3),\mathrm{NS}}$, $A_{gq,Q}^{(3)}$

Harmonic Sums

[Vermaseren '98; Blümlein, Kurth '98]

$$\sum_{i=1}^{N} \frac{1}{i^3} \sum_{j=1}^{i} \frac{1}{j}$$

HPLs

[Remiddi, Vermaseren '99]

$$\int_0^x \frac{d\tau_1}{1+\tau_1} \int_0^{\tau_1} \frac{d\tau_2}{1-\tau_2}$$

$A_{gg,Q}^{(3)}$

Generalized harmonic and binomial sums

[Ablinger, Blülein, Schneider '13]
[Ablinger, Blülein, Raab, Schneider '14]

$$\sum_{i=1}^{N} \frac{4^i (1-\eta)^{-i}}{i\binom{2i}{i}} \sum_{j=1}^{i} \frac{(1-\eta)^j}{j^2}$$

Iterated integrals over root and η valued letters

[Ablinger, Blümlein, Raab, Schneider '14]

$$\int_0^x d\tau_1 \frac{\sqrt{\tau_1(1-\tau_1)}}{1-\tau_1(1-\eta)} \int_0^{\tau_1} \frac{d\tau_2}{\tau_2}$$

$A_{Qq}^{(3),\mathrm{PS}}$

–

Iterated integrals over root valued letters with restricted support

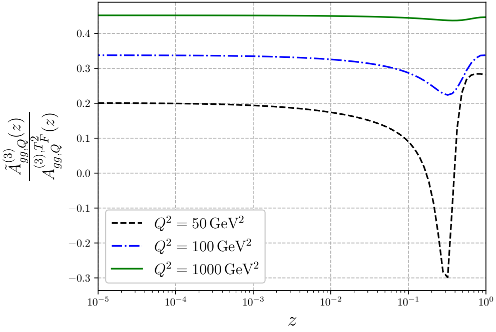
$$\theta(x-\eta_+) \int_0^{x(1-x)/\eta} d\tau \frac{\sqrt{1-4\tau}}{\tau}$$

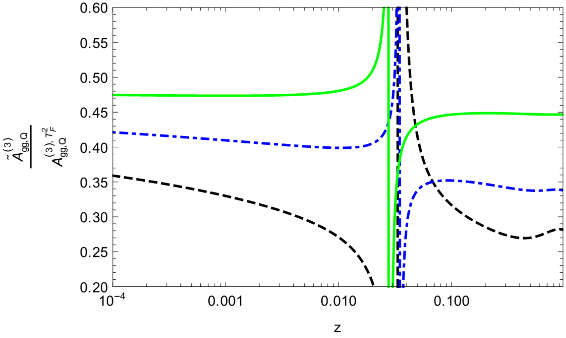
Results: $A_{gg,Q}^{(3)}$

$$
\begin{aligned}
\tilde{a}_{gg,Q}^{(3)}(N) &= \frac{1}{2}\left(1+(-1)^N\right)\Biggl\{T_F^3\Biggl\{\frac{32}{3}\left(L_1^3+L_2^3\right)+\frac{64}{3}L_1L_2\left(L_1+L_2\right)+32\zeta_2\left(L_1+L_2\right)+\frac{128}{9}\zeta_3\Biggr\} \\
&+ C_F T_F^2\Biggl\{\cdots+32\left(\mathrm{H}_0^2(\eta)-\frac{1}{3}S_2\right)S_1+\frac{128}{3}S_{2,1}-\frac{64}{3}S_{1,1,1}\left(\frac{1}{1-\eta},1-\eta,1,N\right) \\
&\quad -\frac{4P_{41}}{3(N-1)N^3(N+1)^2(N+2)(2N-3)(2N-1)}\left(\frac{\eta}{1-\eta}\right)^N\Biggl[\mathrm{H}_0^2(\eta) \\
&\quad -2\mathrm{H}_0(\eta)S_1\left(\frac{\eta-1}{\eta},N\right)-2S_2\left(\frac{\eta-1}{\eta},N\right)+2S_{1,1}\left(\frac{\eta-1}{\eta},1,N\right)\Biggr]+\ldots\Biggr\} \\
&+ C_A T_F^2\Biggl\{\cdots+\Biggl[\frac{8P_{65}}{3645\eta(N-1)N^3(N+1)^3(N+2)(2N-3)(2N-1)} \\
&\quad +\frac{8P_{37}\mathrm{H}_0(\eta)}{45\eta(N-1)N^2(N+1)^2(N+2)}+\frac{2P_{23}\mathrm{H}_0^2(\eta)}{9\eta(N-1)N(N+1)^2}+\frac{32}{27}\mathrm{H}_0^3(\eta)+\frac{128}{9}\mathrm{H}_{0,0,1}(\eta) \\
&\quad +\frac{64}{9}\mathrm{H}_0^2(\eta)\mathrm{H}_1(\eta)-\frac{128}{9}\mathrm{H}_0(\eta)\mathrm{H}_{0,1}(\eta)\Biggr]S_1 \\
&\quad +\frac{2^{-1-2N}P_{47}}{45\eta^2(N-1)N(N+1)^2(N+2)(2N-3)(2N-1)}\binom{2N}{N}\sum_{i=1}^{N}\frac{4^i\left(\frac{\eta}{\eta-1}\right)^i}{i\binom{2i}{i}}\Biggl\{\frac{1}{2}\mathrm{H}_0^2(\eta) \\
&\quad S_{1,1}\left(\frac{\eta-1}{\eta},1,i\right)\Biggr\}+\ldots\Biggr\}
\end{aligned}
$$

Results: $A_{gg,Q}^{(3)}$

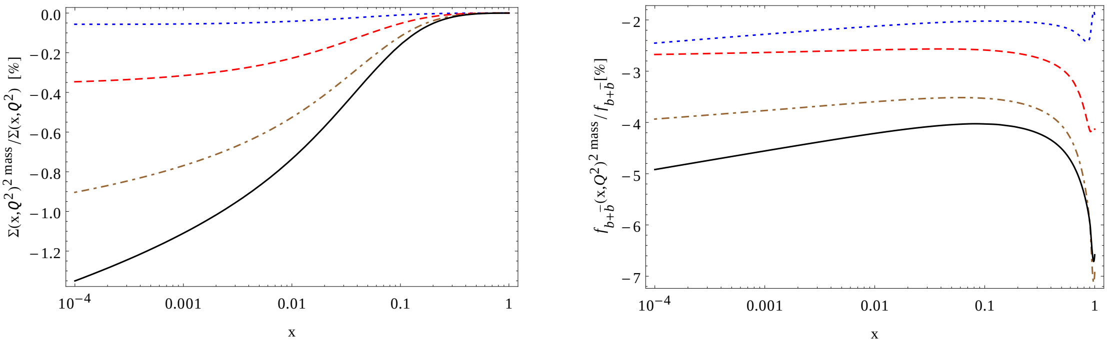
The two mass contributions over the whole T_F^2- contributions to the OME $A_{gg,Q}^{(3)}$:





The variable flavor number scheme at NLO

$\Sigma(x, Q^2)^{\text{2-mass}} / \Sigma(x, Q^2)$ $\qquad$ $f_{b+\bar{b}}(x, Q^2)^{\text{2-mass}} / f_{b+\bar{b}}(x, Q^2)$



- The ratio of the 2-mass contributions to the singlet parton distribution $\Sigma(x, Q^2)$ (left) and the heavy flavor parton distribution $f_{b+\bar{b}}(x, Q^2)$ (right) over their full form in percent for $m_c = 1.59$ GeV, $m_b = 4.78$ GeV in the on-shell scheme. Dash-dotted line: $Q^2 = 30$ GeV^2; Dotted line: $Q^2 = 30$ GeV^2; Dashed line: $Q^2 = 100$ GeV^2; Dash-dotted line: $Q^2 = 1000$ GeV^2; Full line: $Q^2 = 10000$ GeV^2.
- For the PDFs the NNLO variant of ABMP16 with $N_f = 3$ flavors was used.
  Alekhin et al., Phys. Rev. D 96 (2017) 1

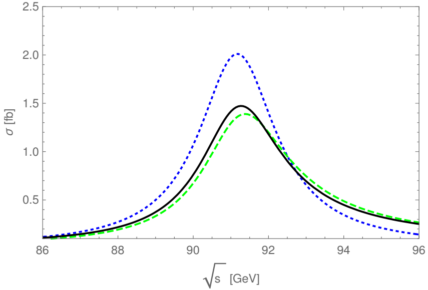
Massive 3-Loop Formfactors

- Quarkonic corrections finished
- Using difference eq. technologies; all first order factorizing contributions have been obtained.
- Non-first order factorizing contributions are analytically continued using differential equations and the matching method.
- Here the problem can be traced back to a series of new special numbers thanks to the use of computer algebra.
- Gluonic corrections are in preparation.
- The latter case can be solved in a similar way.
- The calculation of these quantities are of importance for the precision determination of the mass of the top quark, as fundamental parameter of the Standard Model, at future high energy e^+e^- colliders.

Precision QED Corrections to $e^+e^- \to \gamma^*/Z^*$

- In the future the properties of the Z-boson shall be measured down to a few MeV, where $M_Z = 91187.6$ MeV and $\Gamma_Z = 2495.2$ MeV for the energy-dependent scattering cross section.
- This requires to calculate the initial state QED radiative corrections to very high orders.
- Errors on the level of the $O(\alpha^2)$ corrections had to be corrected [several years of intense work, developing new computation methods.]
- Correction now known at $O(\alpha^2)$ completely and all the first three logarithmic terms in higher orders up to $O(\alpha^6 L^5)$, with $L = \ln(M_Z^2/m_e^2)$.
- Note that $m_e^2 \approx M_Z^2 10^{-10}$ and numerics in this small parameter has to be exactly controlled before expanding in m_e^2/M_Z^2.
- Two-valued root letter alphabets emerge, including incomplete elliptic integrals.
- The forward-backward asymmetry A_{FB} has also been calculated to high order in the leading log approximation.
- It allows to measure the fine structure constant α_{QED} at high energies at high precision.
- This provides to measure non-perturbative hadronic contributions.

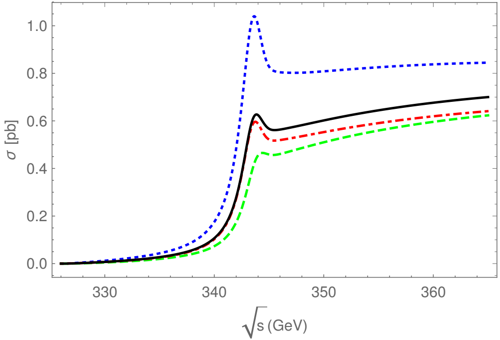
Z peak

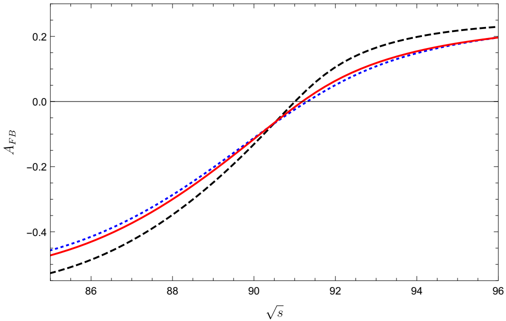


	Fixed width		s dep. width	
	Peak (MeV)	Width (MeV)	Peak (MeV)	Width (MeV)
$O(\alpha)$ correction	185.638	539.408	181.098	524.978
$O(\alpha^2L^2)$:	– 96.894	–177.147	– 95.342	–176.235
$O(\alpha^2L)$:	6.982	22.695	6.841	21.896
$O(\alpha^2)$:	0.176	– 2.218	0.174	– 2.001
$O(\alpha^3L^3)$:	23.265	38.560	22.968	38.081
$O(\alpha^3L^2)$:	– 1.507	– 1.888	– 1.491	– 1.881
$O(\alpha^3L)$:	– 0.152	0.105	– 0.151	– 0.084
$O(\alpha^4L^4)$:	– 1.857	0.206	– 1.858	0.146
$O(\alpha^4L^3)$:	0.131	– 0.071	0.132	– 0.065
$O(\alpha^4L^2)$:	0.048	– 0.001	0.048	0.001
$O(\alpha^5L^5)$:	0.142	– 0.218	0.144	– 0.212
$O(\alpha^5L^4)$:	– 0.000	0.020	– 0.001	0.020
$O(\alpha^5L^3)$:	– 0.008	0.009	– 0.008	0.008
$O(\alpha^6L^6)$:	– 0.007	0.027	– 0.007	0.027
$O(\alpha^6L^5)$:	– 0.001	0.000	– 0.001	0.000

Table 1: Shifts in the Z-mass and the width due to the different contributions to the ISR QED radiative corrections for a fixed width of $\Gamma_Z = 2.4952$ GeV and s-dependent width using $M_Z = 91.1876$ GeV and $s_0 = 4m_\tau^2$.

Top-threshold corrections and A_{FB}





dotted: $O(\alpha^0)$, dashed: $O(\alpha)$, dash-dotted: $O(\alpha^2)$, full: $O(\alpha^2)$ + soft resummation

dotted: $O(\alpha^0)$, dashed: $O(\alpha)$, full line: including $O(\alpha^6 L^6)$

3–loop anomalous dimensions and massless Wilson coefficients

- All non-singlet anomalous dimensions were calculated in a fully automated way.
- Also the polarized singlet anomalous dimensions were computed.
- The method used are off-shell gauge variant massless operator matrix elements, using our method of arbitrary high Mellin moments.
- In the unpolarized case the complete singlet anomalous dimensions are obtained at 2-loop order, correcting errors in the foregoing literature.
- First 3–loop DIS Wilson coefficients have been computed and the others are forthcoming.
- The anomalous dimensions for transversity were calculated first.
- In earlier calculations many special assumptions have been made, which we all did not assume. Those were now verified for the first time by the present calculations.
- These calculations form the preparatory phase to compute 4-loop anomalous dimensions in the future.
- The Wilson coefficients are needed to form scheme-invariant combinations to remedy potential gauge artefacts.

The evolution equations

The different anomalous dimensions

- 3 non–singlet anomalous dimensions (starting at 1-, 2-, and 3-loop) $\gamma_{qq}^{\rm NS,\pm}, \gamma_{qq}^{\rm NS,s}$
- 4 singlet anomalous dimensions $\gamma_{qq}^{\rm PS}, \gamma_{qg}, \gamma_{gq}, \gamma_{gg}$ [$\gamma_{qq}^{\rm PS}$ contributes from 2-loops.]
- both in the unpolarized and polarized case: $\gamma_{ij} \rightarrow \Delta\gamma_{ij}$
- + expansion coefficients for the so called alien operators, including new anomalous dimensions.

Evolution equations

$$
\begin{aligned}
\Sigma(N) &= \sum_{k=1}^{N_F} q_k(N) + \bar{q}_k(N) \\
\gamma_{ij} &= \sum_{k=0}^{\infty} a_s^{k+1} \gamma_{ij} \\
\frac{dq^{\rm NS,\pm,(s)}(N,a_s)}{d\ln(\mu^2)} &= \sum_{k=0}^{\infty} a_s^{k+1} \gamma_{qq}^{(k),\rm NS,\pm(s)}(N) \cdot q^{\rm NS,\pm(s)}(N,a_s) \\
\frac{d}{d\ln(\mu^2)} \begin{pmatrix} \Sigma(N) \\ G(N) \end{pmatrix} &= \sum_{k=0}^{\infty} a_s^{k+1} \begin{pmatrix} \gamma_{qq}^{(k)} & \gamma_{qg}^{(k)} \\ \gamma_{gq}^{(k)} & \gamma_{gg}^{(k)} \end{pmatrix} \cdot \begin{pmatrix} \Sigma(N) \\ G(N) \end{pmatrix}
\end{aligned}
$$

The non-singlet anomalous dimension $\gamma_{qq}^{+,\mathrm{NS}}$

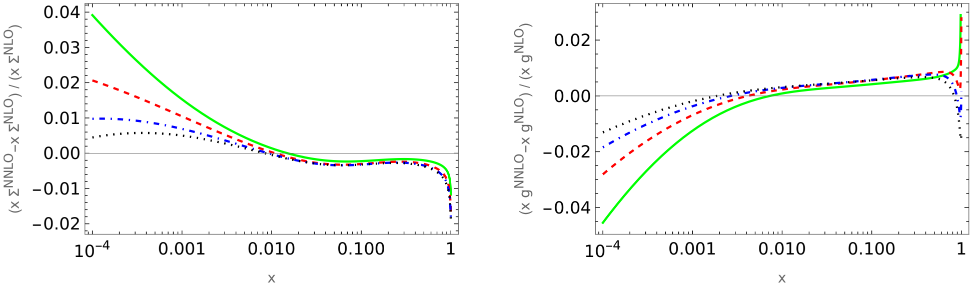
$$\gamma_{\mathrm{NS}}^{(2),+} = \frac{1}{2}\left[1+(-1)^N\right]$$
$$\times\Big\{C_F^2\Big\{C_A\Big[\frac{72P_3}{N^2(1+N)^2}\zeta_3 + \frac{32P_{15}}{9N^2(1+N)^2}S_{-2,1} - \frac{16P_{17}}{9N^2(1+N)^2}S_3 + \frac{P_{33}}{18N^4(1+N)^4}$$
$$+\Big(-\frac{16P_{29}}{9N^4(1+N)^4} - \frac{4288}{9}S_2 + \frac{64(-12+31N+31N^2)}{3N(1+N)}S_3 + 320S_4 - 1024S_{3,1}$$
$$+\frac{64(-84+31N+31N^2)}{3N(1+N)}S_{-2,1} + 3712S_{-2,2} + 3840S_{-3,1} - 7168S_{-2,1,1}\Big)S_1 + \Big(256S_3$$
$$+1792S_{-2,1}\Big)S_1^2 + \Big(\frac{4P_{19}}{9N^2(1+N)^2} - 832S_3 - 5248S_{-2,1}\Big)S_2 + \frac{352}{3}S_2^2$$
$$+\frac{16(-30+151N+151N^2)}{3N(1+N)}S_4 + \Big(-\frac{16P_{22}}{9N^2(1+N)^3} + (-\frac{64P_9}{9N^2(1+N)^2} - 256S_2\Big)S_1$$
$$+\frac{32(12+31N+31N^2)S_2}{3N(1+N)} + 64S_3 + 5376S_{2,1} - 384S_{-2,1} + 576\zeta_3\Big)S_{-2}$$
$$+\Big(-\frac{32(8+3N+3N^2)}{N(1+N)} + 512S_1\Big)S_{-2}^2 + \Big(\frac{32(108+31N+31N^2)}{3N(1+N)}S_1 - \frac{16P_{16}}{9N^2(1+N)^2}$$
$$-1152S_1^2 + 2624S_2 + 960S_{-2}\Big)S_{-3} + \Big(\frac{16(138+35N+35N^2)}{3N(1+N)} - 1472S_1\Big)S_{-4}$$
$$+2304S_{-5} + 768S_{2,3} + 2688S_{2,-3} - \frac{64(-24+29N+29N^2)}{3N(1+N)}S_{3,1} - 768S_{4,1}$$
$$+\frac{32(-174+31N+31N^2)}{3N(1+N)}S_{-2,2} - 3648S_{-2,3} - \frac{1920}{N(1+N)}S_{-3,1} + 1728S_{-4,1}$$
$$-5376S_{2,1,-2} + 1536S_{3,1,1} - \frac{128(-84+31N+31N^2)}{3N(1+N)}S_{-2,1,1} - 1536S_{-2,1,-2}$$
$$-5376S_{-2,2,1} - 5376S_{-3,1,1} + 10752S_{-2,1,1,1}\Big] + T_FN_F\Big[-\frac{16P_5}{9N^2(1+N)^2}S_2 + \frac{4P_{34}}{9N^4(1+N)^4}$$
$$+\Big(-\frac{8P_{13}}{9N^2(1+N)^2} + \frac{1280}{9}S_2 - \frac{512}{3}S_3 - \frac{512}{3}S_{-2,1} + 128\zeta_3\Big)S_1 - \frac{128}{3}S_2^2$$
$$+\frac{64(12+29N+29N^2)}{9N(1+N)}S_3 - \frac{512}{3}S_4 + \Big(-\frac{128(-3+10N+16N^2)}{9N^2(1+N)^2} + \frac{2560}{9}S_1 - \frac{256}{3}S_2\Big)$$
$$\times S_{-2} + \Big(\frac{128(3+10N+10N^2)}{9N(1+N)} - \frac{256}{3}S_1\Big)S_{-3} - \frac{256}{3}S_{-4} - \frac{256(-3+10N+10N^2)}{9N(1+N)}S_{-2,1}$$
$$+\frac{256}{3}S_{3,1} - \frac{256}{3}S_{-2,2} + \frac{1024}{3}S_{-2,1,1} - \frac{32(2+3N+3N^2)}{N(1+N)}\zeta_3\Big]\Big\}$$
$$+C_F\Big\{T_F^2N_F^2\Big[\frac{8P_{28}}{27N^3(1+N)^3} - \frac{128}{27}S_1 - \frac{640}{27}S_2 + \frac{128}{9}S_3\Big] + C_A^2\Big[-\frac{24P_3}{N^2(1+N)^2}\zeta_3$$
$$\frac{32P_{11}}{9N^2(1+N)^2}S_{-2,1} + \frac{8P_{18}}{9N^2(1+N)^2}S_3 + \frac{P_{32}}{54N^3(1+N)^3} + \Big(\frac{4P_{35}}{3N^4(1+N)^4}$$
$$-\frac{16(-8+11N+11N^2)}{N(1+N)}S_3 - 256S_4 + 512S_{3,1} - \frac{64(-24+11N+11N^2)}{3N(1+N)}S_{-2,1}$$
$$-1024S_{-2,2} - 1024S_{-3,1} + 2048S_{-2,1,1}\Big)S_1 + \Big(-128S_3 - 512S_{-2,1}\Big)S_1^2 + \Big(-\frac{8344}{27}$$
$$+384S_3 + 1536S_{-2,1}\Big)S_2 - \frac{16(-24+55N+55N^2)}{3N(1+N)}S_4 + 64S_5 + \Big(\frac{32P_{10}}{9N^2(1+N)^2}S_1$$
$$+\frac{16P_{27}}{9N^3(1+N)^3} - \frac{352}{3}S_2 - 64S_3 - 1536S_{2,1} + 128S_{-2,1} - 192\zeta_3\Big)S_{-2}$$
$$+\Big(\frac{48(2+N+N^2)}{N(1+N)} - 192S_1\Big)S_{-2}^2 + \Big(\frac{16P_{12}}{9N^2(1+N)^2} - \frac{32(24+11N+11N^2)}{3N(1+N)}S_1$$
$$+256S_1^2 - 768S_2 - 320S_{-2}\Big)S_{-3} + \Big(-\frac{16(30+13N+13N^2)}{3N(1+N)} + 320S_1\Big)S_{-4}$$
$$-704S_{-5} - 384S_{2,3} - 768S_{2,-3} + \frac{64(-12+11N+11N^2)}{3N(1+N)}S_{3,1} + 384S_{4,1}$$
$$-\frac{32(-48+11N+11N^2)}{3N(1+N)}S_{-2,2} + 1088S_{-2,3} + \frac{512}{N(1+N)}S_{-3,1} - 448S_{-4,1}$$
$$+1536S_{2,1,-2} - 768S_{3,1,1} + \frac{128(-24+11N+11N^2)}{3N(1+N)}S_{-2,1,1} + 512S_{-2,1,-2} + 1536[S_{-2,2,1}$$
$$+S_{-3,1,1}] - 3072S_{-2,1,1,1}\Big] + C_AT_FN_F\Big[-\frac{8P_{31}}{27N^3(1+N)^3} + \Big(-\frac{16P_{30}}{27N^3(1+N)^3} + 64S_3$$
$$+\frac{256}{3}S_{-2,1} - 128\zeta_3\Big)S_1 + \frac{5344}{27}S_2 - \frac{32(3+14N+14N^2)}{3N(1+N)}S_3 + \frac{320}{3}S_4 + \Big(-\frac{1280}{9}S_1$$
$$+\frac{64(-3+10N+16N^2)}{9N^2(1+N)^2} + \frac{128}{3}S_2\Big)S_{-2} + \Big(-\frac{64(3+10N+10N^2)}{9N(1+N)} + \frac{128}{3}S_1\Big)S_{-3}$$
$$+\frac{128}{3}S_{-4} - \frac{256}{3}S_{3,1} + \frac{128(-3+10N+10N^2)}{9N(1+N)}S_{-2,1} + \frac{128}{3}S_{-2,2} - \frac{512}{3}S_{-2,1,1}$$
$$+\frac{32(2+3N+3N^2)}{N(1+N)}\zeta_3\Big]\Big\} + C_F^3\Big\{-\frac{48P_3}{N^2(1+N)^2}\zeta_3 + \frac{8P_4}{N^2(1+N)^2}S_3 + \frac{P_{36}}{N^5(1+N)^5}$$
$$+\Big(\frac{8P_{26}}{N^4(1+N)^4} - \frac{128(1+2N)}{N^2(1+N)^2}S_2 + 128S_2^2 - 384S_3 + 128S_4 + 512S_{3,1} - 3328S_{-2,2}$$
$$-\frac{384(-4+N+N^2)}{N(1+N)}S_{-2,1} - 3584S_{-3,1} + 6144S_{-2,1,1}\Big)S_1 + \Big(-\frac{64(1+3N+3N^2)}{N^3(1+N)^3}$$
$$-1536S_{-2,1}\Big)S_1^2 + \Big(\frac{4P_{25}}{N^3(1+N)^3} + 512S_3 + 4352S_{-2,1}\Big)S_2 - \frac{32(2+3N+3N^2)}{N(1+N)}S_2^2$$
$$\frac{32(2+15N+15N^2)}{N(1+N)}S_4 + (\frac{32P_{24}}{N^3(1+N)^3} + \Big(-\frac{128(5+7N+3N^2)}{N^2(1+N)^2} + 512S_2\Big)S_1$$
$$-\frac{64(4+3N+3N^2)}{N(1+N)}S_2 + 128S_3 - 4608S_{2,1} + 256S_{-2,1} - 384\zeta_3\Big)S_{-2} + \Big(\frac{128}{N(1+N)}$$
$$-256S_1\Big)S_{-2}^2 + \Big(\frac{32(8+5N+9N^2)}{N^2(1+N)^2} - \frac{64(20+3N+3N^2)}{N(1+N)}S_1 + 1280S_1^2 - 2176S_2$$
$$-640S_{-2}\Big)S_{-3} + \Big(-\frac{32(26+3N+3N^2)}{N(1+N)} + 1664S_1\Big)S_{-4} - 1792S_{-5} - 384S_{2,3}$$
$$-2304S_{2,-3} + \frac{128(-2+3N+3N^2)}{N(1+N)}S_{3,1} + 384S_{4,1} - \frac{64(4-N+3N^2)}{N^2(1+N)^2}S_{-2,1}$$
$$-\frac{64(-26+3N+3N^2)}{N(1+N)}S_{-2,2} + 2944S_{-2,3} + \frac{1792}{N(1+N)}S_{-3,1} - 1664S_{-4,1}$$
$$+4608S_{2,1,-2} - 768S_{3,1,1} + \frac{768(-4+N+N^2)}{N(1+N)}S_{-2,1,1} + 1024S_{-2,1,-2}$$
$$+4608[S_{-2,2,1} + S_{-3,1,1}] - 9216S_{-2,1,1,1}\Big\}\Big\},$$

The polarized singlet anomalous dimension: $\Delta\gamma_{gg}^{(2)}$

$$
\begin{aligned}
\Delta\gamma_{gg}^{(2)} = {}& C_A T_F^2 N_F^2 \Bigg[-\frac{16 P_8}{27 N^2 (1+N)^2} S_1 - \frac{4 P_{48}}{27 N^3 (1+N)^3} \Bigg] + C_F \Bigg[T_F^2 N_F^2 \Bigg[-\frac{8 P_{59}}{27 N^4 (1+N)^4} \\
& + \frac{64 (N-1)(2+N)\big(-6 - 8N + N^2\big)}{9 N^3 (1+N)^3} S_1 + \frac{32 (N-1)(2+N)}{3 N^2 (1+N)^2} S_1^2 \\
& - \frac{32 (N-1)(2+N)}{N^2 (1+N)^2} S_2 \Bigg] + C_A T_F N_F \Bigg[\frac{8 P_6}{N^3 (1+N)^3} S_2 - \frac{8 P_9}{3 N^3 (1+N)^3} S_1^2 \\
& + \frac{2 P_{77}}{27 (N-1) N^5 (1+N)^5 (2+N)} + \Bigg(-\frac{8 P_{67}}{9(-1+N) N^4 (1+N)^4 (2+N)} \\
& - \frac{32 (N-1)(2+N)}{N^2 (1+N)^2} S_2 + 128 \zeta_3 \Bigg) S_1 + \frac{32 (N-1)(2+N)}{3 N^2 (1+N)^2} S_1^3 - \frac{32\big(34 + N + N^2\big)}{3 N^2 (1+N)^2} \\
& \times S_3 + \Bigg(\frac{128 P_2}{(N-1) N^2 (1+N)^2 (2+N)} S_1 - \frac{32 P_{23}}{(N-1) N^2 (1+N)^3 (2+N)} \Bigg) S_{-2} \\
& - \frac{192\big(4 - N - N^2\big)}{N^2 (1+N)^2} S_{-3} + \frac{64 (N-1)(2+N)}{N^2 (1+N)^2} S_{2,1} - \frac{128\big(-8 + N + N^2\big)}{N^2 (1+N)^2} S_{-2,1} \\
& - \frac{64(-3+N)(4+N)}{N^2 (1+N)^2} \zeta_3 \Bigg] \Bigg] + C_A^3 \Bigg[\frac{64 P_{16}}{9 N^2 (1+N)^2} S_{-2,1} - \frac{32 P_{18}}{9 N^2 (1+N)^2} S_3 \\
& + \frac{P_{74}}{27 (N-1) N^5 (1+N)^5 (2+N)} + \Bigg(\frac{4 P_{69}}{9 (N-1) N^4 (1+N)^4 (2+N)} \\
& - \frac{64 P_{17}}{9 N^2 (1+N)^2} S_2 + 128 S_2^2 + \frac{16\big(-96 + 11N + 11N^2\big)}{3 N (1+N)} S_3 + 192 S_4 \\
& + \frac{1024}{N(1+N)} S_{-2,1} - 640 S_{-2,2} - 768 S_{-3,1} + 1024 S_{-2,1,1} \Bigg) S_1 \\
& + \Bigg(-\frac{256\big(1 + 3N + 3N^2\big)}{N^3 (1+N)^3} + 128 S_3 - 256 S_{-2,1} \Bigg) S_1^2 + \Bigg(-\frac{16 P_{41}}{9 N^3 (1+N)^3} \\
& + 64 S_3 + 640 S_{-2,1} \Bigg) S_2 - \frac{256}{N(1+N)} S_2^2 - \frac{384}{N(1+N)} S_4 + 64 S_5 \\
& + \Bigg(\frac{32 P_{52}}{9 (N-1) N^3 (1+N)^3 (2+N)} + \Big(-\frac{64 P_{32}}{9(-1+N) N (1+N)^2 (2+N)} + 256 S_2 \Bigg) \\
& \times S_1 - \frac{512}{N(1+N)} S_2 + 128 S_3 - 768 S_{2,1} \Bigg) S_{-2} + \Bigg(-\frac{16\big(24 + 11N + 11N^2\big)}{3 N (1+N)} \\
& + 64 S_1 \Bigg) S_{-2}^2 + \Bigg(-\frac{32 P_{15}}{9 N^2 (1+N)^2} - \frac{1536}{N(1+N)} S_1 + 384 S_1^2 - 320 S_2 \Bigg) S_{-3} \\
& + \Bigg(-\frac{1024}{N(1+N)} + 512 S_1 \Bigg) S_{-4} - 192 S_{-5} - 384 S_{2,-3} + \frac{1280}{N(1+N)} S_{-2,2} \\
& + 384 S_{-2,3} + \frac{1536}{N(1+N)} S_{-3,1} - 384 S_{-4,1} + 768 S_{2,1,-2} - \frac{2048}{N(1+N)} S_{-2,1,1}
\end{aligned}
$$

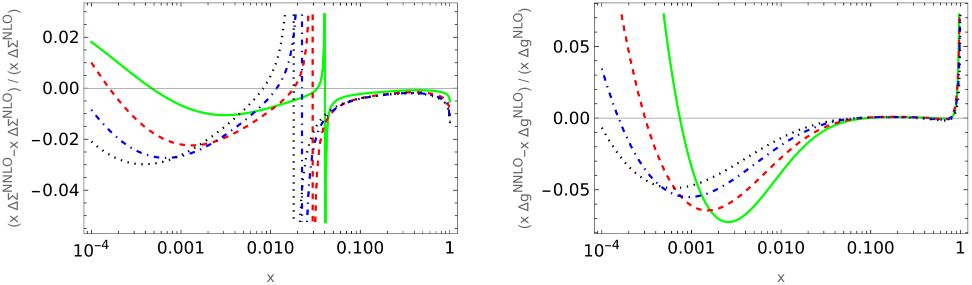
$$
\begin{aligned}
& + 768 [S_{-2,2,1} + S_{-3,1,1}] - 1536 S_{-2,1,1,1} \Bigg] \\
& + C_F^2 T_F N_F \Bigg[-\frac{4 P_{75}}{(N-1) N^5 (1+N)^5 (2+N)} + \Bigg(\frac{32 (N-1)(2+N) S_2}{N^2 (1+N)^2} \\
& - \frac{16 P_{42}}{N^4 (1+N)^4} \Bigg) S_1 + \frac{8 (N-1)(2+N)\big(2 + 3N + 3N^2\big)}{N^3 (1+N)^3} S_1^2 - \frac{32 (N-1)(2+N)}{3 N^2 (1+N)^2} \\
& \times S_1^3 - \frac{8 (2+N)\big(2 - 11N - 16N^2 + 9N^3\big)}{N^3 (1+N)^3} S_2 + \frac{32\big(10 + 7N + 7N^2\big)}{3 N^2 (1+N)^2} S_3 \\
& + \Bigg(-\frac{64\big(10 + N + N^2\big)}{(N-1) N (1+N)(2+N)} + \frac{512}{N^2 (1+N)^2} S_1 \Bigg) S_{-2} + \frac{256}{N^2 (1+N)^2} S_{-3} \\
& - \frac{64 (N-1)(2+N)}{N^2 (1+N)^2} S_{2,1} - \frac{512}{N^2 (1+N)^2} S_{-2,1} + \frac{192\big(-2 - N - N^2\big)}{N^2 (1+N)^2} \zeta_3 \Bigg] \\
& + C_A^2 T_F N_F \Bigg[\frac{32 P_4}{9 N^2 (1+N)^2} S_2 + \frac{32 P_{11}}{9 N^2 (1+N)^2} S_{-3} - \frac{64 P_{11}}{9 N^2 (1+N)^2} S_{-2,1} \\
& + \frac{16 P_{13}}{9 N^2 (1+N)^2} S_3 + \frac{2 P_{76}}{27 (N-1) N^5 (1+N)^5 (2+N)} + \Bigg(\frac{1280}{9} S_2 - \frac{64}{3} S_3 \\
& - \frac{8 P_{68}}{27(-1+N) N^4 (1+N)^4 (2+N)} - 128 \zeta_3 \Bigg) S_1 + \frac{64}{3} S_{-2}^2 \\
& + \Bigg(\frac{64 P_{45}}{9 (N-1) N^2 (1+N)^2 (2+N)} S_1 - \frac{32 P_{50}}{9 (N-1) N^3 (1+N)^3 (2+N)} \Bigg) S_{-2} \\
& + \frac{128\big(-3 + 2N + 2N^2\big)}{N^2 (1+N)^2} \zeta_3 \Bigg]
\end{aligned}
$$

The unpolarized NNLO evolution



$Q^2 = 10, 10^2, 10^3, 10^4$ GeV^2 dotted, dash-dotted, dashed, full lines.

The polarized NNLO evolution



Topical Workshops

- Elliptic integrals and modular forms in QFT (2017) [KMPB Berlin]
- Analytic integration methods in quantum field theory (2020) [Wolfgang-Pauli Center]

Other workshops:

- Loops and Legs in Quantum Field Theory (since 1992, bi-annually); last: Ettal,D 2022.
- RADCOR (bi-annually); last Tallahassee,FL, 2021.
- Review-Workshop: Loopsummit, Cadenabbia, I; last: 2021 (next planned for 2024).

Further Plans

- Completion of the calculation of the single–mass and two–mass corrections to deep-inelastic scattering at 3-loop order; partial use of (tuneable) analytic approximation methods and creation of an associated numerical code.
- Completion of the (semi)analytic calculation of the massive 3-loop form factors.
- Refinement of the technologies to calculate massless anomalous dimensions and deep-inelastic Wilson coefficients at 3-loop order and other inclusive scattering cross sections.
- Further development of technologies to calculate 4-loop anomalous dimensions.They are instrumental for the precision physics at the high-luminosity LHC.
- Further research topics will result from the temporal development.
- It is necessary to maintain the qftquad cluster to be able to perform the planned analytic calculations using methods of computer algebra. They concern most advanced topics of research in this field.