Relations between Nested Harmonic Sums and MZV's

Johannes Blümlein, DESY

- Introduction
- Algebraic Relations
- Structural Relations
- Representation of some Observables
- Factorial Series
- The Basis
- News about alternating MZV's
- Conclusions

Refs. J.B. DESY 07–042, and in preparation;
J.B., D. Broadhurst, J. Vermaseren, in preparation.

1. Introduction

Why are no- and single scale quantities in Quantum Field Theory related to ζ-values, nested harmonic sums and related objects ?

- The former quantities can be obtained from the latter putting the Mellin variable N either to fixed values or $N \rightarrow \infty$.

Perturbation Theory [fixed order]

Scalar Propagators:

$$\frac{i}{p^2 + i\epsilon}$$

Combine Momenta using the Feynman Trick

$$\frac{1}{A \cdot B} = \int_0^1 \frac{dx}{[xA + (1-x)B]^2}$$

Feynman parameter integrals substitute all non-trivial angular integrals in $D = 4 + \varepsilon$–dimensions.

Introduction

Momentum Integrals yield rational functions of Γ–functions

$$\left.\frac{\Gamma(n_1+\alpha_1\varepsilon)...\Gamma(n_k+\alpha_k\varepsilon)}{\Gamma(m_1+\beta_1\varepsilon)...\Gamma(m_l+\beta_k\varepsilon)}\right|_{m_i,n_i\epsilon\mathbf{Z},\alpha_i,\beta_i\epsilon\mathbf{Q}}$$

- The scale-ratio in the diagrams factors form the Feynman parameter integrals for single scale processes

The Feynman parameter integrals can be transformed into Mellin-Barnes Integrals

$$\frac{1}{(A+B)^q}=\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}d\sigma A^\sigma B^{-q-\sigma}\frac{\Gamma(-\sigma)\Gamma(q+\sigma)}{\Gamma(q)}$$

Then all Feynman parameter integrals can again be integrated $\Longrightarrow$ rational function of Γ–functions

One or more Feynman parameters contain N as power $x_i^N, ...$ in the numerators. $\Longrightarrow$ the Γ–functions contain N as argument.

The Mellin-Barnes Integrals can be carried out using the Residue Theorem.
$\Longrightarrow$ several infinite sums over the rational functions of Γ–functions.

Introduction

- Seek compact representations for these in terms of

(Generalizations of) generalized hypergeometric functions.

- The Mellin variable N is a discrete quantity in the first place for physical reasons

$\Longrightarrow$ Light-cone expansion; cut vertex method + dispersion relations

- Perform the ε–expansion.
- The respective coefficients obey Difference Equations of finite order.
- The ε–expansion of Pochhammer-Symbols & Γ–functions leads to products of single finite harmonic sums and MZV's.
- The infinite sums over the Mellin–Barnes parameters lead to the respective Nesting.
- Observation: Most of the sums occurring are Nested Finite Harmonic Sums.
- However, other related sums are possible too for individual Feynman diagrams. [Vermaseren et al. (2005)]
- General solution formalisms like, `Sigma`, will reveal this uniquely. cf. C. Schneider.

Introduction

- Single scale processes in massless Quantum Field Theories or being considered in the limit $m^2/Q^2 \to 0$ exhibit significant simplifications when calculated in Mellin space.

- This is, to some extent, due to structure of Feynman parameter intergrals which posess a Mellin symmetry.

- Harmonic sums form the appropriate language to derive compact expressions in the respective calculations.

- We will line out the relations of the harmonic sums, resp. their continuations to $N \epsilon \mathbf{Q}, \mathbf{R}, \mathbf{C}$.

x-space results :

Nielsen-type integrals, resp. harmonic polylogarithms (E. Remiddi and J. Vermaseren (1999))

$$S_{n,p,q}(x) = \frac{(-1)^{n+p+q-1}}{\Gamma(n)p!q!} \int_0^1 \frac{dz}{z} \ln^{(n-1)}(z) \ln^p(1-zx) \ln^q(1+zx)$$

2 Loop Wilson Coefficients

Order α_s^2 contributions to the deep inelastic Wilson coefficient

W.L. van Neerven and E.B. Zijlstra
Instituut-Lorentz, University of Leyden, P.O. Box 9506, NL-2300 RA Leyden, The Netherlands

Volume 272, number 1,2 PHYSICS LETTERS B 28 November 1991

$$
\begin{aligned}
C_2^{(2),(+)}(x,1) = C_F^2\Big[& \frac{1+x^2}{1-x}\{4\ln^3(1-x)-(14\ln x+9)\ln^2(1-x) \\
& -[4\,\mathrm{Li}_2(1-x)-12\ln^2 x-12\ln x+16\zeta(2)+\tfrac{27}{2}]\ln(1-x)-\tfrac{4}{3}\ln^3 x-\tfrac{3}{2}\ln^2 x \\
& +[-24\,\mathrm{Li}_2(-x)+24\zeta(2)+\tfrac{61}{2}]\ln x+12\,\mathrm{Li}_3(1-x)-12S_{1,2}(1-x) \\
& +48\,\mathrm{Li}_3(-x)-6\,\mathrm{Li}_2(1-x)+32\zeta(3)+18\zeta(2)+\tfrac{51}{4}\} \\
& +(1+x)\{2\ln x\ln^2(1-x)+4[\mathrm{Li}_2(1-x)-\ln^2 x]\ln(1-x) \\
& -4[\mathrm{Li}_2(1-x)+\zeta(2)]\ln x+\tfrac{5}{3}\ln^3 x-4\,\mathrm{Li}_3(1-x)\} \\
& +\left(40+8x-48x^2-\tfrac{72}{5}x^3+\frac{8}{5x^2}\right)[\mathrm{Li}_2(-x)+\ln x\ln(1+x)] \\
& +(-8+40x)[\ln x\,\mathrm{Li}_2(-x)+S_{1,2}(1-x)-2\,\mathrm{Li}_3(-x)-\zeta(2)\ln(1-x)]+(5+9x)\ln^2(1-x) \\
& +\tfrac{1}{2}(-91+141x)\ln(1-x)-(28+44x)\ln x\ln(1-x)-(14+30x)\,\mathrm{Li}_2(1-x) \\
& +(\tfrac{29}{2}+\tfrac{25}{2}x+24x^2+\tfrac{36}{5}x^3)\ln^2 x+\frac{1}{10}\left(13-407x+144x^2-\frac{16}{x}\right)\ln x+(-10+6x-48x^2-\tfrac{72}{5}x^3)\zeta(2) \\
& +\tfrac{407}{20}-\tfrac{1917}{20}x+\tfrac{72}{5}x^2+\frac{8}{5x}+[6\zeta(2)^2-78\zeta(3)+69\zeta(2)+\tfrac{331}{8}]\delta(1-x)\Big] \\
+C_A C_F\Big[& \frac{1+x^2}{1-x}\{-\tfrac{11}{3}\ln^2(1-x)+[4\,\mathrm{Li}_2(1-x)+2\ln^2 x+\tfrac{44}{3}\ln x-4\zeta(2)+\tfrac{367}{18}]\ln(1-x) \\
& -\ln^3 x-\tfrac{55}{6}\ln^2 x+[4\,\mathrm{Li}_2(1-x)+12\,\mathrm{Li}_2(-x)-\tfrac{239}{6}]\ln x-12\,\mathrm{Li}_3(1-x)+12S_{1,2}(1-x)-24\,\mathrm{Li}_3(-x) \\
& +\tfrac{22}{3}\mathrm{Li}_2(1-x)+2\zeta(3)+\tfrac{22}{3}\zeta(2)-\tfrac{3155}{108}\} \\
& +4(1+x)[\mathrm{Li}_2(1-x)+\ln x\ln(1-x)]+\left(-20-4x+24x^2+\tfrac{36}{5}x^3-\frac{4}{5x^2}\right)[\mathrm{Li}_2(-x)+\ln x\ln(1+x)] \\
& +(4-20x)[\ln x\,\mathrm{Li}_2(-x)+S_{1,2}(1-x)-2\,\mathrm{Li}_3(-x)-\zeta(2)\ln(1-x)]+(\tfrac{133}{6}-\tfrac{1113}{18}x)\ln(1-x) \\
& +(-2+2x-12x^2-\tfrac{18}{5}x^3)\ln^2 x+\frac{1}{30}\left(13+1753x-216x^2+\frac{24}{x}\right)\ln x+(-2-10x+24x^2+\tfrac{36}{5}x^3)\zeta(2) \\
& -\tfrac{9687}{540}+\tfrac{59157}{540}x-\tfrac{36}{5}x^2-\frac{4}{5x}+[\tfrac{71}{5}\zeta(2)^2+\tfrac{140}{3}\zeta(3)-\tfrac{251}{3}\zeta(2)-\tfrac{5465}{72}]\delta(1-x)\Big] \\
+n_f C_F\Big(& \frac{1+x^2}{1-x}[\tfrac{2}{3}\ln^2(1-x)-(\tfrac{8}{3}\ln x+\tfrac{29}{9})\ln(1-x)-\tfrac{4}{3}\mathrm{Li}_2(1-x)+\tfrac{5}{3}\ln^2 x+\tfrac{19}{3}\ln x-\tfrac{4}{3}\zeta(2)+\tfrac{247}{54}] \\
& +\tfrac{1}{3}(1+13x)\ln(1-x)-\tfrac{1}{3}(7+19x)\ln x-\tfrac{23}{18}-\tfrac{27}{2}x+[\tfrac{4}{3}\zeta(3)+\tfrac{38}{3}\zeta(2)+\tfrac{457}{36}]\delta(1-x)\Big),
\end{aligned} \tag{9}
$$

where C_A, C_F denote the colour factors and n_f stands for the number of flavours. Here we have put $\mu^2=Q^2$. The more general case ($\mu^2\neq Q^2$) can be easily derived using renormalization group methods (see ref. [14]). In the above expression the terms of the type $\ln^i(1-x)/(1-x)$ have to be understood in the distributional sense [12]. The latter and the coefficient of the delta function can be derived from eq. (16) in ref. [13]. The second part in (8) is given by

$$
\begin{aligned}
C_2^{(2),G}(x,1) = n_f C_F\Big[& 8(1+x)^2 \\
& \times[-4S_{1,2}(-x)-4\ln(1+x)\,\mathrm{Li}_2(-x)-2\zeta(2)\ln(1+x)-2\ln x\ln^2(1+x)+\ln^2 x\ln(1+x)] \\
& +4(1-x)^2\{\tfrac{5}{6}\ln^3(1-x)-(2\ln x+\tfrac{13}{4})\ln^2(1-x)+[2\,\mathrm{Li}_2(1-x)+2\ln^2 x+4\ln x+\tfrac{7}{2}]\ln(1-x)-\tfrac{5}{12}\ln^3 x \\
& +[\mathrm{Li}_2(1-x)-4\,\mathrm{Li}_2(-x)+3\zeta(2)]\ln x-4\,\mathrm{Li}_3(1-x)-S_{1,2}(1-x)+12\,\mathrm{Li}_3(-x)+13\zeta(3)+\tfrac{13}{2}\zeta(2)\} \\
& +x^2\{\tfrac{10}{3}\ln^3(1-x)-12\ln x\ln^2(1-x)+[16\ln^2 x-16\zeta(2)]\ln(1-x)-5\ln^3 x \\
& +[12\,\mathrm{Li}_2(1-x)+20\zeta(2)]\ln x-8\,\mathrm{Li}_3(1-x)+12S_{1,2}(1-x)\} \\
& +\left(48+\tfrac{64}{3}x+\tfrac{96}{5}x^3+\frac{8}{15x^2}\right)[\mathrm{Li}_2(-x)+\ln x\ln(1+x)]+(14x-23x^2)\ln^2(1-x) \\
& +(-12x+10x^2)\ln(1-x)+(-24x+56x^2)\ln x\ln(1-x)+64x\,\mathrm{Li}_3(-x)+(-10+24x)\mathrm{Li}_2(1-x) \\
& +(-\tfrac{3}{2}+\tfrac{22}{3}x-36x^2-\tfrac{48}{5}x^3)\ln^2 x+\frac{1}{15}\left(-236+339x-648x^2-\frac{8}{x}\right)\ln x+(64x+36x^2)\zeta(3) \\
& +(-\tfrac{20}{3}x+46x^2+\tfrac{96}{5}x^3)\zeta(2)-\tfrac{647}{15}+\tfrac{239}{5}x-\tfrac{36}{5}x^2+\frac{8}{15x}\Big] \\
+n_f C_A\Big\{ & 4(1+x)^2[S_{1,2}(1-x)-2\,\mathrm{Li}_3(-x)+4S_{1,2}(-x)-2\ln x\,\mathrm{Li}_2(1-x)+4\ln(1+x)\,\mathrm{Li}_2(-x) \\
& +2\ln x\,\mathrm{Li}_2(-x)+2\zeta(2)\ln(1+x)+2\ln x\ln^2(1+x)+\ln^2 x\ln(1+x)] \\
& +8(1+2x+2x^2)\left[\mathrm{Li}_3\left(\frac{1-x}{1+x}\right)-\mathrm{Li}_3\left(-\frac{1-x}{1+x}\right)-\ln(1-x)\,\mathrm{Li}_2(-x)-\ln x\ln(1-x)\ln(1+x)\right] \\
& +\left(-24+\tfrac{80}{3}x^2-\frac{16}{3x}\right)[\mathrm{Li}_2(-x)+\ln x\ln(1+x)]+x^2[-4S_{1,2}(1-x)+16\,\mathrm{Li}_3(-x)+8\ln x\,\mathrm{Li}_2(1-x) \\
& +8\ln^2 x\ln(1+x)]+\tfrac{2}{3}(1-2x+2x^2)\ln^3(1-x)+(24x-8x^2)\ln x\ln^2(1-x) \\
& +\left(-2+36x-\tfrac{122}{3}x^2+\frac{8}{3x}\right)\ln^2(1-x)+(-4-32x+8x^2)\ln^2 x\ln(1-x) \\
& +(8-144x+148x^2)\ln x\ln(1-x)+(4+40x-8x^2)\ln(1-x)\,\mathrm{Li}_2(1-x) \\
& +(-20+24x-32x^2)\zeta(2)\ln(1-x)+\frac{1}{9}\left(-186-1362x+1570x^2+\frac{104}{x}\right)\ln(1-x) \\
& +(-4-72x+8x^2)\,\mathrm{Li}_3(1-x)+\frac{1}{3}\left(12-192x+176x^2+\frac{16}{x}\right)\mathrm{Li}_2(1-x)+\tfrac{1}{3}(10+28x)\ln^3 x \\
& +(-1+88x-\tfrac{194}{3}x^2)\ln^2 x+(-48x+16x^2)\zeta(2)\ln x+(58+\tfrac{584}{3}x-\tfrac{2090}{9}x^2)\ln x-(10+12x+12x^2)\zeta(3) \\
& +\frac{1}{3}\left(12-240x+268x^2-\frac{32}{x}\right)\zeta(2)+\tfrac{239}{9}+\tfrac{1072}{9}x-\tfrac{4493}{27}x^2+\frac{344}{27x}\Big\},
\end{aligned} \tag{5}
$$

W.L. van Neerven et al.: (1992) 79 functions 80 objects would be maximal.

- The high complexity is partly caused applying the the IBP–Method.
- x–space usually is not the best space to work in.

3 Loop Anomalous Dimensions & Wilson Coefficients

- $\Longrightarrow$ Harmonic Sums in linear representation.
- Still high complexity of terms.
- Compactification possible applying algebraic and structural relations.
- Observation : In all single scale calculations the same Basic Functions occur in the resp. weight.

- $\Longrightarrow$ Derive these Universal Functions and their complex analysis.

2. Algebraic Relations

cf. J.Blümlein, Comput. Phys. Commun. **159** (2004) 19

Number of harmonic sums up to weight w : 3^{w-1}.

Harmonic sums form a quasi-shuffle algebra through $\sqcup\!\sqcup$. (M.E. Hoffman, J. Algebraic Combin. **11** (2000) 49)

$$\begin{aligned} S_{a_1,a_2} \sqcup\!\sqcup\, S_{a_3,a_4} &= S_{a_1,a_2,a_3,a_4} + S_{a_1,a_3,a_2,a_4} + S_{a_1,a_2,a_4,a_2} \\ &+ S_{a_3,a_4,a_1,a_2} + S_{a_3,a_1,a_4,a_2} + S_{a_3,a_1,a_2,a_4} \quad etc. \end{aligned}$$

Solve all the linear equations possible for the harmonic sums $\Longrightarrow$ algebraic basis.

Let $\{a, a, a, ..., b, b, ..., ..., z, z\}$ a set of n_1 a's, n_2 b's etc. The number of basis elements corresponding to all words formed by ALL the above letters is:

$$l_n(n_1, ..., n_q) = \frac{1}{n} \sum_{d|n_i} \mu(d) \frac{(n/d)!}{(n_1/d)! ... (n_d/d)!}, \quad \sum_i n_i = n$$

(E. Witt, 1937) $\Longrightarrow$ # Lyndon words

w	1	2	3	4	5	6
$\#_c$	2	8	26	80	242	728
$\#_r$	0	1	7	23	69	183

Algebraic Relations

Observation in Quantum Field Theory :

At least up to $O(\alpha_s^3)$ the contributing harmonic sums exhibit never any index $a_k = -1$ applying a compact representation.

The number of sums of this type is

$$N_{\neg\{-1\}}(w) = \frac{1}{2}\left[\left(1-\sqrt{2}\right)^w + \left(1+\sqrt{2}\right)^w\right]$$

$$N^{\text{basic}}_{\neg\{-1\}}(\mathsf{w}) \; = \; \frac{2}{\mathsf{w}} \sum_{d|\mathsf{w}} \mu\left(\frac{\mathsf{w}}{d}\right) N^{\text{basic}}_{\neg\{-1\}}(d) \; .$$

w	1	2	3	4	5	6
$\#_c$	1	4	11	28	69	168
$\#_r$	1	3	7	14	30	60

- Here $\#_c$ is smaller than $\#_r$ in the general case.

Algebraic Relations

Remark:

Harmonic, Generalized Harmonic Polylogarithms and Multiple Polylogarithms also form shuffle algebras. As shuffle algebras are sub-sets of the quasi-shuffle algebra studied above, the respective algebraic relations can be derived directly.

- Form the index alphabet.
- Solve the shuffle-relations $\Longrightarrow$ Basis

As the relations in J.B., Comput. Phys. Commun. **159** (2004) 19 are of arbitrary weight (general alphabet) and depth $d \leq 6$ the corresponding relations can be read off there.

Algorithms to extend this scenario are available.

3. Structural Relations

w = 1:

$$\frac{1}{1-x} \qquad \& \qquad \frac{1}{1+x}$$

$$\frac{1}{1-x^2} = \frac{1}{2}\left[\frac{1}{1-x} + \frac{1}{1+x}\right]$$

$$\mathbf{M}\left[\left(\frac{1}{1-x}\right)_+\right]\left(\frac{N}{2}\right) = \mathbf{M}\left[\left(\frac{1}{1-x}\right)_+\right](N) + \mathbf{M}\left[\frac{1}{1+x}\right](N) + \ln(2)$$

$$-\psi\left(\frac{N}{2}\right) - \gamma_E \quad = \quad -\psi(N) - \gamma_E + \beta(N) + \ln(2); \qquad \beta(N) = \frac{1}{2}\left[\psi\left(\frac{N+1}{2}\right) - \psi\left(\frac{N}{2}\right)\right]$$

- $S_{-1}(N)$ depends on $S_1(N)$ for $N \,\epsilon\, \mathbf{Q}$

Structural Relations

<u>$N \epsilon \mathbf{R}$:</u>

$$S_2(N) = -\frac{d}{dN} S_1(N) + \zeta_2 \qquad \text{(etc.)}$$

For $N \epsilon \mathbf{R}$: only one independent single sum occurs.

$$S_1(N) = \sum_{k=1}^{N} \frac{1}{k} = \psi(N+1) + \gamma_E$$

Single Harmonic sums $\cup$ $\zeta_{k_1,\dots k_n}$ are closed under differentiation.

<u>w = 2:</u>

$$\mathbf{M}\left[\frac{\ln(1-x)}{1+x}\right](N) = -\mathbf{M}\left[\frac{\ln(1+x)}{1+x}\right](N) - [\psi(N) + \gamma_E + \ln(2)]\beta(N) + \beta'(N)$$

$$F_1(N) := \mathbf{M}\left[\frac{\ln(1+x)}{1+x}\right](N) \rightarrow S_{1,-1}(N)$$

Structural Relations

The relations for w $= 2$ were explored by N. Nielsen (1906).

$$\xi(N) = \mathbf{M}\left[\left(\frac{\ln(1-x)}{1+x}\right)_+\right](N); \qquad \eta(N) = \mathbf{M}\left[\frac{\ln(1+x)-\ln(2)}{1-x}\right](N)$$

$$\xi_1(N) = \mathbf{M}\left[\frac{\ln(1+x)}{1+x}\right](N); \qquad -\xi_2(N) = \mathbf{M}\left[\frac{\ln(1-x)}{1+x}\right](N)$$

$$\begin{aligned}
[\psi(z)+\gamma_E][\psi(1-z)+\gamma_E] &= 2\zeta_2 - \xi(z) - \xi(1-z) \\
\beta(z)[\psi(z)+\gamma_E] &= \beta'(z) + \beta(z)\ln(2) - \xi_1(z) + \xi_2(z) \\
\beta(z)\beta(1-z) &= \eta(z) + \eta(1-z) \\
\beta^2(z) &= \psi'(z) - 2\eta(z)
\end{aligned}$$

Structural Relations

If half-integer arguments in N are allowed
$\mathbf{M}[\mathrm{Li}_k(-x)/(x \pm 1)](N)$ are not independent functions :

$$\frac{1}{2^{k-2}} \frac{\mathrm{Li}_k(x^2)}{1-x^2} = \frac{\mathrm{Li}_k(x)}{1-x} + \frac{\mathrm{Li}_k(x)}{1+x} + \frac{\mathrm{Li}_k(-x)}{1-x} + \frac{\mathrm{Li}_k(-x)}{1+x} \rightarrow \frac{\mathrm{Li}_k(-x)}{1-x}$$

- There always exists another IBP relation to express also $\mathrm{Li}_k(-x)/(1+x)$

$$\begin{aligned}
(-1)^N \mathbf{M}\left[\frac{\mathrm{Li}_2(-x)}{1+x}\right](N) &= -S_{2,-1}(N) - \ln(2)[S_2(N) - S_{-2}(N)] \\
&\quad -\frac{1}{2}\zeta_2 S_{-1}(N) + \frac{1}{4}\zeta_3 - \frac{1}{2}\zeta_2 \ln(2) \\
(-1)^N \mathbf{M}\left[\frac{-\mathrm{Li}_2(x) - \ln(x)\ln(1-x) + \zeta_2}{1+x}\right](N) &= -S_{-1,2}(N) + \zeta_2 S_{-1}(N) - \zeta_3 + \frac{3}{2}\zeta_2 \ln(2) \\
S_{-1,2}(N) + S_{2,-1}(N) &= S_{-1}(N) S_2(N) + S_{-3}(N)
\end{aligned}$$

Structural Relations

$$
\begin{aligned}
(-1)^{(N+1)}\mathbf{M}\left[\frac{\mathrm{Li}_3(-x)}{1+x}\right](N) &= -S_{3,-1}(N)-\ln(2)[S_3(N)-S_{-3}(N)] \\
& \quad -\frac{1}{2}\zeta_2 S_{-2}(N)+\frac{3}{4}\zeta_3 S_{-1}(N)-\frac{1}{8}\zeta_2^2+\frac{3}{4}\ln(2)\zeta_3 \\
(-1)^{N}\mathbf{M}\left[\frac{S_{1,2}(1-x)}{1+x}\right](N) &= -S_{-1,3}(N)+\zeta_3 S_{-1}(N)-\frac{19}{40}\zeta_2^2+\frac{7}{4}\zeta_3\ln(2) \\
S_{1,2}(1-x) &= -\mathrm{Li}_3(x)+\log(x)\mathrm{Li}_2(x)+\frac{1}{2}\log(1-x)\log^2(x)+\zeta_3 \\
S_{-1,3}(N)+S_{3,-1}(N) &= S_{-1}(N)S_3(N)+S_{-4}(N)
\end{aligned}
$$

- At even w there exists an algebraic relation

$$S_{\mathsf{w}/2,\mathsf{w}/2}(N)=\frac{1}{2}\left[S^2_{\mathsf{w}/2}(N)+S_{\mathsf{w}}(N)\right]$$

which yields an additional relation for $\mathrm{Li}_k(x)/(1+x)$.

<u>w = 3:</u>

$$\rightarrow \frac{\mathrm{Li}_2(x)}{x\pm 1}, \qquad \frac{\ln^2(1+x)}{x\pm 1}$$

Double Sums in General

- Applying differential operators one may show :

For $N \epsilon \mathbf{R}$ double harmonic sums can always be represented by one basic function for even weight and two basic functions for odd weight.

$$\Longrightarrow \quad \frac{\mathrm{Li}_k(x)}{1+x}, \qquad \frac{\mathrm{Li}_k(x)}{1 \pm x}$$

Examples, which reduce :

$$S_{2,3}(N) = \mathbf{M}\left[\left(\frac{\ln(x)\left[S_{1,2}(1-x)-\zeta_3\right]+3\left[S_{1,3}(1-x)-\zeta_4\right]}{x-1}\right)_+\right](N)+3\zeta_4 S_1(N)$$

$$S_{-4,-2}(N) = -\mathbf{M}\left[\left(\frac{4\mathrm{Li}_5(-x)-\ln(x)\mathrm{Li}_4(-x)}{x-1}\right)_+\right](N)$$
$$+\frac{1}{2}\zeta_2\left[S_4(N)-S_{-4}(N)\right]-\frac{3}{2}\zeta_3 S_3(N)+\frac{21}{8}\zeta_4 S_2(N)-\frac{15}{4}\zeta_5 S_1(N)$$

$$S_{1,3}(1-x) = -\mathrm{Li}_4(x)+\log(x)\mathrm{Li}_3(x)-\frac{1}{2}\log^2(x)\mathrm{Li}_2(x)-\frac{1}{6}\log^3(x)\log(1-x)+\zeta_4$$

Structural Relations

w $= 4;\ i \neq -1$:

$$\frac{\mathrm{Li}_3(x)}{x+1}, \qquad \frac{S_{1,2}(x)}{x \pm 1}$$

The Mellin transform of

$$\left(\frac{\mathrm{Li}_3(x)}{x-1}\right)_+$$

reads

$$\mathbf{M}\left[\left(\frac{\mathrm{Li}_3(x)}{x-1}\right)_+\right](N) = \frac{1}{2}\left\{\frac{d}{dN}\mathbf{M}\left[\left(\frac{\mathrm{Li}_2(x)+\zeta_2}{x-1}\right)_+\right](N)\right.$$
$$\left. -S_{2,2}(N-1) + \zeta_2 S_2(N-1) + 2\zeta_3 S_1(N-1)\right\}$$

and can be traced back to that of $(\mathrm{Li}_2(x)/(x-1))_+$

Structural Relations

$\underline{\mathrm{w} = 5;\ i \neq -1\ \ :}$

$$\frac{\mathrm{Li}_4(x)}{x \pm 1} \qquad \frac{S_{1,3}(x)}{x+1} \qquad \frac{S_{2,2}(x)}{x \pm 1} \qquad \frac{\mathrm{Li}_2^2(x)}{x+1} \qquad \frac{\ln(x) S_{1,2}(-x) - \mathrm{Li}_2^2(-x)/2}{x \pm 1}$$

$$\frac{\mathrm{Li}_2^2(x)}{x-1} \qquad \frac{S_{1,3}(x)}{x-1} \qquad [\mathrm{Occur\ in\ the\ 3-loop\ Wils.\ Coeff.\ only}]$$

$\underline{\mathrm{w} = 6;\ i \neq -1\ \ :}$

$$\frac{\mathrm{Li}_5(x)}{x+1} \qquad \frac{S_{3,2}(x)}{x \pm 1} \qquad \frac{S_{2,3}(x)}{x \pm 1} \qquad \frac{S_{1,4}(x)}{x \pm 1} \qquad \frac{\mathrm{Li}_2(x)\mathrm{Li}_3(x)}{x \pm 1}$$

$$\frac{S_{1,2}(x)\mathrm{Li}_2(x)}{x+1} \qquad \frac{A_1(x)}{x+1} \qquad \frac{A_2(x)}{x \pm 1} \qquad \frac{A_3(x)}{x+1} \qquad \frac{H_{0,-1,0,1,1}(x)}{x-1}$$

$$\left.\frac{A_1(-x) + N_\alpha(x)}{x+1}\right|_{\alpha=1..3}$$

New numerator functions :

$$A_1(x) = \int_0^x \frac{dy}{y} \mathrm{Li}_2^2(y), \qquad A_2(x) = \int_0^x \frac{dy}{y} \ln(1-y) S_{1,2}(y), \qquad A_3(x) = \int_0^x \frac{dy}{y} [\mathrm{Li}_4(1-y) - \zeta_4]$$

Harmonic Polylogarithms

- iterated integrals over the alphabet

$$f_a(x) = \frac{1}{x}, \qquad \frac{1}{1-x}, \qquad \frac{1}{1+x}$$

$$H_0(x) = \int_0^x \frac{dx}{x}, \qquad H_1(x) = \int_0^x \frac{dx}{1-x}, \qquad H_{-1}(x) = \int_0^x \frac{dx}{1+x}$$

$$H_{a,\vec{b}}(x) = \int_0^x dz f_a(z) H_{\vec{b}}(z)$$

Structural Relations

$\mathrm{w} = 5;\ i \neq -1\ \ :$

$$\frac{\mathrm{Li}_4(x)}{x \pm 1} \qquad \frac{S_{1,3}(x)}{x+1} \qquad \frac{S_{2,2}(x)}{x \pm 1} \qquad \frac{\mathrm{Li}_2^2(x)}{x+1} \qquad \frac{\ln(x) S_{1,2}(-x) - \mathrm{Li}_2^2(-x)/2}{x \pm 1}$$

$$\frac{\mathrm{Li}_2^2(x)}{x-1} \qquad \frac{S_{1,3}(x)}{x-1} \qquad [\text{Occur in the } 3-\text{loop Wils. Coeff. only}]$$

$\mathrm{w} = 6;\ i \neq -1\ \ :$

$$\frac{\mathrm{Li}_5(x)}{x+1} \qquad \frac{S_{3,2}(x)}{x \pm 1} \qquad \frac{S_{2,3}(x)}{x \pm 1} \qquad \frac{S_{1,4}(x)}{x \pm 1} \qquad \frac{\mathrm{Li}_2(x)\mathrm{Li}_3(x)}{x \pm 1}$$

$$\frac{S_{1,2}(x)\mathrm{Li}_2(x)}{x+1} \qquad \frac{A_1(x)}{x+1} \qquad \frac{A_2(x)}{x \pm 1} \qquad \frac{A_3(x)}{x+1} \qquad \frac{H_{0,-1,0,1,1}(x)}{x-1}$$

$$\frac{A_1(-x) + N_\alpha(x)}{x+1}\Big|_{\alpha=1..3}$$

New numerator functions :

$$A_1(x) = \int_0^x \frac{dy}{y} \mathrm{Li}_2^2(y), \qquad A_2(x) = \int_0^x \frac{dy}{y} \ln(1-y) S_{1,2}(y), \qquad A_3(x) = \int_0^x \frac{dy}{y} [\mathrm{Li}_4(1-y) - \zeta_4]$$

Representation of Observables

- Unpolarized and Polarized Drell-Yan an Higgs-Boson Production Cross Section $O(\alpha_s^2)$,
  w $= 4$ JB and V. Ravindran, Nucl. Phys. **B716** (2005) 128.
- Unpolarized and Polarized Time-like Anomalous Dimensions and Wilson Coefficients
  $O(\alpha_s^2)$, w $= 4$ JB and V. Ravindran, Nucl. Phys. **B749** (2006) 1.
- Anomalous Dimensions and Wilson Coefficients $O(\alpha_s^3)$, w $= 5, 6$,

  from: S. Moch, J. Vermaseren, A. Vogt, Nucl. Phys. **B688** (2004) 101; **691** (2004) 129; **B724** (2005) 3 → J.B., DESY 07-042
- Polarized and Unpolarized Wilson Coefficients $O(\alpha_s^2)$, w $= 4$ J.B. and S. Moch
- Polarized and Unpolarized asymptotic Heavy Flavor Wilson Coefficients $O(\alpha_s^{2(3)})$, w $=$ 4,5,

  J.B., A. de Freitas, W. van Neerven, S. Klein, Nucl. Phys. **B755** (2006) 272; I. Bierenbaum, J.B., S. Klein, DESY 07-026,

  DESY 07-027; DESY-08-029;
- Virtual and soft corrections to Bhabha Scattering $O(\alpha^2)$, w $= 4$,

  J.B. and S. Klein, arXiv:0706.2426 [hep-ph]

Example: Bhabha s+v

$$
\begin{aligned}
T_0 \quad = \quad & \frac{248+15\,N^2+N^4}{2(N-2)(N-1)N(N+1)(N+2)}S_{1,1,1,1}(N)+\frac{-2}{(N-1)(N+1)}\mathbf{S_{2,1,1}}(N) \\
&+\frac{-340+120\,N+17\,N^2+18\,N^3-31\,N^4}{2(N-2)(N-1)N(N+1)(N+2)}S_{3,1}(N)+\frac{1344-502\,N-69\,N^2-2\,N^3+57\,N^4}{8(N-2)(N-1)N(N+1)(N+2)}S_4(N) \\
&+\frac{304-328\,N-500\,N^2+330\,N^3-6\,N^4+6\,N^5-2\,N^6+4\,N^7}{(N-2)^2(N-1)^2N^2(N+1)(N+2)}\mathbf{S_{2,1}}(N) \\
&+\frac{-112-4\,N^2-4\,N^4}{(N-2)(N-1)N(N+1)(N+2)}S_{2,1}(N)\mathbf{S_1}(N)+\frac{-48+8\,N+6\,N^2+7\,N^3}{(N-1)N(N+1)(N+2)}S_3(N)S_1(N) \\
&+\frac{-1840+292\,N+5532\,N^2+827\,N^3-1978\,N^4-274\,N^5+36\,N^6+19\,N^7-22\,N^8}{4(N-2)^2(N-1)^2N^2(N+1)^2(N+2)}S_{1,1,1}(N) \\
&+\frac{128-56\,N-252\,N^2+54\,N^3+177\,N^4-91\,N^5+19\,N^6+9\,N^7}{2(N-2)(N-1)^2N^2(N+1)^2(N+2)}S_3(N) \\
&+\frac{4032-2048\,N-14200\,N^2+5036\,N^3+23610\,N^4+2521\,N^5-12342\,N^6}{4(N-2)^3(N-1)^3N^3(N+1)^3(N+2)}S_{1,1}(N) \\
&+\frac{-3365\,N^7+2148\,N^8+903\,N^9+14\,N^{10}-167\,N^{11}+50\,N^{12}}{4(N-2)^3(N-1)^3N^3(N+1)^3(N+2)}S_{1,1}(N) \\
&+\frac{-124+16\,N+24\,N^2-4\,N^3-14\,N^4}{(N-2)(N-1)N(N+1)(N+2)}S_{1,1}(N)\zeta(2)+\frac{424-118\,N+9\,N^2-2\,N^3+23\,N^4}{4(N-2)(N-1)N(N+1)(N+2)}S_2(N)S_{1,1}(N) \\
&+\frac{224+144\,N-1216\,N^2-56\,N^3+1786\,N^4+641\,N^5-406\,N^6}{4(N-2)^2(N-1)^3N^3(N+1)^3(N+2)}S_2(N) \\
&+\frac{+17\,N^7-308\,N^8+141\,N^9-56\,N^{10}+N^{11}}{4(N-2)^2(N-1)^3N^3(N+1)^3(N+2)}S_2(N)+\frac{58+21\,N+N^2+15\,N^3+10\,N^4}{(N-2)(N-1)N(N+1)(N+2)}S_2(N)\zeta(2)
\end{aligned}
$$

Example: Bhabha s+v

$$+\frac{232-384\,N^2-17\,N^3+286\,N^4-128\,N^5-14\,N^6+N^7}{4(N-2)(N-1)^2N^2(N+1)^2(N+2)}S_2(N)S_1(N)$$
$$+\frac{-560-26\,N-31\,N^2-10\,N^3-33\,N^4}{8(N-2)(N-1)N(N+1)(N+2)}S_2(N)^2$$
$$+\frac{576+1088\,N-3280\,N^2-5136\,N^3+11764\,N^4+20392\,N^5-17385\,N^6-30114\,N^7}{4(N-2)^3(N-1)^4N^4(N+1)^4(N+2)}S_1(N)$$
$$+\frac{+5984\,N^8+17228\,N^9-1228\,N^{10}-2754\,N^{11}-112\,N^{12}-8\,N^{13}+33\,N^{14}-24\,N^{15}}{4(N-2)^3(N-1)^4N^4(N+1)^4(N+2)}S_1(N)$$
$$+\frac{-56+336\,N+522\,N^2+424\,N^3-53\,N^4-500\,N^5+60\,N^6+28\,N^7-5\,N^8}{2(N-2)^2(N-1)^2N^2(N+1)^2(N+2)}S_1(N)\zeta(2)$$
$$+\frac{64+6\,N^2+N^3}{(N-2)(N-1)N(N+1)}S_1(N)\zeta(3)+\frac{2112+608\,N+76\,N^2-140\,N^3+107\,N^4}{10(N-2)(N-1)N(N+1)(N+2)}\zeta(2)^2$$
$$+\frac{-224-136\,N+1688\,N^2+1290\,N^3-1998\,N^4-1997\,N^5+198\,N^6}{2(N-1)^3N^3(N-2)^2(N+2)(N+1)^3}\zeta(2)$$
$$+\frac{+405\,N^7+376\,N^8-119\,N^9+56\,N^{10}+5\,N^{11}}{2(N-1)^3N^3(N-2)^2(N+2)(N+1)^3}\zeta(2)$$
$$+\frac{-552+144\,N+1654\,N^2-370\,N^3-361\,N^4+19\,N^5+35\,N^6-25\,N^7}{2(N-2)^2(N-1)^2N^2(N+1)^2}\zeta(3)$$
$$+\frac{320-64\,N-1920\,N^2+1600\,N^3+6524\,N^4-14872\,N^5-19036\,N^6+31543\,N^7-43960\,N^8-13935\,N^9}{16(N-1)^5(N+1)^5(N-2)^3N^5(N+2)}$$
$$+\frac{+65372\,N^{10}+26822\,N^{11}-44576\,N^{12}-9558\,N^{13}+9840\,N^{14}+339\,N^{15}+428\,N^{16}-371\,N^{17}+128\,N^{18}}{16(N-1)^5(N+1)^5(N-2)^3N^5(N+2)}$$
$$+4\frac{N^4-N^2+12}{(N-2)(N-1)N(N+1)(N+2)}f_{0,2}+(-2)\frac{N^4-N^2+12}{(N-2)(N-1)N(N+1)(N+2)}f_{0,1}^2$$

z-space: A. Penin, (2005)

$\Longrightarrow$ 3 basic sums only; no alternating sums.

5. Factorial Series

Consider

$$\Omega(z) = \int_0^1 dt\ t^{z-1}\ \varphi(t); \qquad \varphi(1-t) = \sum_{k=0}^{\infty} a_k t^k$$

$$Re(z) > 0, \qquad \Omega(z) = \sum_{k=0}^{\infty} \frac{a_{k+1} k!}{z(z+1)\dots(z+k)}$$

- $\Omega(z)$ is meromorphic in $z \epsilon \mathbf{C}$, obeys a recursion $z \rightarrow z+1$ and has an analytic asymptotic representation.
- The poles are situated at the non-positive integers.

<u>Examples:</u>

$$\begin{aligned}
F_5(z) &= \mathbf{M}\left[\frac{\mathrm{Li}_2(z)}{1+z}\right](z) \\
F_5(z+1) &= -F_5(z) + \frac{1}{z}\left[\zeta_2 - \frac{\psi(z+1)+\gamma_E}{z}\right] \\
\text{Asymp. ser. : } \mathrm{Li}_2(z) &\rightarrow -\mathrm{Li}_2(1-z) - \ln(z)\ln(1-z) + \zeta_2 \\
\mathbf{M}\left[\frac{\mathrm{Li}_2(1-z)}{1+z}\right](N) &\sim \frac{1}{2N^2} + \frac{1}{4N^3} - \frac{7}{24}\frac{1}{N^4} - \frac{1}{3}\frac{1}{N^5} + \frac{73}{120}\frac{1}{N^6}\dots
\end{aligned}$$

Factorial Series

$$
\begin{aligned}
F_{13}(z) &= \mathbf{M}\left[\left(\frac{\mathrm{Li}_2^2(z)}{z-1}\right)_+\right](z) \\
F_{13}(z+1) &= -F_{13}(z) + \frac{\zeta_2^2}{z} + \frac{4\zeta_3}{z^2} + \frac{2\zeta_2}{z^2} S_1(z) + \frac{2S_{2,1}(z)}{z^2} + \frac{2}{z^3}\left[S_1^2(z) + S_2(z)\right] \\
\text{Asymp. ser.}: \mathrm{Li}_2^2(z) &\rightarrow \mathrm{Li}_2^2(1-z) + \ln^2(z)\ln^2(1-z) + \zeta_2^2 + 2\mathrm{Li}_2(1-z)\ln(z)\ln(1-z) + \dots \\
\mathbf{M}\left[\left(\frac{\mathrm{Li}_2^2(1-z)}{z-1}\right)_+\right](N) &\sim \frac{1}{z^2} - \frac{7}{24}\frac{1}{z^4} + \frac{1}{12}\frac{1}{z^5} + \frac{223}{1080}\frac{1}{z^6} - \frac{7}{45}\frac{1}{z^7} - \frac{3767}{15120}\frac{1}{z^8} + \frac{38}{105}\frac{1}{z^9} \\
&\quad + \frac{14327}{31500}\frac{1}{z^{10}} - \frac{198}{175}\frac{1}{z^{11}} - \frac{138673}{118800}\frac{1}{z^{12}} + \frac{3263}{693}\frac{1}{z^{13}} + \frac{5265804043}{1324323000}\frac{1}{z^{14}} \\
&\quad - \frac{13399637}{525525}\frac{1}{z^{15}} - \frac{143341487}{8408400}\frac{1}{z^{16}} + \frac{25092}{143}\frac{1}{z^{17}} + \frac{34809672614}{402026625}\frac{1}{z^{18}} \\
&\quad - \frac{5749693892}{3828825}\frac{1}{z^{19}} + O\left(\frac{1}{z^{20}}\right)
\end{aligned}
$$

6. The Basis

$w=1$	$1/(x-1)_+$		
$w=2$	$\ln(1+x)/(x+1)$		
$w=3$	$\mathrm{Li}_2(x)/(x\pm 1)$		
$w=4$	$\mathrm{Li}_3(x)/(x+1)$	$S_{1,2}(x)/(x\pm 1)$	
$w=5$	$\mathrm{Li}_4(x)/(x\pm 1)$	$S_{1,3}(x)/(x\pm 1)$	$S_{2,2}(x)/(x\pm 1)$
	$\mathrm{Li}_2^2(x)/(x\pm 1)$	$[\ln(x)S_{1,2}(-x)-\mathrm{Li}_2^2(-x)/2]/(x\pm 1)$	
$w=6$	$\mathrm{Li}_5(x)/(x+1)$	$S_{1,4}(x)/(x\pm 1)$	$S_{2,3}(x)/(x\pm 1)$
	$S_{3,2}(x)/(x\pm 1)$	$\mathrm{Li}_2(x)\mathrm{Li}_3(x)/(x\pm 1)$	$S_{1,2}(x)\mathrm{Li}_2(x)/(x+1)$
	$A_1(x)/(x+1)$	$A_2(x)/(x\pm 1)$	$A_3(x)/(x+1)$
	$H_{0,-1,0,1,1}(x)/(x-1)$	$[A_1(-x)+N_\alpha(x)]/(x+1)\vert_{\alpha=1..3}$	

- $O(\alpha)$ Wilson Coefficients/anom. dim. #1
- $O(\alpha^2)$ Anomalous Dimensions #2
- $O(\alpha^2)$ Wilson Coefficients $\# \leq 5$
- $O(\alpha^3)$ Anomalous Dimensions #15
- $O(\alpha^3)$ Wilson Coefficients #35

7. News about alternating MZV's

Evaluated to :

multiple ζ-values:

Bigotte et al.	1998	weight 12
Broadhurst	2000	weight 9
Vermaseren	2000	weight 9
Minh, Petitot	2000	weight 10
Vermaseren	2003	weight 16
Minh, Petitot	2003	weight 16
J.B., D. Broadhurst, J. Vermaseresn	2007	weight 18

$N = 2$ colored multiple ζ-values:

Gastmans & Troost	1981	weight 4
J.B. & S. Kurth	1998	weight 4 completed
Vermaseren	2000	weight 9
Bigotte et al.	2002	weight 7
J.B., D. Broadhurst, J. Vermaseresn	2007	weight 12

News about alternating MZV's

Basic Numbers : (non–alternating case)

- Kreimer & Broadhurst, 1996, D. Zagier conjecture :

w	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$\#_b$	0	1	1	0	1	0	1	1	1	1	2	2	3	3	4	5	7	8
$\#_c$	0	1	2	2	3	3	4	5	6	7	9	11	14	17	21	26	33	41

$N = 18$ 131.072 initial constants

$$N(w) = \frac{1}{w} \sum_{d|w} \mu\left(\frac{w}{d}\right) P_d, \quad \text{Perrin numbers.}$$

$$P_1 = 0, P_2 = 2, P_3 = 3; \quad P_d = P_{d-2} + P_{d-3}, d \geq 3$$

$$\zeta_2; \ \zeta_3; \ \zeta_5; \ \zeta_7; \ldots$$

The above conjecture is true up to $N = 18$

News about alternating MZV's

- Broadhurst, 1996 conjecture :

Basic Numbers : (alternating case)

w	1	2	3	4	5	6	7	8	9	10	11	12
$\#_b$	2	1	1	1	2	2	4	5	8	11	18	25
$\#_c$	2	3	4	5	7	9	13	18	26	37	55	80

$$(\sigma_1(\infty),\ \ln(2));\ \zeta_2;\ \zeta_3;\ \mathrm{Li}_4(1/2);\ (\zeta_5,\ \mathrm{Li}_5(1/2));$$

$$(\mathrm{Li}_6(1/2),\ \sigma_{-5,-1});(\zeta_7,\ \mathrm{Li}_7(1/2),\ \sigma_{-5,1,1},\ \sigma_{5,-1,-1});$$

$$(\mathrm{Li}_8(1/2),\ \sigma_{5,3},\ \sigma_{-7,-1},\ \sigma_{-5,-1,-1,-1},\ \sigma_{-5,-1,1,1};);$$

$$(\zeta_9, \mathrm{Li}_9(1/2),\ \sigma_{7,-1,-1},\ \sigma_{-7,-1,1},\ \sigma_{-6,-2,-1},\ \sigma_{-5,-1,1,1,1},\ \sigma_{-5,-1,-1,-1,1},\ \sigma_{-5,-1,-1,1,-1};);\ldots$$

The above conjecture is true up to $N = 12$

The original systems contains 354.294 Variables.

The data base has a size of $\sim$ 2.4 Gbyte

News about alternating MZV's

w = 12 Result: $\longrightarrow$ "MZV Data Mine"

w = 20, d=4 solved as well + a series of cases within between.

• Which relations were used ?

$\Longrightarrow$ shuffle, stuffle, argument doubling relations + more special relations

• shuffle :

$$\int_0^1 d\Omega_1 \ldots \Omega_n \int_0^1 d\Omega_{n+1} \ldots \Omega_{n+m} = \sum \int_0^1 \Omega_{\sigma(1)} \ldots \Omega_{\sigma(n+m)}$$

Example:

$$\zeta(2,1)\zeta(2) = 6\zeta(3,1,1) + 3\zeta(2,2,1) + \zeta(2,1,1)$$

• stuffle :Example:

$$\zeta(r,s)\zeta(t) = \zeta(r,s,t) + \zeta(r,s\&t) + \zeta(r,t,s) + \zeta(r\&t,s) + \zeta(t,r,s); \quad a\&b = sign(a)sign(b)|a| \cdot |b|$$

News about alternating MZV's

- doubling :

$$S_{n_1,\ldots,n_p}(N) = 2^{n_1+\ldots+n_p-p} \sum_{\pm} S_{\pm n_1,\ldots \pm n_p}(2N), \qquad N \to \infty$$

- more special relations:

based on partial fractioning in different ways; lengthy expressions.

The latter relations were instrumental to express all alternating MZV's in terms of a basis counting á la Broadhurst, Kreimer, Zagier to $w = 12$

News about alternating MZV's

Solutions at fixed Depth (non–alternating case)

$d = 4$ up to $w = 20$ initial constants = 16950 $\Longrightarrow$ 45 constants Output: $\sim$ 1 Gbyte
$d = 5$ up to $w = 17$ initial constants = 68226 $\Longrightarrow$ 55 constants Output: $\sim$ 2 Gbyte
$d = 6$ up to $w = 12$ initial constants = 43254 $\Longrightarrow$ 20 constants Output: $\sim$ .14 Gbyte

$$\begin{aligned} 2^5 \cdot 3^3 \zeta_{4,4,2,2} \quad = \quad & 2^5 \cdot 3^2 \zeta_3^4 + 2^6 \cdot 3^3 \cdot 5 \cdot 13 \cdot \zeta_9 \cdot \zeta_3 + 2^6 \cdot 3^3 \cdot 7 \cdot 13 \zeta_7 \cdot \zeta_5 + 2^7 \cdot 3^5 \cdot \zeta_7 \cdot \zeta_3 \cdot \zeta_2 \\ & + 2^6 \cdot 3^5 \cdot \zeta_5^2 \cdot \zeta_2 - 2^6 \cdot 3^3 \cdot 5 \cdot 7 \cdot \zeta_5 \cdot \zeta_4 \cdot \zeta_3 - 2^8 \cdot 3^2 \cdot \zeta_6 \cdot \zeta_3^2 \\ & - \frac{13177 \cdot 15991}{691} \zeta_{12} + 2^4 \cdot 3^3 \cdot 5 \cdot 7 \cdot \zeta_{6,2} \cdot \zeta_4 \\ & - 2^7 \cdot 3^3 \cdot \zeta_{8,2} \cdot \zeta_2 - 2^6 \cdot 3^2 \cdot 11^2 \cdot \zeta_{10,2} + 2^{14} \zeta_{-9,-3} \end{aligned}$$

This relation, numerically conjectured by Broadhurst (1996), and other relations of this kind could be proven.

News about alternating MZV's

Testable Suggestions & Conjectures are Very Welcome to exploit the MZV Data Mine !

8. Conclusions

- We considered mathematical structures which determine no scale and single scale quantities in Quantum Field Theories.
- The former correspond to integrated cross sections, expansion coefficients of the β–function, or anomalous dimensions at fixed moments, etc.
- The latter correspond to differential scattering cross sections of one variable, N–dependent anomalous dimensions, coefficient functions, etc.
- The single-scale quantities in Quantum Field Theories to 3 Loop Order $\Leftrightarrow$ $\mathrm{w} = 6$ can be represented in a polynomial ring spanned by a few Mellin transforms of the above basic functions, which are the same for all known processes. This points to their general nature.
- The basic Mellin transforms are meromorphic functions with single poles at the non-positive integers.
- The total amount of harmonic sums reduces due to algebraic relations [index structure], and structural relations $\mathrm{N} \in \mathbf{Q}$, $\mathrm{N} \in \mathbf{R}$.

- They can be represented in terms of factorial series up to simple "soft components". This allows an exact analytic continuation.
- Up to $w = 6$ physical (pseudo-) observables are free of harmonic sums with index $= \{-1\}$. Up to $w = 5$ all numerator functions are Nielsen integrals.
- We calculated all (alternating) MZV's up to weight w=12 and selected depths to higher weight, e.g. w=20 d=4.
- The Fibonacci-type counting of the basis elements requires more relations than just the stuffle, shuffle, and argument doubling relations.
- Several relation conjectured by D. Broadhurst with numerical methods (PSLQ) were confirmed algebraically.
- The data base should contain more non-trivial new relations.