Mathematical Structure of QCD Wilson Coefficients and Anomalous Dimensions

Johannes Blümlein

DESY

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1. Introduction

Consider hard scattering processes in massless field theories:

QCD, QED, $m_i \to 0$

Factorization Theorem Leading Twist:

The cross section σ factorizes as

$$\sigma = \sum_k \sigma_{k,W} \otimes f_k$$

σ_W perturbative Wilson Coefficient

f non-perturbative Parton Density

$\otimes$ Mellin convolution

$$\begin{aligned} [A \otimes B](x) &= \int_0^1 dx_1 \int_0^1 dx_2 \delta(x - x_1 x_2) A(x_1) B(x_2) \\ \mathbf{M}[A \otimes B](N) &= \mathbf{M}[A](N) \cdot \mathbf{M}[B](N) \end{aligned}$$

with the Mellin transform :

$$\mathbf{M}[f(x)](N) = \int_0^1 dx x^{N-1} f(x), \quad Re[N] > c$$

Observation :

Feynman Amplitudes seem to obey the **Mellin Symmetry**

i.e. to significantly simplify in **Mellin Space**

2. x Space Results

Usual Starting Point of Higher Order Calculations :

$\Longrightarrow$ Nielsen type Integrals and their Generalization

$$S_{n,p,q}(x) = \frac{(-1)^{n+p+q-1}}{(n-1)!p!q!} \int_0^1 \frac{dz}{z} \ln^{(n-1)}(z) \ln^p(1-zx) \ln^q(1+zx)$$

Special Cases:

$$\begin{aligned}
\mathrm{Li}_n(x) &= S_{n-1,1}(x) & w = n \\
\frac{d\mathrm{Li}_2(\pm x)}{d\ln(x)} &= -\ln(1 \mp x) & w = 1 \\
\mathrm{Li}_0(x) &= \frac{x}{1-x} & w = 0
\end{aligned}$$

van Neerven, Zijlstra 1992

$$c^{(2)}_{2,-}(x) = C_F\,(C_F - C_A/2) \times$$
$$\left\{\frac{1+x^2}{1-x}\left[\left[4\ln^2(x) - 16\ln(x)\ln(1+x) - 16\mathrm{Li}_2(-x) - 8\zeta_2\right]\ln(1-x)\right.\right.$$
$$+\left[-2\ln^2(x) + 20\ln(x)\ln(1+x) - 8\ln^2(1+x) + 8\mathrm{Li}_2(1-x) + 16\mathrm{Li}_2(-x) - 8\right]\ln(x)$$
$$-16\ln(1+x)\mathrm{Li}_2(-x) - 8\zeta_2\ln(1+x) - 16\left[\mathrm{Li}_3\left(-\frac{1-x}{1+x}\right) - \mathrm{Li}_3\left(\frac{1-x}{1+x}\right)\right]$$
$$\left.-16\mathrm{Li}_2(1-x) + 8S_{1,2}(1-x) + 8\mathrm{Li}_3(-x) - 16S_{1,2}(-x) + 8\zeta_3\right]$$
$$+(4+20x)\left[\ln^2(x)\ln(1+x) - 2\ln(x)\ln^2(1+x) - 2\zeta_2\ln(1+x) - 4\ln(1+x)\mathrm{Li}_2(-x)\right.$$
$$\left.+2\mathrm{Li}_3(-x) - 4S_{1,2}(-x) + 2\zeta_3\right] + \left(32 + 32x + 48x^2 - \frac{72}{5}x^3 + \frac{8}{5x^2}\right)$$
$$\times\left[\mathrm{Li}_2(-x) + ln(x)\ln(1+x)\right] + 8(1+x)\left[\mathrm{Li}_s(1-x) + \ln(x)\ln(1-x)\right] + 16(1-x)\ln(1-x)$$
$$+\left(-4 - 16x - 24x^2 + \frac{36}{5}x^3\right)\ln^2(x) + \frac{1}{5}\left(-26 - 106x + 72x^2 - \frac{8}{x}\right)\ln(x)$$
$$\left.+\left(-4 + 20x + 48x^2 - \frac{72}{5}x^3\right)\zeta_2 + \frac{1}{5}\left(-162 + 82x + 72x^2 + \frac{8}{x}\right)\right\}$$

.... several other pages for $c_2^{(+)}(x), c_2^G(x), c_L^{(q,G)}(x)$

$\Longrightarrow$ 77 Functions @ 2 Loops

- $\Longrightarrow$ partly rather complicated arguments
- $\Longrightarrow$ relations are not directly visible ...

The 77 functions do roughly correspond in number to the number of all possible harmonic sums up to weight w=4: 80.

x Space Results

No.	$f(z)$	$M[f](N) = \int_0^1 dz z^{N-1} f(z)$
1	$\delta(1-z)$	1
2	z^r	$\frac{1}{N+r}$
3	$\left(\frac{1}{1-z}\right)_+$	$-S_1(N-1)$
4	$\frac{1}{1+z}$	$(-1)^{N-1}[\log(2) - S_1(N-1)]$ $+\frac{1+(-1)^{N-1}}{2} S_1\left(\frac{N-1}{2}\right) - \frac{1-(-1)^{N-1}}{2} S_1\left(\frac{N-2}{2}\right)$
5	$z^r \log^n(z)$	$\frac{(-1)^n}{(N+r)^{n+1}} \Gamma(n+1)$
6	$z^r \log(1-z)$	$-\frac{S_1(N+r)}{N+r}$
7	$z^r \log^2(1-z)$	$\frac{S_1^2(N+r) + S_2(N+r)}{N+r}$
8	$z^r \log^3(1-z)$	$-\frac{S_1^3(N+r) + 3S_1(N+r)S_2(N+r) + 2S_3(N+r)}{N+r}$
9	$\left[\frac{\log(1-z)}{1-z}\right]_+$	$\frac{1}{2} S_1^2(N-1) + \frac{1}{2} S_2(N-1)$
10	$\left[\frac{\log^2(1-z)}{1-z}\right]_+$	$-\left[\frac{1}{3} S_1^3(N-1) + S_1(N-1)S_2(N-1) + \frac{2}{3} S_3(N-1)\right]$
11	$\left[\frac{\log^3(1-z)}{1-z}\right]_+$	$\frac{1}{4} S_1^4(N-1) + \frac{3}{2} S_1^2(N-1)S_2(N-1)$ $+\frac{3}{4} S_2^2(N-1) + 2S_1(N-1)S_3(N-1)$ $+\frac{3}{2} S_4(N-1)$
12	$\frac{\log^n(z)}{1-z}$	$(-1)^{n+1}\Gamma(n+1)\left[S_{n+1}(N-1) - \zeta(n+1)\right]$

Only single sums.

No.	$f(z)$	$M[f](N)$
64	$\dfrac{\mathrm{Li}_3(-z)}{1+z}$	$(-1)^{N-1}\Big\{S_{3,-1}(N-1)+\Big[S_3(N-1)-S_{-3}(N-1)\Big]\log 2$ $+\frac{1}{2}\zeta(2)S_{-2}(N-1)-\frac{3}{4}\zeta(3)S_{-1}(N-1)$ $+\frac{1}{8}\zeta^2(2)-\frac{3}{4}\zeta(3)\log 2\Big\}$
65	$\mathrm{Li}_3(1-z)$	$\frac{1}{N}\left[S_1(N)S_2(N)-\zeta(2)S_1(N)+S_3(N)\right.$ $\left.-S_{2,1}(N)+\zeta(3)\right]$
66	$\dfrac{\mathrm{Li}_3(1-z)}{1-z}$	$-S_{1,1,2}(N-1)+\frac{1}{2}\zeta(2)S_1^2(N-1)+\frac{1}{2}\zeta(2)S_2(N-1)$ $-\zeta(3)S_1(N-1)+\frac{2}{5}\zeta^2(2)$
67	$\dfrac{\mathrm{Li}_3(1-z)}{1+z}$	$(-1)^{N-1}\Big[S_{-1,1,2}(N-1)-\zeta(2)S_{-1,1}(N-1)$ $+\zeta(3)S_{-1}(N-1)+\mathrm{Li}_4\left(\frac{1}{2}\right)-\frac{9}{20}\zeta^2(2)$ $+\frac{7}{8}\zeta(3)\log 2+\frac{1}{2}\zeta(2)\log^2 2+\frac{1}{24}\log^4 2\Big]$
68	$\mathrm{Li}_3\left(\dfrac{1-z}{1+z}\right)$ $-\mathrm{Li}_3\left(-\dfrac{1-z}{1+z}\right)$	$\frac{(-1)^N}{N}\Big[-S_{-1,2}(N)-S_{-2,1}(N)+S_1(N)S_{-2}(N)$ $+S_{-3}(N)$ $+\zeta(2)S_{-1}(N)+\frac{1}{2}\zeta(2)S_1(N)-\frac{7}{8}\zeta(3)+\frac{3}{2}\zeta(2)\log 2\Big]$ $+\frac{1}{N}\Big[-S_{-1,-2}(N)-S_{2,1}(N)+S_1(N)S_2(N)+S_3(N)$ $-\frac{1}{2}\zeta(2)S_{-1}(N)-\zeta(2)S_1(N)+\frac{21}{8}\zeta(3)-\frac{3}{2}\zeta(2)\log 2\Big]$
69	$\dfrac{1}{1+z}\left[\mathrm{Li}_3\left(\dfrac{1-z}{1+z}\right)\right.$ $\left.-\mathrm{Li}_3\left(-\dfrac{1-z}{1+z}\right)\right]$	$(-1)^{N-1}\Big\{S_{1,1,-2}(N-1)-S_{1,-1,2}(N-1)$ $+S_{-1,1,2}(N-1)-S_{-1,-1,-2}(N-1)$ $+2\zeta(2)S_{1,-1}(N-1)+\frac{1}{4}\zeta(2)S_1^2(N-1)-\frac{1}{4}\zeta(2)S_{-1}^2(N-1)$ $-\zeta(2)S_1(N-1)S_{-1}(N-1)-\zeta(2)S_{-2}(N-1)$ $-\left[\frac{7}{8}\zeta(3)-\frac{3}{2}\zeta(2)\log 2\right]S_1(N-1)$ $+\left[\frac{21}{8}\zeta(3)-\frac{3}{2}\zeta(2)\log 2\right]S_{-1}(N-1)$ $-2\mathrm{Li}_4\left(\frac{1}{2}\right)+\frac{19}{40}\zeta^2(2)+\frac{1}{2}\zeta(2)\log^2 2-\frac{1}{12}\log^4 2\Big\}$

2 loop coefficient functions $\Longrightarrow$ Nested Harmonic Sums of Weight $w=4$

x Space Results

$$S_{-1,-1,-2}(N) =$$
$$(-1)^{N+1}\mathbf{M}\left\{\frac{1}{1+x}\left[F_1(x)+\log(1-x)\mathrm{Li}_2(-x)\right]\right\}(N)$$
$$+(-1)^{N+1}\mathbf{M}\left\{\frac{1}{1+x}\left[\frac{1}{2}\mathrm{S}_{1,2}(x^2)-\mathrm{S}_{1,2}(x)-\mathrm{S}_{1,2}(-x)\right]\right\}(N)$$
$$+\frac{1}{2}\zeta(2)\left[S_{-1,1}(N)-S_{-1,-1}(N)\right]$$
$$+\left[\frac{9}{8}\zeta(3)-\frac{3}{2}\zeta(2)\log(2)-\frac{1}{6}\log^3(2)\right]S_{-1}(N)$$
$$-\frac{1}{10}\zeta(2)^2+\frac{17}{8}\zeta(3)\log(2)-\frac{7}{4}\zeta(2)\log^2(2)-\frac{1}{6}\log^4(2)$$

with

$$\begin{aligned} F_1(x) \quad = \quad & S_{1,2}\left(\frac{1-x}{2}\right)+S_{1,2}(1-x)-S_{1,2}\left(\frac{1-x}{1+x}\right) \\ & +S_{1,2}\left(\frac{1}{1+x}\right)-\ln(2)\left(\frac{1-x}{2}\right) \\ & +\frac{1}{2}\ln^2(2)\ln\left(\frac{1+x}{2}\right)-\ln(2)\mathrm{Li}_2\left(\frac{1-x}{1+x}\right) \end{aligned}$$

$F_1(x)$, although of complicated structure, it reduces completely via algebraic relations

$\Rightarrow$ Mellin polynomial of simpler objects

These objects can be very complicated integrals. J.B., van Neerven, Ravindran, Kawamura 2000, 2003

3. Multiple Harmonic Sums to Level 6

The simplest example :

$$P_{qq}(x) = \left(\frac{1+x^2}{1-x}\right)_+ = \frac{2}{(1-x)_+} + \dots$$

$$\int_0^1 dx \frac{x^{N-1}}{(1-x)_+} = -\sum_{k=0}^{N-2} \int_0^1 dx x^k = -\sum_{k=1}^{N-1} \frac{1}{k} = -S_1(N-1)$$

Alternating sums :

$$S_{-1}(N-1) = (-1)^{N-1}\mathbf{M}\left[\frac{1}{1+x}\right](N) - \ln(2) = \int_0^1 dx \frac{x^{N-1}}{(1-x)_+} = \sum_{k=1}^{N-1} \frac{(-1)^k}{k}$$

(Finite for $N \to \infty$.)

General case :

$$S_{a_1,\ldots,a_l}(N) = \sum_{k_1=1}^{N} \frac{(\mathrm{sign}(a_1))^{k_1}}{k_1^{|a_1|}} \sum_{k_2=1}^{k_1} \frac{(\mathrm{sign}(a_2))^{k_2}}{k_2^{|a_2|}} \dots$$

Vermaseren, 1997

All Mellin transforms occurring in massless Field Theories for 1-Parameter Quantities can be represented by Harmonic Sums

(at least to 3–loop order).

Algebraic Relations

First relation:

L. Euler, 1775

$$S_{m,n} + S_{n,m} = S_m \cdot S_n + S_{m+n}, \qquad m, n > 0$$

Generalized to alternating sums by

$$\begin{aligned} S_{m,n} + S_{n,m} &= S_m \cdot S_n + S_{m \wedge n}, \\ m \wedge n &= [|m| + |n|]\, \mathrm{sign}(m)\mathrm{sign}(n) \end{aligned}$$

Ternary relations: Sita Ramachandra Rao, 1984,

4-ary relation: J.B., Kurth, 1998.

These & other relations hold widely independent
of their **Value** and **Type**.

Determined by :
- Index Structure
- Multiplication Relation

The Formalism applies as well to the Harmonic Polylogarithms.
Remiddi, Vermaseren, 1999.

Application to QED: T. Riemann et al., 2004

J.B., Comput. Phys. Commun. **159** (2004) 19.

Linear Representations of Mellin Transform by Harmonic Sums:

$$\mathbf{M}[F_w(x)](N) = S^w_{k_1,\ldots,k_m}(N) + P\left(S^{\tau'}_{k_1,\ldots,k_r}, \sigma^{\tau''}_{k_1,\ldots,k_p}\right)$$

$w = \sum_{i=1}^m |k_i|$ Weight

$\tau', \tau'' < w$ P is a polynomial.

w	#	Σ	
1	2	2	
2	6	8	
3	18	26	2 Loop anom. Dimensions
4	54	80	2 Loop Wilson Coefficients
5	162	242	3 Loop anom. Dimensions
6	486	728	3 Loop Wilson Coefficients
	$2 \cdot 3^{w-1}$	$3^w - 1$	

Shuffle Products

Depth 2:

$$S_{a_1}(N) ⧢ S_{a_2}(N) = S_{a_1,a_2}(N) + S_{a_2,a_1}(N)$$

Depth 3:

$$S_{a_1}(N) ⧢ S_{a_2,a_3}(N) = S_{a_1,a_2,a_3}(N) + S_{a_2,a_1,a_3}(N) + S_{a_2,a_3,a_1}(N)$$

Depth 4:

$$\begin{aligned}
S_{a_1}(N) ⧢ S_{a_2,a_3,a_4}(N) &= S_{a_1,a_2,a_3,a_4}(N) + S_{a_2,a_1,a_3,a_4}(N) + S_{a_2,a_3,a_1,a_4}(N) \\
&+ S_{a_2,a_3,a_4,a_1}(N) \\
S_{a_1,a_2}(N) ⧢ S_{a_3,a_4}(N) &= S_{a_1,a_2,a_3,a_4}(N) + S_{a_1,a_3,a_2,a_4}(N) + S_{a_1,a_3,a_4,a_2}(N) \\
&+ S_{a_3,a_4,a_1,a_2}(N) + S_{a_3,a_1,a_4,a_2}(N) + S_{a_3,a_1,a_2,a_4}(N)
\end{aligned}$$

Depth 5:

$$\begin{aligned}
S_{a_1}(N) ⧢ S_{a_2,a_3,a_4,a_5}(N) &= S_{a_1,a_2,a_3,a_4,a_5}(N) + S_{a_2,a_1,a_3,a_4,a_5}(N) \\
&+ S_{a_2,a_3,a_1,a_4,a_5}(N) + S_{a_2,a_3,a_4,a_1,a_5}(N) \\
&+ S_{a_2,a_3,a_4,a_5,a_1}(N) \\
S_{a_1,a_2}(N) ⧢ S_{a_3,a_4,a_5}(N) &= S_{a_1,a_2,a_3,a_4,a_5}(N) + S_{a_1,a_3,a_2,a_4,a_5}(N) \\
&+ S_{a_1,a_3,a_4,a_2,a_5}(N) + S_{a_1,a_3,a_4,a_5,a_2}(N) \\
&+ S_{a_3,a_1,a_2,a_4,a_5}(N) + S_{a_3,a_1,a_4,a_2,a_5}(N) \\
&+ S_{a_3,a_1,a_4,a_5,a_2}(N) + S_{a_3,a_4,a_5,a_1,a_2}(N) \\
&+ S_{a_3,a_4,a_1,a_5,a_2}(N) + S_{a_3,a_4,a_1,a_2,a_5}(N)
\end{aligned}$$

Depth 6: .....

Algebraic Equations

Depth 2:

$$S_{a_1}(N) ⧢ S_{a_2}(N) - S_{a_1}(N) S_{a_2}(N) - S_{a_1 \wedge a_2}(N) = 0$$

Depth 3:

$$S_{a_1}(N) ⧢ S_{a_2,a_3}(N) \quad - \quad S_{a_1}(N) S_{a_2,a_3}(N) - S_{a_1 \wedge a_2, a_3}(N) - S_{a_2, a_1 \wedge a_3}(N) = 0$$

Depth 4:

$$\begin{aligned} S_{a_1}(N) ⧢ S_{a_2,a_3,a_4}(N) \quad &- \quad S_{a_1}(N) S_{a_2,a_3,a_4}(N) - S_{a_1 \wedge a_2, a_3, a_4}(N) \\ &- \quad S_{a_2, a_1 \wedge a_3, a_4}(N) - S_{a_2, a_3, a_1 \wedge a_4}(N) = 0 \\ S_{a_1,a_2}(N) ⧢ S_{a_3,a_4}(N) \quad &- \quad S_{a_1,a_2}(N) S_{a_3,a_4}(N) - S_{a_1, a_2 \wedge a_3, a_4}(N) \\ &- \quad S_{a_1, a_3, a_2 \wedge a_4}(N) - S_{a_3, a_1 \wedge a_4, a_2}(N) \\ &- \quad S_{a_3, a_1, a_2 \wedge a_4}(N) - S_{a_1 \wedge a_3, a_2, a_4}(N) \\ &- \quad S_{a_1 \wedge a_3, a_4, a_2}(N) + S_{a_1 \wedge a_3, a_2 \wedge a_4} = 0 \end{aligned}$$

Depth 5: .....

Basic Sums = # Permutations - # Independent Equations

Some Solution for $d = 6$

$$
\begin{aligned}
&S_{a,a,a,a,b,b} = \\
&-\frac{1}{4}S_aS_{b,a,a,a,b} + \frac{3}{4}S_{a\wedge b,a,a,a,b} - \frac{1}{4}S_{b,a,a,a,a\wedge b} + \frac{1}{12}S_aS_{a,a,b,b,a} \\
&+S_{a,a,a,a,b\wedge b} - \frac{1}{12}S_{a\wedge a,b,b,a,a} - \frac{1}{12}S_{a,b,b,a\wedge a,a} - \frac{1}{12}S_{a,b,b,a,a\wedge a} \\
&-\frac{1}{4}S_{b,a\wedge a,a,a,b} - \frac{1}{4}S_{b,a,a\wedge a,a,b} - \frac{1}{4}S_{b,a,a,a\wedge a,b} - \frac{1}{4}S_{a,a\wedge a,a,b,b} \\
&-\frac{1}{4}S_{a,a,a\wedge a,b,b} - \frac{1}{4}S_{a,b,a\wedge a,a,b} - \frac{1}{4}S_{a,b,a,a\wedge a,b} + \frac{1}{12}S_{a\wedge a,b,a,b,a} \\
&+\frac{1}{12}S_{a,b,a\wedge a,b,a} - \frac{1}{4}S_{a,a,a,b,a\wedge b} - \frac{1}{4}S_{a,a,b,a,a\wedge b} + \frac{1}{12}S_{a,a,a\wedge b,b,a} \\
&+\frac{3}{4}S_{a,a,a\wedge b,a,b} - \underline{S_{b,b,a,a,a,a}} + \frac{1}{4}S_{b,b,a,a\wedge a,a} - \frac{1}{4}S_{a\wedge a,a,a,b,b} \\
&+\frac{1}{12}S_{a,b,a,b,a\wedge a} + \frac{1}{12}S_{b,a\wedge a,a,b,a} + \frac{1}{12}S_{b,a,a\wedge a,b,a} + \frac{1}{12}S_{b,a,a,b,a\wedge a} \\
&-\frac{1}{12}S_{b,a\wedge a,b,a,a} - \frac{1}{12}S_{b,a,b,a\wedge a,a} + \frac{1}{4}S_{b,b,a\wedge a,a,a} + \frac{1}{4}S_{b,b,a,a,a\wedge a} \\
&-\frac{1}{4}S_{a\wedge a,a,b,a,b} - \frac{1}{4}S_{a,a\wedge a,b,a,b} - \frac{1}{4}S_{a,a,b,a\wedge a,b} + \frac{1}{12}S_{a\wedge a,a,b,b,a} \\
&+\frac{1}{12}S_{a,a\wedge a,b,b,a} + \frac{1}{12}S_{a,a,b,b,a\wedge a} - \frac{1}{4}S_{a\wedge a,b,a,a,b} - \frac{1}{12}S_{b,a,b,a,a\wedge a} \\
&+\frac{1}{12}S_{a\wedge b,a,a,b,a} + \frac{1}{12}S_{b,a,a,a\wedge b,a} - \frac{1}{12}S_{a\wedge b,a,b,a,a} + \frac{1}{4}S_{a\wedge b,b,a,a,a} \\
&+\frac{1}{4}S_{b,a\wedge b,a,a,a} - \frac{1}{12}S_{b,a,a\wedge b,a,a} + \frac{3}{4}S_{a,a,a,a\wedge b,b} + \frac{1}{12}S_{a,a,b,a\wedge b,a} \\
&+\frac{3}{4}S_{a,a\wedge b,a,a,b} - \frac{1}{4}S_{a,b,a,a,a\wedge b} + \frac{1}{12}S_{a,a\wedge b,a,b,a} + \frac{1}{12}S_{a,b,a,a\wedge b,a} \\
&-\frac{1}{12}S_{a,a\wedge b,b,a,a} - \frac{1}{12}S_{a,b,a\wedge b,a,a} - \frac{1}{4}S_aS_{a,a,a,b,b} - \frac{1}{12}S_aS_{a,b,b,a,a} \\
&+\frac{1}{12}S_aS_{a,b,a,b,a} - \frac{1}{12}S_aS_{b,a,b,a,a} + \frac{1}{12}S_aS_{b,a,a,b,a} - \frac{1}{4}S_aS_{a,b,a,a,b} \\
&+S_bS_{a,a,a,a,b} + \frac{1}{4}S_aS_{b,b,a,a,a} - \frac{1}{4}S_aS_{a,a,b,a,b}
\end{aligned}
$$

DEPENDENCE UP TO 2 BASIC SUMS.

Depth $d = 3$

Index Set	Number	Dep. Sums of Depth 3	min. Weight	Fraction of fund. Sums
$\{a, a, a\}$	1	1	3	0
$\{a, a, b\}$	3	2	3	1/3
$\{a, b, c\}$	6	4	4	1/3

Depth $d = 4$

Index Set	Number	Dep. Sums of Depth 4	min. Weight	Fraction of fund. Sums
$\{a, a, a, a\}$	1	1	4	0
$\{a, a, a, b\}$	4	3	4	1/4
$\{a, a, b, b\}$	6	5	4	1/6
$\{a, a, b, c\}$	12	9	5	1/4
$\{a, b, c, d\}$	24	18	6	1/4

Depth $d = 6$

Index Set	Number	Dep. Sums of Depth 6	min. Weight	Fraction of fund. Sums
$\{a, a, a, a, a, a\}$	1	1	6	0
$\{a, a, a, a, a, b\}$	6	5	6	1/6
$\{a, a, a, a, b, b\}$	15	13	6	2/15
$\{a, a, a, b, b, b\}$	20	17	6	3/20
$\{a, a, a, a, b, c\}$	30	25	7	1/6
$\{a, a, a, b, b, c\}$	60	50	7	1/6
$\{a, a, b, b, c, c\}$	90	76	8	7/45
$\{a, a, a, b, c, d\}$	120	100	8	1/6
$\{a, a, b, b, c, d\}$	180	150	8	1/6
$\{a, a, b, c, d, e\}$	360	300	10	1/6
$\{a, b, c, d, e, f\}$	720	600	12	1/6

Theory of Words

Can we count the Basis in simpler way ? $\Longrightarrow$ YES.

Free Algebras and Elements of the Theory of Codes
$\Longrightarrow$ **Particle Physics**

Only the multiplication relation and the Index structure matters

$\mathfrak{A} = \{a, b, c, d, \ldots\}$ **Alphabet**

$a < b < c < d < \ldots$ **ordered**

$\mathfrak{A}^*(\mathfrak{A})$ **Set of all words W**

$W = a_1 \cdot a_2 \cdot a_{27} \ldots a_{532} \equiv$ **concatenation product (nc)**

$W = p \cdot x \cdot s$ **p = prefix; s = suffix**

Definition:

A Lyndon word is smaller than any of its suffixes.

Theorem:[Radford, 1979]

The shuffle algebra $K\langle\mathfrak{A}\rangle$ is freely generated by the Lyndon words.
I.e. the number of Lyndon words yields the number of basic elements.

Examples :

$\{a, a, \ldots, a, b\} = aaa \ldots ab$ 1 Lyndon word for these sets

$n \quad a's: \quad n_{basic}/n_{all} = 1/n \qquad n \equiv$ depth of the sums

$\{a, a, a, b, b, b\}$ $aaabbb, aababb, aabbab$ 3 Lyndon words

$n_{basic}/n_{all} = 3/20 < 1/6$. Symmetries lead to a smaller fraction.

Is there a general Counting Relation ?

E. Witt, 1937

$$l_n(n_1, \ldots, n_q) = \frac{1}{n} \sum_{d|n_i} \mu(d) \frac{(n/d)!}{(n_1/d)! \ldots (n_q/d)!}, \qquad \sum_i n_i = n$$

$\mu(k)$ Möbius function

2nd Witt formula.

The Length of the Basis is a function mainly of the Depth.

$$l_6(\{a, a, a, b, b, b\}) = \frac{1}{6} \left[\mu(1) \frac{6!}{3!3!} + \mu(3) \frac{2!}{1!1!} \right] = 3$$

$$n_6(\{a, a, a, b, b, b\}) = \frac{6!}{2!3!} = 20$$

Weight	# Sums	Cum. # Sums	# Basic Sums	Cum. # Basic Sums	Cum. Fraction
1	2	2	0	0	0.0
2	6	8	1	1	0.1250
3	18	26	6	7	0.2692
4	54	80	16	23	0.2875
5	162	242	46	69	0.2851
6	486	728	114	183	0.2513

⇑ 2nd Witt formula

Structural Relations

Seek for further Reduction:

Relations using the Value of the Objects [DESY 04-064]

Use Relations like:

$$\frac{1}{2}\frac{\mathrm{Li}_2(x^2)}{1-x^2} = \frac{\mathrm{Li}_2(x)}{1-x} + \frac{\mathrm{Li}_2(x)}{1+x} + \frac{\mathrm{Li}_2(-x)}{1-x} + \frac{\mathrm{Li}_2(-x)}{1+x}$$

$$\begin{aligned}\frac{1}{8}\mathbf{M}\left[\left(\frac{\mathrm{Li}_4(x)}{1-x}\right)_+\right]\left(\frac{N}{2}\right) \quad = \quad & \mathbf{M}\left[\left(\frac{\mathrm{Li}_4(x)}{1-x}\right)_+\right](N) + \mathbf{M}\left[\left(\frac{\mathrm{Li}_4(-x)}{1-x}\right)_+\right](N) \\ & + \mathbf{M}\left[\frac{\mathrm{Li}_4(x)}{1+x}\right](N) + \mathbf{M}\left[\frac{\mathrm{Li}_4(-x)}{1+x}\right](N) \\ & - \frac{9}{32}\zeta(2)\zeta(3) + \frac{141}{256}\zeta(5)\ .\end{aligned}$$

and similar ones.

Apply Symmetries among Mellin-Transforms of Nielsen Integrals.

Since all harmonic sums are meromorphic functions for $N \in \mathbf{C}$ since they may be represented by **Factorial Series** Derivatives are not essentially new functions.

$$\mathbf{M}[\ln^k(x) f(x)](N) = \frac{\partial^k}{\partial N^k}\mathbf{M}[f(x)](N)$$

Further Reduction due to the Structure of Feynman Amplitudes

The Lord is mercy, after all!

Analytic Continuation

The Harmonic Sums and Mellin Transforms have to be represented such, that the outer summation index can be analytically continued to $N \in \mathbf{C}$ [J.B., 2000]

- Use precise, adaptive Representations in analytic Form
- Refer to the Representation through Factorial Series etc.
- The **Residue Theorem** is used to get back to x space

4. A Quadratic Law ?

The anomalous dimensions and Wilson coefficients for $m_i = 0$ can be expressed in terms of multiple harmonic sums to 3–loop order.

What are the irreducible functions behind this representation ?

We will not count Euler's Γ-function neither all derivations of the functions occurring.

The final set of functions:

Trivial functions:

$$S_{\pm k}(N) \quad \longrightarrow \quad \psi^{(k-1)}(N+1)$$

For $w = 1, 2$ no non–trivial functions contribute to the anomalous dimensions and Wilson coefficients.

Non–trivial functions:

$N = 3$: Two–Loop anomalous dimensions

$$\mathbf{M}\left[\frac{\mathrm{Li}_2(x)}{1+x}\right](N)$$

Yndurain et al., 1980

$N = 4$: Two–Loop Wilson Coefficients

$$\mathbf{M}\left[\frac{\mathrm{Li}_2(x)}{1-x}\right](N), \quad \mathbf{M}\left[\frac{\mathrm{Li}_3(x)}{1+x}\right](N), \quad \mathbf{M}\left[\frac{\mathrm{S}_{1,2}(x)}{1\pm x}\right](N)$$

J.B., S. Moch, 2003,

also: J.B., V. Ravindran, 2004.

$N = 5$: Three–Loop Anomalous Dimensions

$$\mathbf{M}\left[\frac{\ln(1+x)}{1+x}\right](N), \quad \mathbf{M}\left[\frac{\mathrm{Li}_4(x)}{1\pm x}\right](N), \quad \mathbf{M}\left[\frac{S_{1,3}(x)}{1+x}\right](N),$$

$$\mathbf{M}\left[\frac{S_{2,2}(x)}{1\pm x}\right](N), \quad \mathbf{M}\left[\frac{S_{2,2}(-x)-\mathrm{Li}_2^2(-x)/2}{1\pm x}\right](N),$$

$$\mathbf{M}\left[\frac{\mathrm{Li}_2^2(x)}{1+x}\right](N)$$

J.B., S. Moch, 2004.

The number of **Non-trivial Basic Functions** seems to grow as :

$$\mathbf{N_w = \theta(w-2)\cdot[w-2]^2}$$

Essentially 14 Functions seem to rule the single scale processes of massless QCD.

This is a rather small number if compared to the number of possible harmonic sums $3^w - 1$.

5. The 16th Moment of the 3-Loop Non–Singlet Anomalous Dimension of $F_{2,L}(x, Q^2)$

Seek for another, blind check of the complete calculation of the NS–anomalous dimension by Moch, Vermaseren, and Vogt, 2004.

The calculation was started far before the complete calculation was completed and is based on the `MINCER` algorithm used before by Larin, Noguiera, van Ritbergen, Vermaseren and Retey, 1994–2000

Moment	CPU time [days]	
	$g_{\mu\nu}$	$P_\mu P_\nu$
2	0.002567	0.002190
4	0.012562	0.020027
6	0.057144	0.059320
8	0.303415	0.332731
10	1.108047	1.219046
.	.	.
16	236.236	327.542

J.B., J. Vermaseren, 2004

Set-up used: 1 XEON-Dual PC 3 GHz (Drittmittel) and 1 XEON-Dual 2.6 GHz PC (borrowed from AMANDA) linked to a 4.2 Tbyte RAID system; partly 1 64-bit OPTERON 4Gbyte (dual)

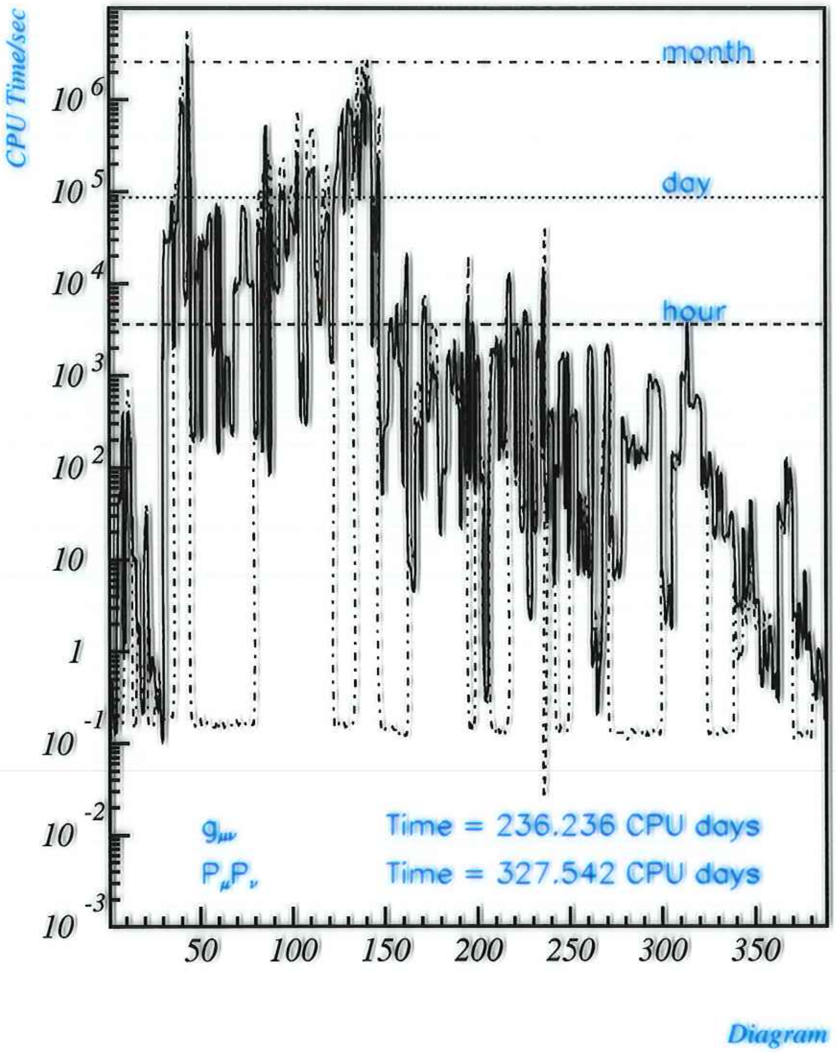
Results after 564 CPU days

Many Thanks to: P. Wegner, S. Wiesand, U. Gensch & C. Spiering for supporting this project.

⇒ **DESY-Z** urgently needs a Parallel PC-Facility ⇐

for FORM Formula Manipulation

Run-time statistics :



Several diagrams stayed in the CPU for 25–40 days each.

Results :

$$\gamma_{16}^{(0)} = \frac{64419601}{6126120}\, \mathrm{C_F}$$

$$\gamma_{16}^{(1)} = -\frac{1176525373840303}{112588038763200}\, \mathrm{C_F\, N_F} + \frac{21546159166129889}{484994628518400}\, \mathrm{C_F\, C_A} - \frac{36890244529287813828 77}{4598185573520098560 00}\, \mathrm{C_F}^2$$

$$\gamma_{16}^{(2)} = \left(\frac{59290512768143}{1563722760600}\,\zeta_3 - \frac{58552930270652300886778705063429867}{34513379706124525343170966732800 00}\right)\mathrm{C_F}^3$$
$$+\left(-\frac{150184218240603886594365 59}{579371382263532418560000} - \frac{64419601}{765765}\,\zeta_3\right)\mathrm{C_F\, C_A\, N_F}$$
$$+\left(\frac{1670423728083984207878825467}{648895948135156308787200 0} + \frac{59290512768143}{3127445521200}\,\zeta_3\right)\mathrm{C_F\, C_A}^2$$
$$-\frac{555946634983457315725 1}{206918350808404435200 0}\,\mathrm{C_F\, N_F}^2$$
$$+\left(-\frac{12297946460007757811278560644 77}{303358855753185574358016000 00} - \frac{59290512768143}{1042481840400}\,\zeta_3\right)\mathrm{C_F}^2\mathrm{C_A}$$
$$+\left(-\frac{715435996779851553425513554 51}{938967886855098206346240000} + \frac{64419601}{765765}\,\zeta_3\right)\mathrm{C_F}^2\mathrm{N_F}$$

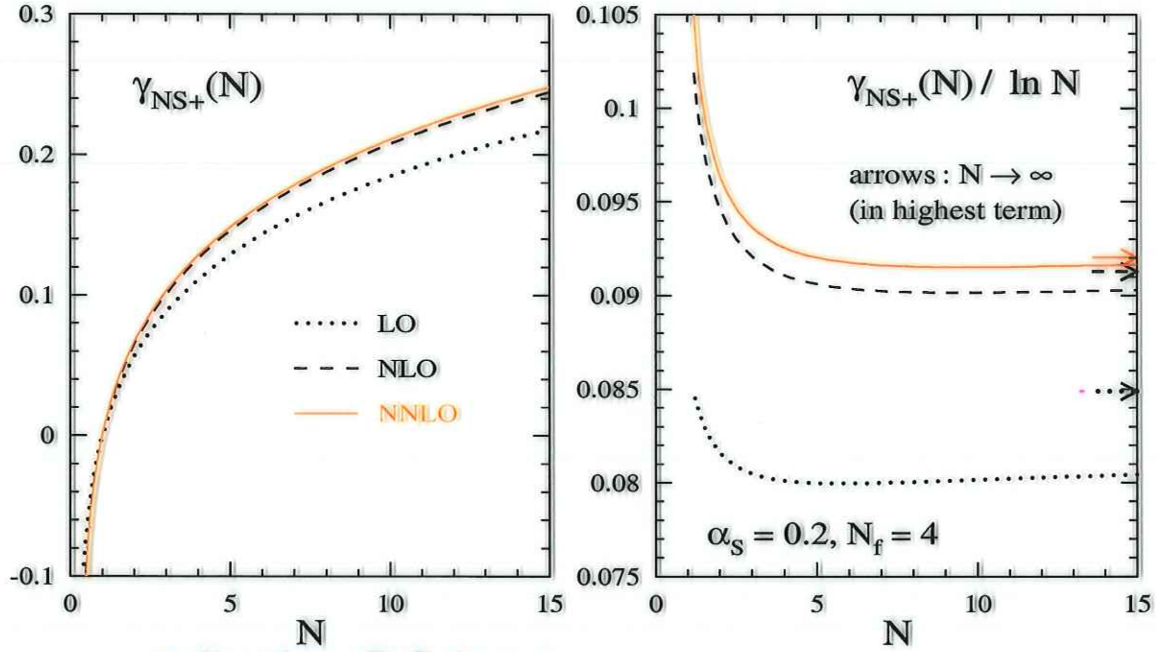
Agreement with : Moch, Vermaseren, Vogt, hep-ph/0403192.

$$
\begin{aligned}
C_2^{\mathrm{NS},16}(x, a_s) &= \frac{4047739719}{190590400}\, C_F\, a_s \\
&+ \Bigg(\Bigg(\frac{4442667416304442887936697012 7}{321931846921747956461568000}\, \frac{24439538}{255255}\, \zeta_3\Bigg) C_F{}^2 \\
&+ \Bigg(\frac{179183084084982942227830 87}{59422705873182812160000} - \frac{113298677}{1021020}\, \zeta_3\Bigg) C_F\, C_A \\
&- \frac{14356837276190747211117 7}{27589113441120591360 00}\, C_F\, N_F\Bigg) a_s{}^2 \\
&+ \Bigg(\Bigg(\frac{59290512768143}{3127445521200}\, \zeta_4 - \frac{27643576}{21879}\, \zeta_5 \\
&+ \frac{30368133975995097250846772938425059765591616 89}{80344580160407759334216478634033479680000 00} \\
&+ \frac{1494341926940450865387403}{595674040206012768000}\, \zeta_3\Bigg) C_F{}^3 \\
&+ \Bigg(\frac{59290512768143}{6254891042400}\, \zeta_4 + \frac{2628653778834757265588009355150331903 33}{56646805852503848671021043712000000} \\
&+ \frac{47187263}{51051}\, \zeta_5 - \frac{1535505046917148231 3}{4991403051835200}\, \zeta_3\Bigg) C_F\, C_A{}^2 \\
&+ \Bigg(\frac{7227384935999670312318789884999}{76056398835262954714045440000} + \frac{64419601}{20675655}\, \zeta_3\Bigg) C_F\, N_F{}^2 \\
&+ \Bigg(\frac{775002662711876875284509176089005146524274 1}{16525006203292422734310258871664640000 00} \\
&- \frac{2849482004138921491531}{6741167121672984000}\, \zeta_3 + \frac{983963}{21879}\, \zeta_5 \\
&- \frac{59290512768143}{2084963680800}\, \zeta_4\Bigg) C_F{}^2 ca + \Bigg(-\frac{552298563960959}{4021001384400}\, \zeta_3 \\
&- \frac{407320724134849319615222207993355752 9}{35297774699445537282788488704000 00} + \frac{64419601}{1531530}\, \zeta_4\Bigg) C_F{}^2 N_F \\
&+ \Bigg(\frac{598788865585667}{1850495446800}\, \zeta_3 - \frac{64419601}{1531530}\, \zeta_4 \\
&- \frac{5828116349215429956471793586985365 47}{4046200418035989190787217408000 00}\Bigg) C_F\, C_A\, N_F\Bigg) a_s{}^3
\end{aligned}
$$

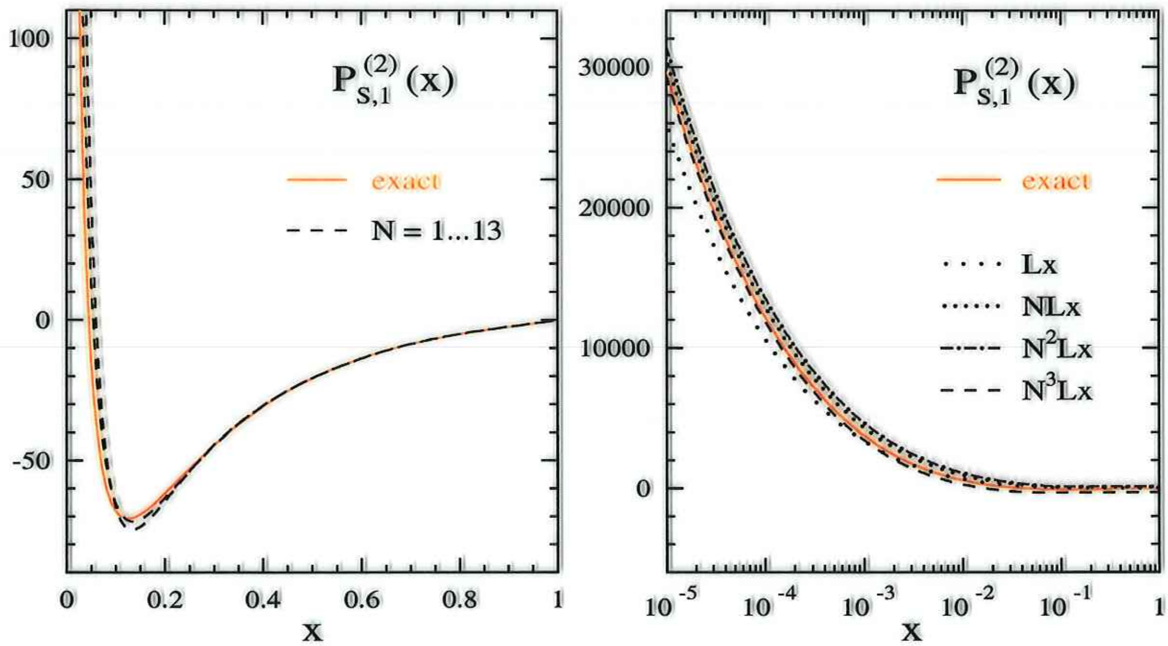
$$
\begin{aligned}
C_L^{\rm NS,16}(x,a_s) &= \frac{4}{17}C_F\,a_s \\
&+\Bigg[-\frac{29393927457809}{44659922042736}{C_F}^2-\frac{39366889}{39054015}C_F\,N_F \\
&-\frac{48}{17}\zeta_3\,C_F\,C_A+\frac{55969347000169}{8209544493150}C_F\,C_A+\frac{96}{17}\zeta_3\,{C_F}^2\Bigg]{a_s}^2 \\
&+\Bigg[\Bigg(\frac{39360}{17}\zeta_5-\frac{1962568998281706 31}{133698296031300}\zeta_3 \\
&-\frac{750828182127677149812644729011 0919}{136478982354388524292425984 00000}\Bigg){C_F}^3 \\
&+\Bigg(\frac{29604550101013356532203920 7159677}{93662046713796046083037440 0000} \\
&-\frac{40160}{17}\zeta_5+\frac{2253147763389895}{1188429298056}\zeta_3\Bigg)C_A\,{C_F}^2 \\
&+\Bigg(\frac{35291373463211704531604 63}{13679602081222293216 0000}-\frac{44651224}{765765}\zeta_3\Bigg)N_F\,{C_F}^2 \\
&+\Bigg(-\frac{1634895686765221}{2673965920626}\zeta_3 \\
&+\frac{1460792499427100139493280371}{825604219725533696448 0000} \\
&+\frac{10240}{17}\zeta_5\Bigg){C_A}^2C_F+\frac{895967716232}{209134250325}C_F\,{N_F}^2 \\
&+\Bigg(-\frac{4495805144658565385501573689}{577922953807873587513 60000} \\
&+\frac{435943306 72}{1249937325}\zeta_3\Bigg)C_A\,N_F\,C_F\Bigg]{a_s}^3
\end{aligned}
$$

6. Phenomenological Results

S. Moch, J. Vermaseren, A. Vogt: non-singlet: hep-ph/0403192 :

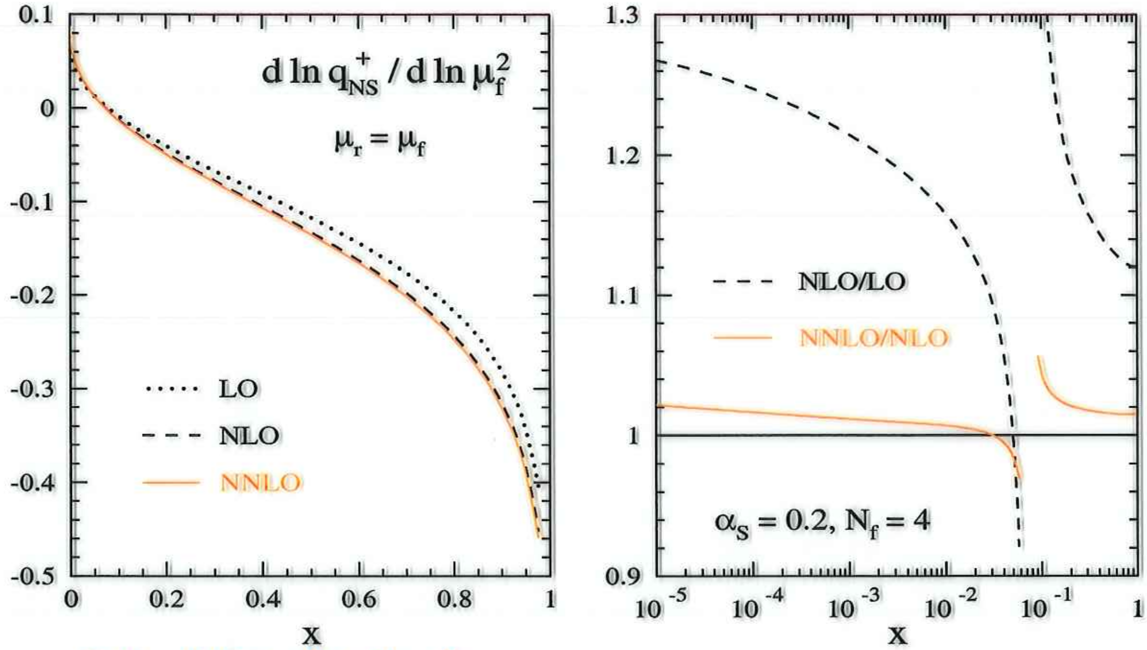


A new contribution @ 3 loops :

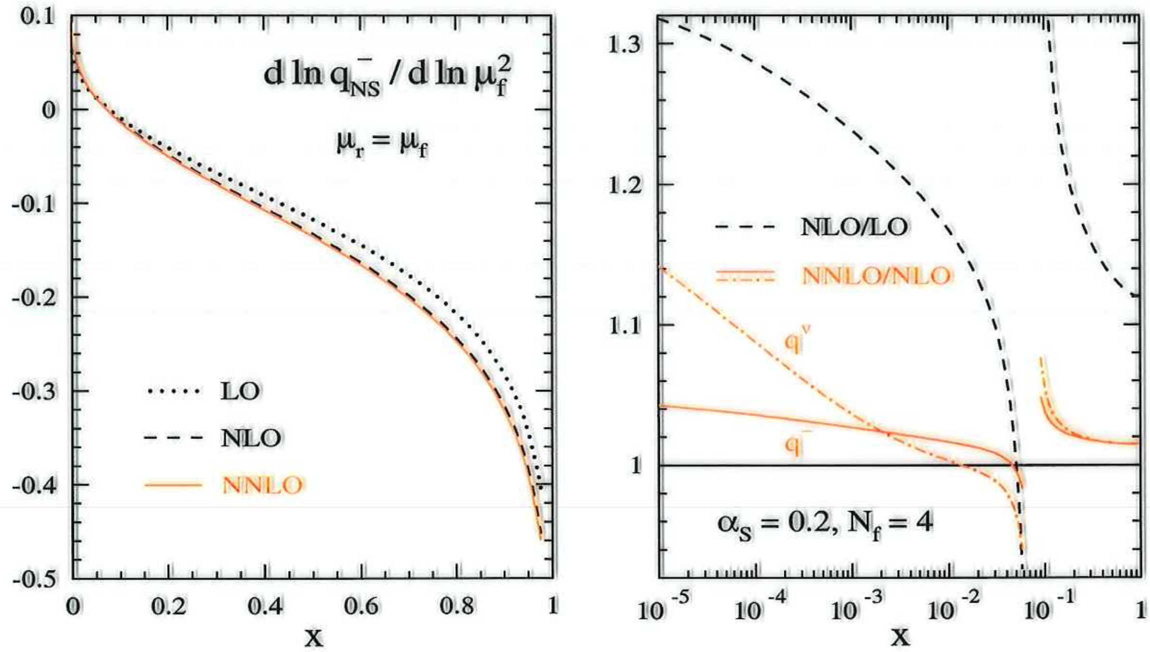


S. Moch, J. Vermaseren, A. Vogt: non-singlet: hep-ph/0403192 :

Slope of the NS$^+$ distribution



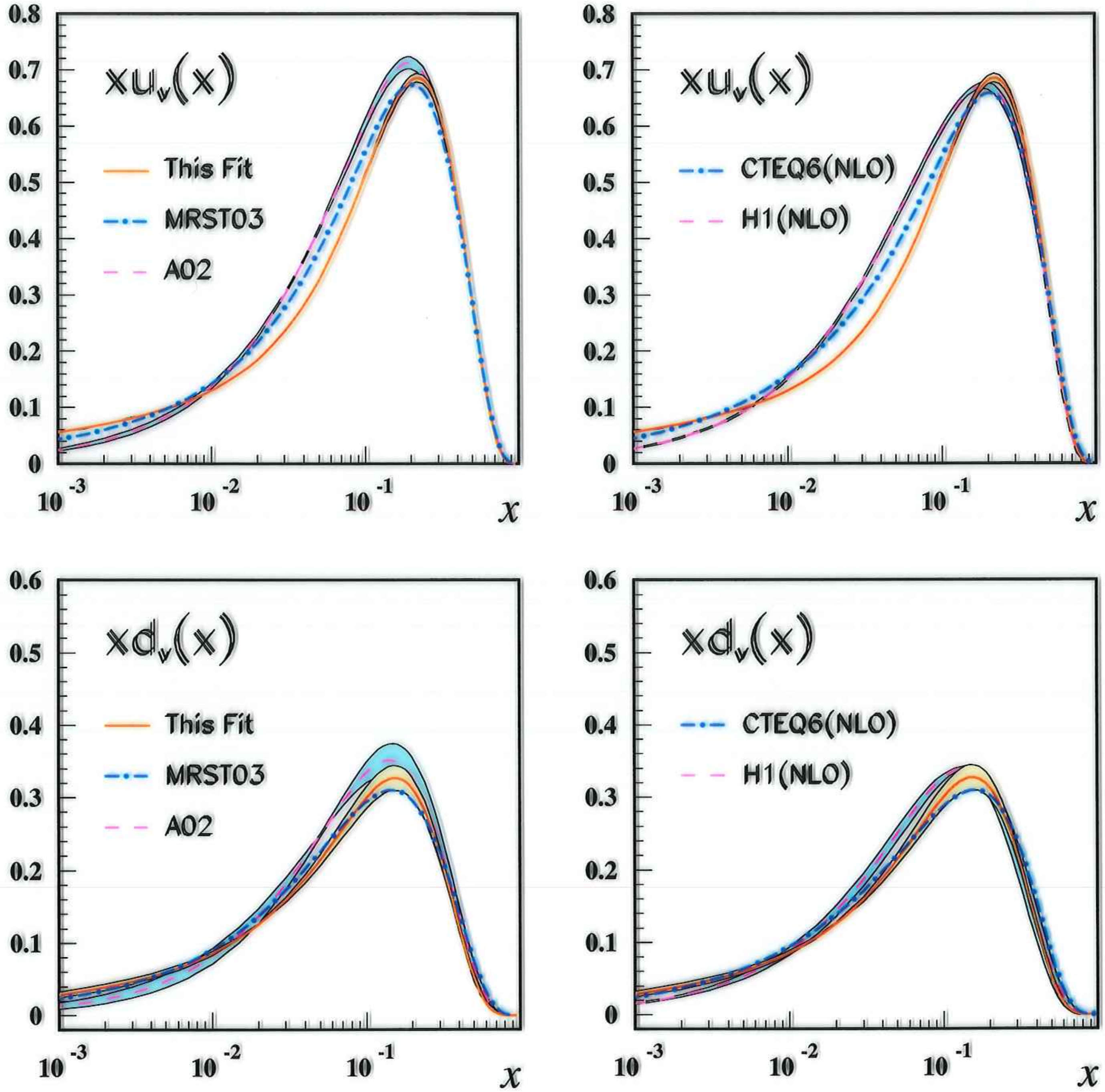
Slope of the NS$^-$ distribution



Valence Parton Densities at $Q_0^2 = 4.0\,GeV^2$

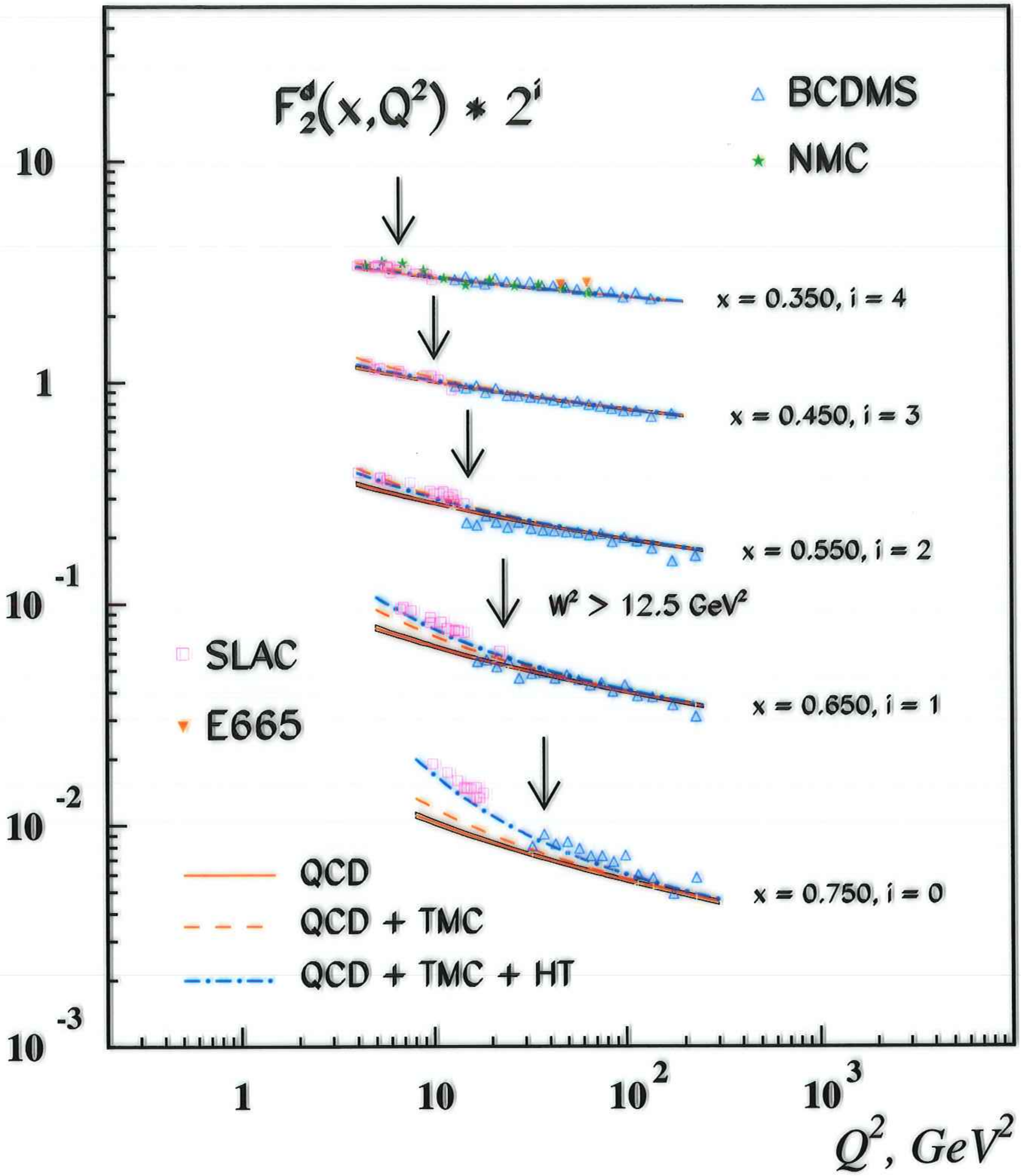
J.B., H. Böttcher, A. Guffanti hep-ph/0407089

- 4+1 parameter non-singlet fit to F_2 data:



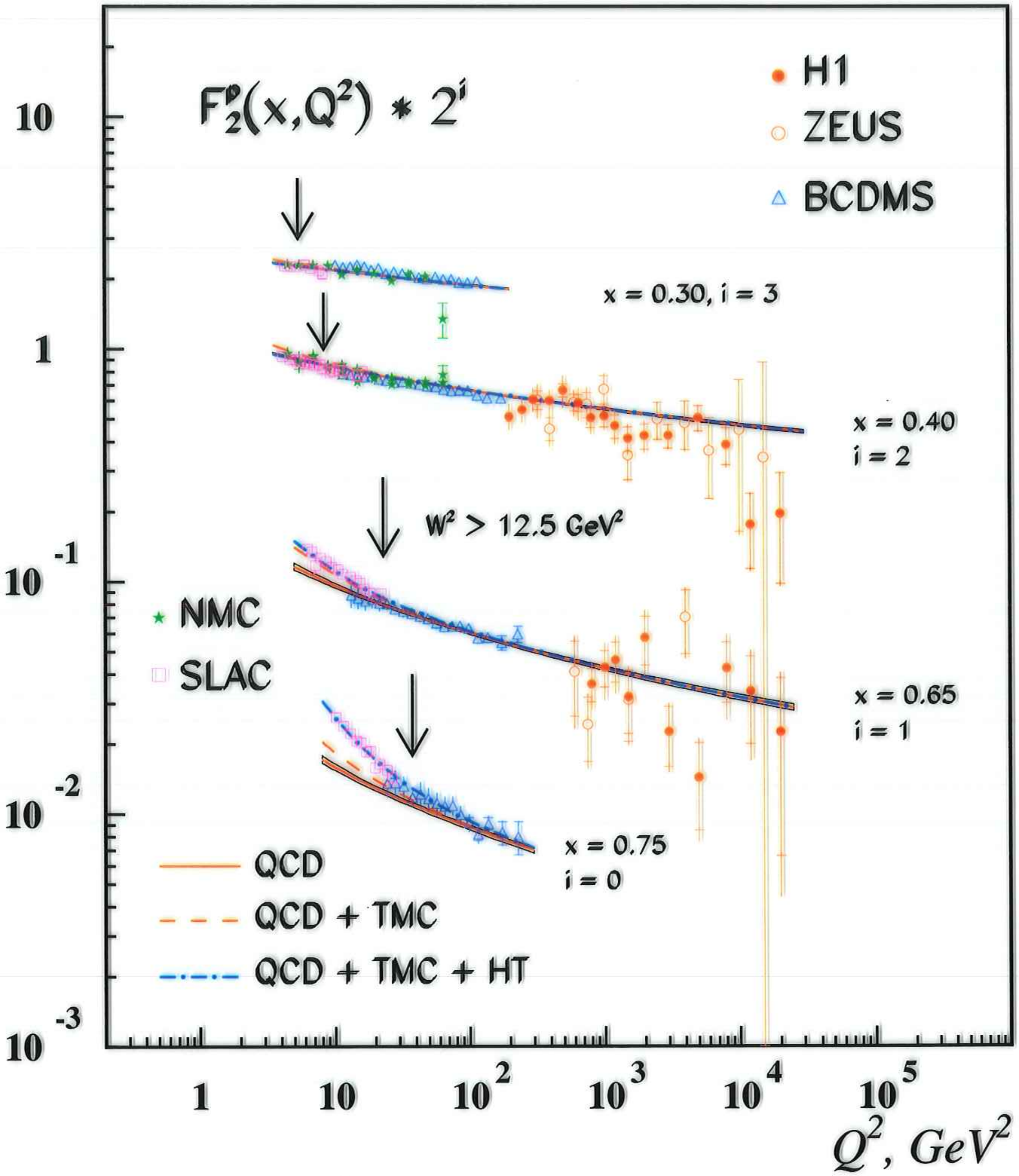
⇒ Yellow error band: Fully correlated 1σ error as given by Gaussian error propagation.

$F_2^d(x)$ versus Q^2



⇒ Yellow error band: Fully correlated 1σ Gaussian error propagation through the evolution equation.

$F_2^p(x)$ versus Q^2



⇒ Yellow error band: Fully correlated 1σ Gaussian error propagation through the evolution equation.

TMC and HT

- **Target Mass Correction** (TMC):

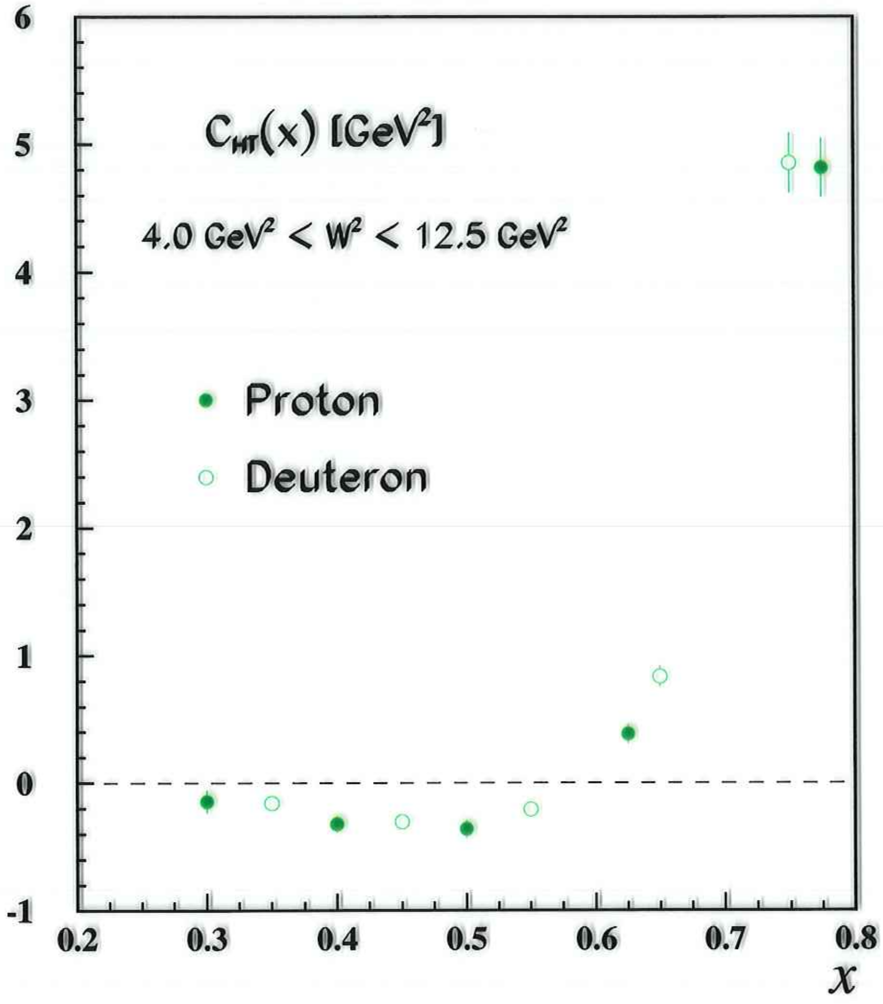
$$F_2^{TMC}(x, Q^2) = F_2^{QCD}(x, Q^2) \times TMC$$

[**Ref**: H.Georgi and H.D.Politzer, Phys.Rev.**D14**(1976)1829.]

- **Higher Twist** (HT):

$$F_2^{HT}(x_i, Q^2) = F_2^{QCD}(x_i, Q^2) \times C_{HT}(x_i)/Q^2$$

[**Ref**: M.Virchaux and A.Milsztajn, Phys.Lett.**B274**(1992)221.]



Fit in intervals of x.
QCD parameters fixed.

$$F_2(x, Q^2) = F_2^{TMC}(x, Q^2) + F_2^{HT}(x, Q^2)$$

$\Lambda_{QCD}^{(4)} \Rightarrow \alpha_s(M_Z^2)$

	$\Lambda_{QCD}^{(4)}$, MeV	$\alpha_s(M_Z^2)$
This Fit	227 ± 30	$0.1135 \; {}^{+0.0023}_{-0.0026}$ (expt)

⇒ $\Lambda_{QCD}^{(4)}$ stable against a variation of the Q^2-cut on the data $(4\,,7\,,10\,GeV^2)$.

⇒ latest world average: 0.1182 ± 0.0027

Ref.: S.Bethke, LL2004, Zinnowitz, April 25-30, 2004

Comparison of $\alpha_s(M_Z^2)$

$\Longrightarrow$ This Fit: $\alpha_s(M_Z^2) = 0.1135 \begin{array}{c} +0.0023 \\ -0.0026 \end{array}$ (expt)

- Comparison with other QCD analyses (significant sea and gluon contributions and correlations):

	$\alpha_s(M_Z^2)$	expt	theory	model	Ref.
NLO					
CTEQ6	0.1165	± 0.0065			[1]
MRST03	0.1165	± 0.0020	± 0.0030		[2]
A02	0.1171	± 0.0015	± 0.0033		[3]
ZEUS	0.1166	± 0.0049		± 0.0018	[4]
H1	0.1150	± 0.0017	± 0.0050	$\begin{array}{c} +0.0009 \\ -0.0005 \end{array}$	[5]
BCDMS	0.110	± 0.006			[6]
BB (pol)	0.113	± 0.004	$\begin{array}{c} +0.009 \\ -0.006 \end{array}$		[7]
NNLO					
MRST03	0.1153	± 0.0020	± 0.0030		[2]
A02	0.1143	± 0.0014	± 0.0009		[3]
SY01(ep)	0.1166	± 0.0013			[8]
SY01(νN)	0.1153	± 0.0063			[8]

[1]: CTEQ Coll.: J.Pumplin et al., JHEP 0207:012 (2002). [2]: MRST Coll.: A.D.Martin et al., hep-ph/0307262. [3]: S.Alekhin, hep-ph/0211096. [4]: ZEUS Coll.: S.Chekanov et al., Phys.Rev.**D67** (2003) 012007. [5]: H1 Coll.: C.Adloff et al., Eur.Phys. **C21** (2001) 33. [6]: BCDMS Coll.: A.C.Benvenuti et al., Phys.Lett. **237** (1990) 592. [7]: J.Blümlein and H.Böttcher, Nucl.Phys. **B636** (2002) 225. [8]: J.Santiago and F.J.Yndurain, Nucl.Phys. **B611** (2001) 447.

Comparison of Moments at $Q^2 = 4.0\, GeV^2$

f	n	This Fit	MRST03	A02
u_v	2	0.289 ± 0.003	0.289	0.304
	3	0.085 ± 0.001	0.084	0.087
	4	0.0324 ± 0.0004	0.032	0.033
d_v	2	0.109 ± 0.004	0.113	0.120
	3	0.025 ± 0.001	0.028	0.028
	4	0.0076 ± 0.0004	0.010	0.010
$u_v - d_v$	2	0.180 ± 0.005	0.176	0.184
	3	0.060 ± 0.001	0.056	0.059
	4	0.0248 ± 0.0006	0.023	0.024

f	n	QCD	Lattice
		This Fit	QCDSF
$u_v - d_v$	2	0.180 ± 0.005	0.191 ± 0.012*)

OVERLAP FERMIONS

$$\Longrightarrow \Gamma_f(Q^2) = \int_0^1 x^{n-1} f(x, Q^2) dx$$

Lattice simulation: Scale $\mu^2 = 1/a^2 \sim 4\, GeV^2$.

$*$) G.Schierholz, private communication. PRELIMINARY

7. Conclusions

- Mellin space expressions of anomalous dimensions and Wilson coefficients are of much simpler structure than the x–space results.

- Index-based algebraic relations of harmonic sums, structural relations of Mellin–transforms of Nielsen–integrals and the specific structure of Feynman amplitudes cause this reduction.

- All algebraic relations are derived in explicit form up to weight $w = 6$ and apply to harmonic sums, harmonic polylogarithms and all other objects in the corresponding equivalence class. The structural relations are worked out up to $w = 4$ and the anomalous dimensions for $w = 5$.

- The number of multiple harmonic sums of weight w is $2 \cdot 3^{w-1}$. The number of the harmonic sums after algebraic reduction is given by the Witt formula(e) yielding a reduction to $\approx 1/4$. Further reductions result from structural relations.

- The number of non–trivial basic functions for $w \leq 5$ which are needed to express the known anomalous dimensions and the (space– and time–like) Wilson coefficients for $m_i = 0$ is given by

$$N_w = \theta(w-2) \cdot [w-2]^2$$

- An independent calculation of the 16th moment of the non–singlet structure function $F_1(x, Q^2)$ shows agreement with the complete calculation.