Mathematical Structure of QCD Wilson Coefficients and Anomalous Dimensions

Johannes Blümlein

DESY

1. **Introduction**
2. **x Space Representations**
3. **The Mellin Symmetry**
4. **Multiple Zeta Values**
5. **Multiple Harmonic Sums**
6. **Theory of Words**
7. **Deeper Relations**
8. **Evolution**

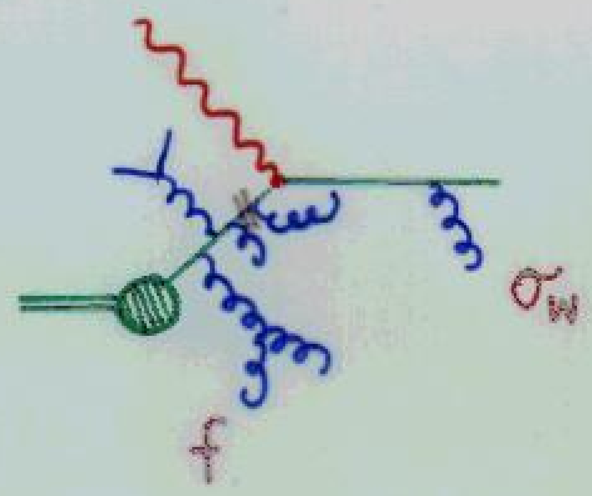
Phys. Rev. **D60** (1999) 014018; CPT **133** (2000) 76-104; DESY 03–134.

1. Introduction

- STUDY OF MASSLESS FIELD THEORIES

QCD, QED $m_i \to 0$

"SIMPLE" PHASE SPACE(S)



$$\sigma = \sigma_W \otimes f$$

$\uparrow$

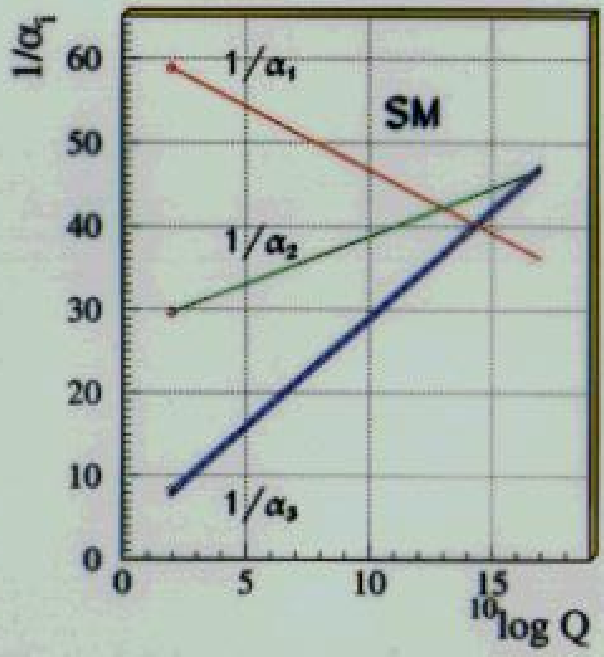
MELLIN CONVOLUTION

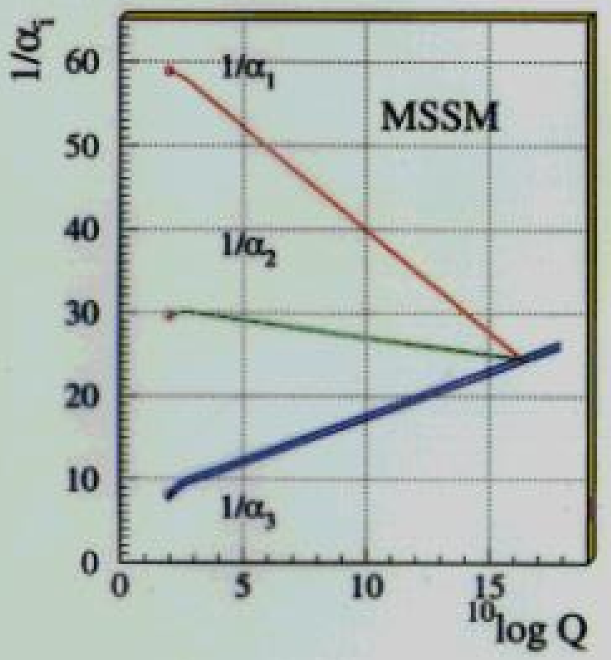
σ_W WILSON COEFFICIENT

f PARTON DENSITY

BOTH RENORMALIZED.

Unification of the Coupling Constants in the SM and the minimal MSSM





de Boer '02

CURRENTLY : $\Delta\alpha_s(M_Z^2)_{TH} = \pm 5\%$

WANTED : 1%

$\longrightarrow$ QCD @ 3LOOPS

MAJOR GOALS:

- ANALYZE QCD SCALING VIOLATIONS
- UNFOLD PARTON DENSITIES

FOR UNPOLARIZED & POLARIZED PROTONS

- PRECISION MEASUREMENTS OF α_s & Λ_{QCD}
- SEARCH FOR NEW QCD PHENOMENA (E.G.: SMALL x ?)

EVOLUTION EQUS.

NS: $$\frac{\partial \hat{F}_i^{NS}(N,a_s)}{\partial \log Q^2} = P_{NS}^{(i)}(a_s, N) \cdot \hat{F}_i^{NS}(N, a_s)$$

S: $$\frac{\partial}{\partial \log Q^2} \begin{pmatrix} \Sigma \\ G \end{pmatrix} = \underbrace{\mathbb{P}_S(a_s)}_{\text{MATRIX}} \cdot \begin{pmatrix} \Sigma \\ G \end{pmatrix}$$

$$P^{(i)}(a_s, N) = a_s P_1^{(i)}(N) + a_s^2 P_2^{(i)} + \underbrace{a_s^3 P_3^{(i)}}_{\text{Needed.}} + \dots$$

2. x–Space Representation

USUAL STARTING POINT OF HIGHER ORDER CALCULATIONS:

$\longrightarrow$ NIELSEN TYPE INTEGRALS

$$S_{n,p}(x) = \frac{(-1)^{n+p-1}}{(n-1)!\,p!} \int_0^1 \frac{dz}{z} \ln^{n-1}(z) \ln^p(1-zx)$$

OR OUR GENERALIZATION:

$$S_{n,p,q}(x) = \frac{(-1)^{n+p+q-1}}{(n-1)!\,p!\,q!} \int_0^1 \frac{dz}{z} \ln^{n-1}(z) \ln^p(1-zx) \ln^q(1+xz)$$

SPECIAL CASES:

$\mathrm{Li}_n(x) = S_{n-1,1}(x)$ WEIGHT n

$\frac{d\mathrm{Li}_2(\pm x)}{d\ln(x)} = -\ln(1\mp x)$ WEIGHT 1

$\mathrm{Li}_0(x) = \frac{x}{1-x}$ WEIGHT 0

$\frac{dx}{1\pm x}, \frac{dx}{x}$ WEIGHT 1

WHAT IS $C_{2,+}^{(2)}(x)^{-1}$?

VAN NEERVEN, ZIJLSTRA 1992

$$
\begin{aligned}
c^{(2)}_{2,+}(x) = {} & C_F^2 \Big\{ \frac{1+x^2}{1-x} \Big[4\ln^3(1-x) - (14\ln(x)+9)\ln^2(1-x) - \frac{4}{3}\ln^3(x) - \frac{3}{2}\ln^2(x) \\
& - \Big[4\mathrm{Li}_2(1-x) - 12\ln^2(x) - 12\ln(x) + 16\zeta(2) + \frac{27}{2} \Big] \ln(1-x) + 48\mathrm{Li}_3(-x) \\
& + \Big[-24\mathrm{Li}_2(-x) + 24\zeta(2) + \frac{61}{2} \Big] \ln(x) + 12\mathrm{Li}_3(1-x) - 12\mathrm{S}_{1,2}(1-x) \\
& + 48\mathrm{Li}_3(-x) - 6\mathrm{Li}_2(1-x) + 32\zeta(3) + 18\zeta(2) + \frac{51}{4} \Big] \\
& + (1+x) \Big[2\ln(x)\ln^2(1-x) + 4\Big[\mathrm{Li}_2(1-x) - \ln^2(x)\Big] \ln(1-x) + \frac{5}{3}\ln^3(x) \\
& - 4\mathrm{Li}_3(1-x) - 4[\mathrm{Li}_2(1-x) + \zeta(2)]\ln(x) \Big] + \Big(40 + 8x - 48x^2 - \frac{72}{5}x^3 + \frac{8}{5x^2} \Big) \\
& \times [\mathrm{Li}_2(-x) + \ln(x)\ln(1+x)] + (5+9x)\ln^2(1-x) + \frac{1}{2}(-91+141x)\ln(1-x) \\
& + (-8+40x)[\ln(x)\mathrm{Li}_2(-x) + \mathrm{S}_{1,2}(1-x) - 2\mathrm{Li}_3(-x) - \zeta(2)\ln(1-x)] \\
& - (28+44x)\ln(x)\ln(1-x) - (14+30x)\mathrm{Li}_2(1-x) + \Big(\frac{29}{2} + \frac{25}{2}x + 24x^2 + \frac{36}{3}x^3 \Big) \\
& \times \ln^2(x) + \frac{1}{10}\Big(13 - 407x + 144x^2 - \frac{16}{x} \Big) \ln(x) + \Big(-10 + 6x - 48x^2 - \frac{72}{5}x^3 \Big) \zeta(2) \\
& + \frac{407}{20} - \frac{1917}{20}x + \frac{72}{5}x^2 + \frac{8}{5x} + \Big[6\zeta^2(2) - 78\zeta(3) + 69\zeta(2) + \frac{331}{8} \Big] \delta(1-x) \Big\} \\
& + C_A C_F \Big\{ \frac{1+x^2}{1-x} \Big[-\frac{11}{3}\ln^2(1-x) + \Big[4\mathrm{Li}_2(1-x) + 2\ln^2(x) + \frac{44}{3}\ln(x) - 4\zeta(2) \\
& + \frac{367}{18} \Big] \ln(1-x) - \ln^3(x) - \frac{55}{6}\ln^2(x) + [4\mathrm{Li}_2(1-x) + 12\mathrm{Li}_2(-x) \\
& - \frac{239}{6} \Big] \ln(x) - 12\mathrm{Li}_3(1-x) + 12\mathrm{S}_{1,2}(1-x) - 24\mathrm{Li}_3(-x) + \frac{22}{3}\mathrm{Li}_2(1-x) + 2\zeta(3) \\
& + \frac{22}{3}\zeta(2) - \frac{3155}{108} \Big] + 4(1+x)[\mathrm{Li}_2(1-x) + \ln(x)\ln(1-x)] \\
& + \Big(-20 - 4x + 24x^2 + \frac{36}{5}x^3 - \frac{4}{5x^2} \Big) [\mathrm{Li}_2(-x) + \ln(x)\ln(1+x)] \\
& + (4-20x)[\ln(x)\mathrm{Li}_2(-x) + \mathrm{S}_{1,2}(1-x) - 2\mathrm{Li}_3(-x) - \zeta(2)\ln(1-x)] \\
& + \Big(\frac{133}{6} - \frac{1113}{18}x \Big) \ln(1-x) + \Big(-2 + 2x - 12x^2 - \frac{18}{5}x^3 \Big) \ln^2(x) \\
& + \frac{1}{30}\Big(13 + 1753x - 216x^2 + \frac{24}{x} \Big) \ln(x) + \Big(-2 - 10x + 24x^2 + \frac{36}{5}x^3 \Big) \zeta(2) \\
& - \frac{9687}{540} + \frac{59157}{540} - \frac{36}{5}x^2 - \frac{4}{5x} \\
& + \Big[\frac{71}{5}\zeta^2(2) + \frac{140}{3}\zeta(3) - \frac{251}{3}\zeta(2) - \frac{5465}{72} \Big] \delta(1-x) \Big\} \\
& + C_F N_F \Big\{ \frac{1+x^2}{1-x} \Big[\frac{2}{3}\ln^2(1-x) - \Big(\frac{8}{3}\ln(x) + \frac{29}{9} \Big) \ln(1-x) - \frac{4}{3}\mathrm{Li}_2(1-x) + \frac{5}{3}\ln^2(x) \\
& + \frac{19}{3}\ln(x) - \frac{4}{3}\zeta(2) + \frac{247}{54} \Big] + \frac{1}{3}(1+13x)\ln(1-x) - \frac{1}{3}(7+19x)\ln(x) - \frac{23}{18} - \frac{27}{2}x
\end{aligned}
$$

$$
+\left[\frac{4}{3}\zeta(3)+\frac{38}{3}\zeta(2)+\frac{457}{36}\right]\delta(1-x)\Bigg\}\,. \tag{1}
$$

$$
\begin{aligned}
c_{2,G}^{(2)}(x) \;=\; & C_F N_F\Big\{8(1+x)^2\left[-4\mathrm{S}_{1,2}(-x)-4\ln(1+x)\mathrm{Li}_2(-x)-2\zeta(2)\ln(1+x)\right.\\
& \left.-2\ln(x)\ln^2(1+x)+\ln^2(x)\ln(1+x)\right]+4(1-x)^2\left\{\frac{5}{6}\ln^3(1-x)\right.\\
& -\left(2\ln(x)+\frac{13}{4}\right)\ln^2(1-x)+\left[2\mathrm{Li}_2(1-x)+2\ln^2(x)+4\ln(x)+\frac{7}{2}\right]\ln(1-x)\\
& -\frac{5}{12}\ln^3(x)+\left[\mathrm{Li}_2(1-x)-4\mathrm{Li}_2(-x)+3\zeta(2)\right]\ln(x)-4\mathrm{Li}_3(1-x)-\mathrm{S}_{1,2}(1-x)\\
& \left.+12\mathrm{Li}_3(-x)+13\zeta(3)+\frac{13}{2}\zeta(2)\right\}+x^2\left\{\frac{10}{3}\ln^3(1-x)-12\ln(x)\ln^2(1-x)\right.\\
& +\left[16\ln^2(x)-16\zeta(2)\right]\ln(1-x)-5\ln^3(x)+\left[12\mathrm{Li}_2(1-x)+20\zeta(2)\right]\ln(x)\\
& \left.-8\mathrm{Li}_3(1-x)+12\mathrm{S}_{1,2}(1-x)\right\}+\left(48+\frac{64}{3}x+\frac{96}{5}x^3+\frac{8}{15x^2}\right)\\
& \times\left[\mathrm{Li}_2(-x)+\ln(x)\ln(1+x)\right]+(14x-23x^2)\ln^2(1-x)-(12x-10x^2)\ln(1-x)\\
& +(-24x+56x^2)\ln(x)\ln(1-x)+64x\mathrm{Li}_3(-x)+(-10+24x)\mathrm{Li}_2(1-x)\\
& +\left(-\frac{3}{2}+\frac{22}{3}x-36x^2-\frac{48}{5}x^3\right)\ln^2(x)+\frac{1}{15}\left(-236+339x-648x^2-\frac{8}{x}\right)\ln(x)\\
& +(64x+36x^2)\zeta(3)+\left(-\frac{20}{3}+46x^2+\frac{96}{5}x^3\right)\zeta(2)-\frac{647}{15}+\frac{239}{5}x-\frac{36}{5}x^2+\frac{8}{15x^2}\Big]\\
& \\
& +C_A N_F\Big\{4(1+x)^2\left[\mathrm{S}_{1,2}(1-x)-2\mathrm{Li}_3(-x)+4\mathrm{S}_{1,2}(-x)-2\ln(x)\mathrm{Li}_2(1-x)\right.\\
& +4\ln(1+x)\mathrm{Li}_2(-x)+2\ln(x)\mathrm{Li}_2(-x)+2\zeta(2)\ln(1+x)+2\ln(x)\ln^2(1+x)\\
& \left.+\ln^2(x)\ln(1+x)\right]+8(1+2x+2x^2)\left[\mathrm{Li}_3\left(\frac{1-x}{1+x}\right)-\mathrm{Li}_3\left(-\frac{1-x}{1+x}\right)\right.\\
& \left.-\ln(1-x)\mathrm{Li}_2(-x)-\ln(x)\ln(1-x)\ln(1+x)\right]+\left(-24+\frac{80}{3}x^2-\frac{16}{3x}\right)\\
& \times\left[\mathrm{Li}_2(-x)+\ln(x)\ln(1+x)\right]+x^2\left[-4\mathrm{S}_{1,2}(1-x)+16\mathrm{Li}_3(-x)+8\ln(x)\mathrm{Li}_2(1-x)\right.\\
& \left.+8\ln^2(x)\ln(1+x)\right]+\frac{2}{3}(1-2x+2x^2)\ln^3(1-x)+(24x-8x^2)\ln(x)\ln^2(1-x)\\
& +\left(-2+36x-\frac{122}{3}x^2+\frac{8}{3x}\right)\ln^2(1-x)+(-4-32x+8x^2)\ln^2x\ln(1-x)\\
& +(8-144x+148x^2)\ln(x)\ln(1-x)+(4+40x-8x^2)\ln(1-x)\mathrm{Li}_2(1-x)\\
& +(-20+24x-32x^2)\zeta(2)\ln(1-x)+\frac{1}{9}\left(-186-1362x+1570x^2+\frac{104}{x}\right)\ln(1-x)\\
& +(-4-72x+8x^2)\mathrm{Li}_3(1-x)+\frac{1}{3}\left(12-192x+176x^2+\frac{16}{x}\right)\mathrm{Li}_2(1-x)\\
& +\frac{1}{3}(10+28x)\ln^3(x)+\left(-1+88x-\frac{194}{3}x^2\right)\ln^2(x)+(-48x+16x^2)\zeta(2)\ln(x)\\
& +\left(58+\frac{584}{3}x-\frac{2090}{9}x^2\right)\ln(x)-(10+12x+12x^2)\zeta(3)\\
& +\frac{1}{3}\left(12-240x+268x^2-\frac{32}{x}\right)\zeta(2)+\frac{239}{9}+\frac{1072}{9}x-\frac{4493}{27}x^2+\frac{344}{27x}\Big\}\,.
\end{aligned} \tag{2}
$$

$$
\begin{aligned}
c_{2,-}^{(2)}(x) = {} & C_F\left(C_F - \frac{1}{2}C_A\right) \times \\
& \left\{\underline{\frac{1+x^2}{1-x}}\left[\left[4\ln^2(x) - 16\ln(x)\ln(1+x) - 16\mathrm{Li}_2(-x) - 8\zeta(2)\right]\ln(1-x)\right.\right. \\
& + \left[-2\ln^2(x) + 20\ln(x)\ln(1+x) - 8\ln^2(1+x) + 8\mathrm{Li}_2(1-x) + 16\mathrm{Li}_2(-x) - 8\right]\ln(x) \\
& - 16\ln(1+x)\mathrm{Li}_2(-x) - 8\zeta(2)\ln(1+x) - \underline{16\mathrm{Li}_3\left(-\frac{1-x}{1+x}\right)} \\
& \left. + \underline{16\mathrm{Li}_3\left(\frac{1-x}{1+x}\right)} - 16\mathrm{Li}_3(1-x) + 8\mathrm{S}_{1,2}(1-x) + 8\mathrm{Li}_3(-x) - 16\mathrm{S}_{1,2}(-x) + 8\zeta(3)\right] \\
& + (4+20x)\left[\ln^2(x)\ln(1+x) - 2\ln(x)\ln^2(1+x) - 2\zeta(2)\ln(1+x) - 4\ln(1+x)\mathrm{Li}_2(-x)\right. \\
& \left. + 2\mathrm{Li}_3(-x) - 4\mathrm{S}_{1,2}(-x) + 2\zeta(3)\right] + \left(32 + 32x + 48x^2 - \frac{72}{5}x^3 + \frac{8}{5x^2}\right) \\
& \times\left[\mathrm{Li}_2(-x) + \ln(x)\ln(1+x)\right] + 8(1+x)\left[\mathrm{Li}(1-x) + \ln(x)\ln(1-x)\right] + 16(1-x)\ln(1-x) \\
& + \left(-4 - 16x - 24x^2 + \frac{36}{5}x^3\right)\ln^2(x) + \frac{1}{5}\left(-26 - 106x + 72x^2 - \frac{8}{x}\right)\ln(x) \\
& \left. + \left(-4 + 20x + 48x^2 - \frac{72}{5}x^3\right)\zeta(2) + \frac{1}{5}\left(-162 + 82x + 72x^2 + \frac{8}{x}\right)\right\}. \qquad (3)
\end{aligned}
$$

→ 77 FUNCTIONS @ 2 LOOPS.

→ RATHER COMPLICATED ARGUMENTS

→ NOT MANY, IF ANY, RELATIONS

.....

KEY PROBLEMS:

- 2 LOOP WILSON COEFFICIENTS DEPEND ON ~ 80 FUNCTIONS
- 3 LOOP ANOM. DIMENSIONS $\lesssim$ 240 FUNCTIONS
- 3 LOOP WILSON COEFFICIENTS ~ 730 FUNCTIONS.

CAN THIS BE MADE TRACTABLE ?

$\longrightarrow$ EVEN MORE INVOLVED: MULTI-JET CROSS SECTIONS.

PRECISION MEASUREMENTS NEED FAST & PRECISE PROGRAMS

$\longrightarrow$ EXP. SYSTEMATICS

CURRENTLY: 1 CPU YEAR ! (NLO) α_s.

A MUCH DEEPER UNDERSTANDING IS NEEDED BEFORE WE CAN GO TO EVEN MORE LOOPS & LEGS.

3. The Mellin Symmetry

COLLINEAR FACTORIZATION ($m_i \to 0$)
IMPLIES THE CONNECTION:

$$\sigma(\hat{s}) = \int_0^1 dx_1 \int_0^1 dx_2\, \sigma_W(x_1)\, f(x_2)\, \delta(x - x_1 x_2)$$

$\hat{s} = xS.$

$$\sigma = \sigma_W \otimes f.$$

$$\mathbf{M}[\sigma(x)](N) := \int_0^1 dx\, x^{N-1}\, \sigma(x).$$

$$\boxed{\mathbf{M}[A \otimes B](N) = \mathbf{M}[A](N) \cdot \mathbf{M}[B](N)}$$

FEYNMAN AMPLITUDES MAY BE SIMPLIFIED BY CONSEQUENT OBSERVATION OF THIS CONNECTION.

WE SHOW THAT :

SPLITTING AND COEFFICIENT FUNCTIONS

UP TO $O(\alpha^2)$ ARE REPRESENTABLE AS

POLYNOMIALS OF FINITE HARMONIC SUMS

IN N-SPACE.

CF. ALSO
MOCH, VERMASEREN
12'99

---

2 LOOP QUANTITIES $F_i(x)$ $\longleftrightarrow$ HARMONIC SUMS UP TO TRANSCENDENTALITY 4

SIMPLIFICATION

ALGEBRA

ANALYTIC CONTINUAT.

MELLIN TRANSFORMS $M[F_i(x)](N)$ $N \in \mathbb{C}$

2 LOOP QUANTITIES $N \in \mathbb{C}$

NON-LINEAR MAPPINGS

PHYSICAL ANOMALOUS DIMENSIONS etc.

SINGLE CONTOUR INTEGRAL

5. Multiple Harmonic Sums

THE SIMPLEST EXAMPLE:

$$P_{qq}(x) = \left(\frac{1+x^2}{1-x}\right)_+ - \frac{2}{(1-x)_+} + \dots$$

$$\int_0^1 dx \frac{x^{N-1}}{(1-x)_+} = -\sum_{k=0}^{N-2}\int_0^1 dx\, x^k = -\sum_{k=1}^{N-1}\frac{1}{k} = -S_1(N-1).$$

ALTERNATING SUMS (→ COLORED ζ-VALUES)

$$S_{-1}(N) = (-1)^N \mathbf{M}\left[\frac{1}{1+x}\right](N) - \ln(2).$$

$$= \sum_{k=1}^{N}\frac{(-1)^k}{k} \qquad \text{(FINITE FOR } N \to \infty).$$

GENERAL CASE:

$$S_{a_1\dots a_\ell}(N) = \sum_{k_1=1}^{N}\frac{(\mathrm{sign}(a_1))^{k_1}}{k_1^{|a_1|}}\sum_{k_2=1}^{k_1}\frac{(\mathrm{sign}(a_2))^{k_2}}{k_2^{|a_2|}}\dots,$$

ALL MELLIN TRANSFORMS IN MASSLESS FIELD THEORIES FOR 1-PAR. QUANTITIES ARE REPRESENTED BY HARMONIC SUMS. OR SIMILAR TYPE.

(KNOWN TO $O(\alpha^3)$).

JB, S.KURTH '98

7 Appendix: Mellin Transforms

No.	$f(z)$	$M[f](N) = \int_0^1 dz z^{N-1} f(z)$
1	$\delta(1-z)$	1
2	z^r	$\frac{1}{N+r}$
3	$\left(\frac{1}{1-z}\right)_+$	$-S_1(N-1)$
4	$\frac{1}{1+z}$	$(-1)^{N-1}[\log(2) - S_1(N-1)]$ $+\frac{1+(-1)^{N-1}}{2} S_1\left(\frac{N-1}{2}\right) - \frac{1-(-1)^{N-1}}{2} S_1\left(\frac{N-2}{2}\right)$
5	$z^r \log^n(z)$	$\frac{(-1)^n}{(N+r)^{n+1}}\Gamma(n+1)$
6	$z^r \log(1-z)$	$-\frac{S_1(N+r)}{N+r}$
7	$z^r \log^2(1-z)$	$\frac{S_1^2(N+r) + S_2(N+r)}{N+r}$
8	$z^r \log^3(1-z)$	$-\frac{S_1^3(N+r) + 3S_1(N+r)S_2(N+r) + 2S_3(N+r)}{N+r}$
9	$\left[\frac{\log(1-z)}{1-z}\right]_+$	$\frac{1}{2}S_1^2(N-1) + \frac{1}{2}S_2(N-1)$
10	$\left[\frac{\log^2(1-z)}{1-z}\right]_+$	$-\left[\frac{1}{3}S_1^3(N-1) + S_1(N-1)S_2(N-1) + \frac{2}{3}S_3(N-1)\right]$
11	$\left[\frac{\log^3(1-z)}{1-z}\right]_+$	$\frac{1}{4}S_1^4(N-1) + \frac{3}{2}S_1^2(N-1)S_2(N-1)$ $+\frac{3}{4}S_2^2(N-1) + 2S_1(N-1)S_3(N-1)$ $+\frac{3}{2}S_4(N-1)$
12	$\frac{\log^n(z)}{1-z}$	$(-1)^{n+1}\Gamma(n+1)\left[S_{n+1}(N-1) - \zeta(n+1)\right]$

Only single sums!

No.	$f(z)$	$M[f](N)$
64	$\frac{\mathrm{Li}_3(-z)}{1+z}$	$(-1)^{N-1}\Big\{S_{3,-1}(N-1)+[S_3(N-1)-S_{-3}(N-1)]\log 2$ $+\frac{1}{2}\zeta(2)S_{-2}(N-1)-\frac{3}{4}\zeta(3)S_{-1}(N-1)$ $+\frac{1}{8}\zeta^2(2)-\frac{3}{4}\zeta(3)\log 2\Big\}$
65	$\mathrm{Li}_3(1-z)$	$\frac{1}{N}[S_1(N)S_2(N)-\zeta(2)S_1(N)+S_3(N)$ $-S_{2,1}(N)+\zeta(3)]$
66	$\frac{\mathrm{Li}_3(1-z)}{1-z}$	$-S_{1,1,2}(N-1)+\frac{1}{2}\zeta(2)S_1^2(N-1)+\frac{1}{2}\zeta(2)S_2(N-1)$ $-\zeta(3)S_1(N-1)+\frac{2}{5}\zeta^2(2)$
67	$\frac{\mathrm{Li}_3(1-z)}{1+z}$	$(-1)^{N-1}\Big[S_{-1,1,2}(N-1)-\zeta(2)S_{-1,1}(N-1)$ $+\zeta(3)S_{-1}(N-1)+\mathrm{Li}_4\left(\frac{1}{2}\right)-\frac{9}{20}\zeta^2(2)$ $+\frac{7}{8}\zeta(3)\log 2+\frac{1}{2}\zeta(2)\log^2 2+\frac{1}{24}\log^4 2\Big]$
68	$\mathrm{Li}_3\left(\frac{1-z}{1+z}\right)$ $-\mathrm{Li}_3\left(-\frac{1-z}{1+z}\right)$	$\frac{(-1)^N}{N}\Big[-S_{-1,2}(N)-S_{-2,1}(N)+S_1(N)S_{-2}(N)+S_{-3}(N)$ $+\zeta(2)S_{-1}(N)+\frac{1}{2}\zeta(2)S_1(N)-\frac{7}{8}\zeta(3)+\frac{3}{2}\zeta(2)\log 2\Big]$ $+\frac{1}{N}\Big[-S_{-1,-2}(N)-S_{2,1}(N)+S_1(N)S_2(N)+S_3(N)$ $-\frac{1}{2}\zeta(2)S_{-1}(N)-\zeta(2)S_1(N)+\frac{21}{8}\zeta(3)-\frac{3}{2}\zeta(2)\log 2\Big]$
69	$\frac{1}{1+z}\Big[\mathrm{Li}_3\left(\frac{1-z}{1+z}\right)$ $-\mathrm{Li}_3\left(-\frac{1-z}{1+z}\right)\Big]$	$(-1)^{N-1}\Big\{S_{1,1,-2}(N-1)-S_{1,-1,2}(N-1)+S_{-1,1,2}(N-1)$ $-S_{-1,-1,-2}(N-1)+2\zeta(2)S_{1,-1}(N-1)+\frac{1}{4}\zeta(2)S_1^2(N-1)$ $-\frac{1}{4}\zeta(2)S_{-1}^2(N-1)-\zeta(2)S_1(N-1)S_{-1}(N-1)-\zeta(2)S_{-2}(N-1)$ $-\left[\frac{7}{8}\zeta(3)-\frac{3}{2}\zeta(2)\log 2\right]S_1(N-1)$ $+\left[\frac{21}{8}\zeta(3)-\frac{3}{2}\zeta(2)\log 2\right]S_{-1}(N-1)$ $-2\mathrm{Li}_4\left(\frac{1}{2}\right)+\frac{19}{40}\zeta^2(2)+\frac{1}{2}\zeta(2)\log^2 2-\frac{1}{12}\log^4 2\Big\}$

$$
\begin{aligned}
&+\zeta(2)S_{1,-1}(N)+\left[\zeta(2)\log(2)-\frac{5}{8}\zeta(3)\right][S_1(N)-S_{-1}(N)]\\
&-\frac{3}{40}\zeta(2)^2+\frac{5}{8}\zeta(3)\log(2)-\frac{1}{2}\zeta(2)\log^2(2)\\
=\;&-2S_{-2,1,1}(N)+S_1(N)S_{-2,1}(N)+S_{-2,2}(N)+S_{-3,1}(N) \qquad (122)
\end{aligned}
$$

$$
\begin{aligned}
S_{1,2,-1}(N)\;=\;&(-1)^N\mathrm{M}\left\{\frac{1}{1+x}\left[\mathrm{Li}_2(-x)\log(1+x)+2\mathrm{S}_{1,2}(-x)\right]\right\}(N)\\
&-\log(2)\left[\mathrm{S}_{1,2}(N)-S_{1,-2}(N)\right]-\frac{1}{2}\zeta(2)S_{1,-1}(N)\\
&+\left[\frac{1}{4}\zeta(3)-\frac{1}{2}\zeta(2)\log(2)\right][S_1(N)-S_{-1}(N)]\\
&+\frac{6}{5}\zeta(2)^2-3\mathrm{Li}_4\left(\frac{1}{2}\right)-\frac{23}{8}\zeta(3)\log(2)+\zeta(2)\log^2(2)-\frac{1}{8}\log^4(2) \qquad (123)
\end{aligned}
$$

$$
\begin{aligned}
S_{1,2,1}(N)\;=\;&\mathrm{M}\left\{\left[\frac{1}{x-1}\left(\mathrm{Li}_2(x)\log(1-x)+2\mathrm{S}_{1,2}(x)\right)\right]_+\right\}(N)+\zeta(2)S_{1,1}(N)\\
=\;&-\mathrm{M}\left[\frac{\mathrm{Li}_3(1-x)}{x-1}\right](N)+\mathrm{M}\left[\left(\frac{1}{x-1}\right)_+\mathrm{S}_{1,2}(x)\right](N)\\
&+S_1(N)S_3(N)+\frac{1}{2}S_1^2(N)S_2(N)+\frac{1}{2}S_2^2(N)-\frac{1}{2}\zeta(2)S_1^2(N)\\
&+S_4(N)-\frac{1}{2}\zeta(2)S_2(N)-\frac{8}{5}\zeta^2(2) \qquad (124)
\end{aligned}
$$

$$
\begin{aligned}
S_{-1,-1,-2}(N)\;=\;&(-1)^{N+1}\mathrm{M}\left\{\frac{1}{1+x}\left[F_1(x)+\log(1-x)\mathrm{Li}_2(-x)\right]\right\}(N)\\
&+(-1)^{N+1}\mathrm{M}\left\{\frac{1}{1+x}\left[\frac{1}{2}\mathrm{S}_{1,2}(x^2)-\mathrm{S}_{1,2}(x)-\mathrm{S}_{1,2}(-x)\right]\right\}(N)\\
&+\frac{1}{2}\zeta(2)\left[S_{-1,1}(N)-S_{-1,-1}(N)\right]+\left[\frac{9}{8}\zeta(3)-\frac{3}{2}\zeta(2)\log(2)-\frac{1}{6}\log^3(2)\right]S_{-1}(N)\\
&-\frac{1}{10}\zeta(2)^2+\frac{17}{8}\zeta(3)\log(2)-\frac{7}{4}\zeta(2)\log^2(2)-\frac{1}{6}\log^4(2)\\
=\;&(-1)^{N+1}\mathrm{M}\left\{\frac{1}{1+x}\left[\mathrm{S}_{1,2}(-x)+\mathrm{Li}_2(-x)\log(1+x)+\mathrm{Li}_2(-x)\log(1-x)\right]\right\}(N)\\
&+S_1(N)S_{2,-1}(N)+S_{2,-2}(N)+S_{3,-1}(N)+S_{-1}(N)S_3(N)\\
&+\frac{1}{2}S_2(N)S_{-2}(N)+\frac{1}{2}S_{-1}^2(N)S_{-2}(N)\\
&+[S_1(N)-S_{-1}(N)][S_2(N)-S_{-2}(N)]\log 2+\frac{1}{2}\zeta(2)S_1(N)S_{-1}(N)\\
&+S_{-4}(N)+2\log(2)[S_3(N)-S_{-3}(N)]+\left[\frac{1}{2}\zeta(2)-\log^2(2)\right]S_2(N)\\
&+S_{-2}(N)\log^2(2)-\left[\frac{1}{4}\zeta(3)-\frac{1}{2}\zeta(2)\log(2)\right]S_1(N)\\
&+\left[\frac{3}{4}\zeta(3)-\frac{1}{2}\zeta(2)\log(2)\right]S_{-1}(N)-4\mathrm{Li}_4\left(\frac{1}{2}\right)+\frac{3}{2}\zeta^2(2)\\
&-\frac{5}{2}\zeta(3)\log(2)+\frac{1}{2}\zeta(2)\log^2(2)-\frac{1}{6}\log^4(2) \qquad (125)
\end{aligned}
$$

$(O(\alpha^2))$

$$F_1(x) = S_{1,2}\left(\frac{1-x}{2}\right) + S_{1,2}(1-x) - S_{1,2}\left(\frac{1-x}{1+x}\right)$$
$$+ S_{1,2}\left(\frac{1}{1+x}\right) - \log(2)\,\mathrm{Li}_2\left(\frac{1-x}{2}\right)$$
$$+ \frac{1}{2}\log^2 2\,\log\left(\frac{1+x}{2}\right) - \log(2)\,\mathrm{Li}_2\left(\frac{1-x}{1+x}\right).$$

$\longrightarrow$ THE MELLIN TRANSFORM OF $F_1(x)$ TURNS OUT TO BE A POLYNOM OF MUCH SIMPLER MELLIN TRANSFORMS.

$\rightarrow$ $\otimes$-PRODUCT REDUCIBLE.

LINEAR REPRESENTATIONS OF MELLIN TRANSFORMS THROUGH HARMONIC SUMS:

$$M[F_w(x)](N) = S^{w}_{k_1,\dots,k_m}(N)$$

ZETA VALUES

$$+ P\left(S^{\tau'}_{k_1\dots k_r},\ \sigma^{\tau''}_{k_1\dots k_p}\right)$$

$w = \sum_{i=1}^{m} |k_i|$ WEIGHT

HARMONIC SUMS

$\tau', \tau'' < w$

P is a polynomial.

NUMBER OF FUNCTIONS $F_w(x)$ & SUMS:

w	#	Σ			
1	2	2			EXPL. KNOWN IN ALL DETAILS
2	6	8			
3	18	26	2 LOOP	ANOM. DIM.	
4	54	80		COEFF. FCT.	
5	162	242	3 LOOP	ANOM. DIM.	ALGEBRA FULLY KNOWN.
6	486	728		COEFF FCT.	
	$2 \cdot 3^{w-1}$	$3^w - 1$			

ALGEBRAIC RELATIONS:

L. EULER (1775):

$$S_{m,n} + S_{n,m} = S_m \cdot S_n + S_{m+n}$$
$$m, n > 0.$$

FIRST ALGEBRAIC RELATION!

ONE MAY GENERALIZE THIS TO $m, n \lessgtr 0$

$$S_{m,n} + S_{n,m} = S_m \cdot S_n + S_{m \wedge n}$$

$$m \wedge n = [|m| + |n|] \, \mathrm{sign}(m) \, \mathrm{sign}(n).$$

TERNARY RELATION: SITA RAMACHANDRA RAO 1984

4-ARY —"— : JB, KURTH 1998.

THESE & OTHER RELATIONS HOLD WIDELY INDEPENDENT OF THE VALUE & TYPE OF THESE OBJECTS.

DETERMINED BY:
- INDEX STRUCTURE
- MULTIPLICATION RELATION

$\longrightarrow$ QUASI-SHUFFLE ALGEBRAS

FREE LIE ALGEBRAS etc.

SHUFFLE PRODUCTS

(MAC LANE 1950)
(LYNDON 1954)

Depth 2 :

$$S_{a_1}(N) \sqcup\!\sqcup S_{a_2}(N) = S_{a_1,a_2}(N) + S_{a_2,a_1}(N)$$

Depth 3 :

$$S_{a_1}(N) \sqcup\!\sqcup S_{a_2,a_3}(N) = S_{a_1,a_2,a_3}(N) + S_{a_2,a_1,a_3}(N) + S_{a_2,a_3,a_1}(N) \tag{2.8}$$

Depth 4 :

$$S_{a_1}(N) \sqcup\!\sqcup S_{a_2,a_3,a_4}(N) = S_{a_1,a_2,a_3,a_4}(N) + S_{a_2,a_1,a_3,a_4}(N) + S_{a_2,a_3,a_1,a_4}(N) + S_{a_2,a_3,a_4,a_1}(N) \tag{2.9}$$

$$\begin{aligned} S_{a_1,a_2}(N) \sqcup\!\sqcup S_{a_3,a_4}(N) &= S_{a_1,a_2,a_3,a_4}(N) + S_{a_1,a_3,a_2,a_4}(N) + S_{a_1,a_3,a_4,a_2}(N) \\ &+ S_{a_3,a_4,a_1,a_2}(N) + S_{a_3,a_1,a_4,a_2}(N) + S_{a_3,a_1,a_2,a_4}(N) \end{aligned} \tag{2.10}$$

Depth 5 :

$$\begin{aligned} S_{a_1}(N) \sqcup\!\sqcup S_{a_2,a_3,a_4,a_5}(N) &= S_{a_1,a_2,a_3,a_4,a_5}(N) + S_{a_2,a_1,a_3,a_4,a_5}(N) + S_{a_2,a_3,a_1,a_4,a_5}(N) \\ &+ S_{a_2,a_3,a_4,a_1,a_5}(N) + S_{a_2,a_3,a_4,a_5,a_1}(N) . \end{aligned} \tag{2.11}$$

$$\begin{aligned} S_{a_1,a_2}(N) \sqcup\!\sqcup S_{a_3,a_4,a_5}(N) &= S_{a_1,a_2,a_3,a_4,a_5}(N) + S_{a_1,a_3,a_2,a_4,a_5}(N) + S_{a_1,a_3,a_4,a_2,a_5}(N) \\ &+ S_{a_1,a_3,a_4,a_5,a_2}(N) + S_{a_3,a_1,a_2,a_4,a_5}(N) + S_{a_3,a_1,a_4,a_2,a_5}(N) \\ &+ S_{a_3,a_1,a_4,a_5,a_2}(N) + S_{a_3,a_4,a_5,a_1,a_2}(N) + S_{a_3,a_4,a_1,a_5,a_2}(N) \\ &+ S_{a_3,a_4,a_1,a_2,a_5}(N) \end{aligned} \tag{2.12}$$

Depth 6 :

$$\begin{aligned} S_{a_1}(N) \sqcup\!\sqcup S_{a_2,a_3,a_4,a_5,a_6}(N) &= S_{a_1,a_2,a_3,a_4,a_5,a_6}(N) + S_{a_2,a_1,a_3,a_4,a_5,a_6}(N) + S_{a_2,a_3,a_1,a_4,a_5,a_6}(N) \\ &+ S_{a_2,a_3,a_4,a_1,a_5,a_6}(N) + S_{a_2,a_3,a_4,a_5,a_1,a_6}(N) + S_{a_2,a_3,a_4,a_5,a_6,a_1}(N) \end{aligned} \tag{2.13}$$

$$\begin{aligned} S_{a_1,a_2}(N) \sqcup\!\sqcup S_{a_3,a_4,a_5,a_6}(N) &= S_{a_1,a_2,a_3,a_4,a_5,a_6}(N) + S_{a_1,a_3,a_2,a_4,a_5,a_6}(N) + S_{a_1,a_3,a_4,a_2,a_5,a_6}(N) \\ &+ S_{a_1,a_3,a_4,a_5,a_2,a_6}(N) + S_{a_1,a_3,a_4,a_5,a_6,a_2}(N) + S_{a_3,a_1,a_2,a_4,a_5,a_6}(N) \\ &+ S_{a_3,a_1,a_4,a_2,a_5,a_6}(N) + S_{a_3,a_1,a_4,a_5,a_2,a_6}(N) + S_{a_3,a_1,a_4,a_5,a_6,a_2}(N) \\ &+ S_{a_3,a_4,a_1,a_2,a_5,a_6}(N) + S_{a_3,a_4,a_1,a_5,a_2,a_6}(N) + S_{a_3,a_4,a_1,a_5,a_6,a_2}(N) \\ &+ S_{a_3,a_4,a_5,a_6,a_1,a_2}(N) + S_{a_3,a_4,a_5,a_1,a_6,a_2}(N) + S_{a_3,a_4,a_5,a_1,a_2,a_6}(N) \end{aligned} \tag{2.14}$$

$$\begin{aligned} S_{a_1,a_2,a_3}(N) \sqcup\!\sqcup S_{a_4,a_5,a_6}(N) &= S_{a_1,a_2,a_3,a_4,a_5,a_6}(N) + S_{a_1,a_2,a_4,a_3,a_5,a_6}(N) + S_{a_1,a_2,a_4,a_5,a_3,a_6}(N) \\ &+ S_{a_1,a_2,a_4,a_5,a_6,a_3}(N) + S_{a_1,a_4,a_2,a_3,a_5,a_6}(N) + S_{a_1,a_4,a_2,a_5,a_3,a_6}(N) \\ &+ S_{a_1,a_4,a_2,a_5,a_6,a_3}(N) + S_{a_1,a_4,a_5,a_6,a_2,a_3}(N) + S_{a_1,a_4,a_5,a_2,a_6,a_3}(N) \\ &+ S_{a_1,a_4,a_5,a_2,a_3,a_6}(N) + S_{a_4,a_5,a_6,a_1,a_2,a_3}(N) + S_{a_4,a_5,a_1,a_6,a_2,a_3}(N) \\ &+ S_{a_4,a_5,a_1,a_2,a_6,a_3}(N) + S_{a_4,a_5,a_1,a_2,a_3,a_6}(N) + S_{a_4,a_1,a_5,a_6,a_2,a_3}(N) \\ &+ S_{a_4,a_1,a_5,a_2,a_6,a_3}(N) + S_{a_4,a_1,a_5,a_2,a_3,a_6}(N) + S_{a_4,a_1,a_2,a_5,a_3,a_6}(N) \\ &+ S_{a_4,a_1,a_2,a_3,a_5,a_6}(N) + S_{a_4,a_1,a_2,a_5,a_6,a_3}(N) . \end{aligned} \tag{2.15}$$

THE ALGEBRAIC EQUATIONS

Depth 2 :

$$S_{a_1}(N) \sqcup\!\sqcup S_{a_2}(N) - S_{a_1}(N)S_{a_2}(N) - S_{a_1\wedge a_2}(N) = 0 \quad [36] \tag{2.17}$$

Depth 3 :

$$S_{a_1}(N) \sqcup\!\sqcup S_{a_2,a_3}(N) - S_{a_1}(N)S_{a_2,a_3}(N) - S_{a_1\wedge a_2,a_3}(N) - S_{a_2,a_1\wedge a_3}(N) = 0 \tag{2.18}$$

Depth 4 :

$$\begin{aligned} S_{a_1}(N) \sqcup\!\sqcup S_{a_2,a_3,a_4}(N) &- S_{a_1}(N)S_{a_2,a_3,a_4}(N) - S_{a_1\wedge a_2,a_3,a_4}(N) - S_{a_2,a_1\wedge a_3,a_4}(N) \\ &- S_{a_2,a_3,a_1\wedge a_4}(N) = 0 \end{aligned} \tag{2.19}$$

$$\begin{aligned} S_{a_1,a_2}(N) \sqcup\!\sqcup S_{a_3,a_4}(N) &- S_{a_1,a_2}(N)S_{a_3,a_4}(N) - S_{a_1,a_2\wedge a_3,a_4}(N) - S_{a_1,a_3,a_2\wedge a_4}(N) \\ &- S_{a_3,a_1\wedge a_4,a_2}(N) - S_{a_3,a_1,a_2\wedge a_4}(N) - S_{a_1\wedge a_3,a_2,a_4}(N) \\ &- S_{a_1\wedge a_3,a_4,a_2}(N) + S_{a_1\wedge a_3,a_2\wedge a_4} = 0 \end{aligned} \tag{2.20}$$

Depth 5 :

$$\begin{aligned} S_{a_1}(N) \sqcup\!\sqcup S_{a_2,a_3,a_4,a_5}(N) &- S_{a_1}(N)S_{a_2,a_3,a_4,a_5}(N) - S_{a_1\wedge a_2,a_3,a_4,a_5}(N) - S_{a_2,a_1\wedge a_3,a_4,a_5}(N) \\ &- S_{a_2,a_3,a_1\wedge a_4,a_5}(N) - S_{a_2,a_3,a_4,a_1\wedge a_5}(N) = 0 \end{aligned} \tag{2.21}$$

$$\begin{aligned} S_{a_1,a_2}(N) \sqcup\!\sqcup S_{a_3,a_4,a_5}(N) &- S_{a_1,a_2\wedge a_3,a_4,a_5}(N) - S_{a_1,a_3,a_2\wedge a_4,a_5}(N) - S_{a_1,a_3,a_4,a_2\wedge a_5}(N) \\ &- S_{a_3,a_1,a_2\wedge a_4,a_5}(N) - S_{a_3,a_1,a_4,a_2\wedge a_5}(N) - S_{a_3,a_4,a_1\wedge a_5,a_2}(N) \\ &- S_{a_3,a_4,a_1,a_2\wedge a_5}(N) - S_{a_3,a_1\wedge a_4,a_2,a_5}(N) - S_{a_3,a_1\wedge a_4,a_5,a_2}(N) \\ &- S_{a_1,a_2}(N)S_{a_3,a_4,a_5}(N) - S_{a_1\wedge a_3,a_2,a_4,a_5}(N) - S_{a_1\wedge a_3,a_4,a_2,a_5}(N) \\ &- S_{a_1\wedge a_3,a_4,a_5,a_2}(N) + S_{a_1\wedge a_3,a_2\wedge a_4,a_5}(N) + S_{a_1\wedge a_3,a_4,a_2\wedge a_5}(N) = 0 \end{aligned} \tag{2.22}$$

Depth 6 :

$$\begin{aligned} S_{a_1}(N) \sqcup\!\sqcup S_{a_2,a_3,a_4,a_5,a_6}(N) &- S_{a_1}(N)S_{a_2,a_3,a_4,a_5,a_6}(N) - S_{a_1\wedge a_2,a_3,a_4,a_5,a_6}(N) \\ &- S_{a_2,a_1\wedge a_3,a_4,a_5,a_6}(N) - S_{a_2,a_3,a_1\wedge a_4,a_5,a_6}(N) \\ &- S_{a_2,a_3,a_4,a_1\wedge a_5,a_6}(N) - S_{a_2,a_3,a_4,a_5,a_1\wedge a_6}(N) = 0 \end{aligned} \tag{2.23}$$

$$\begin{aligned} S_{a_1,a_2}(N) \sqcup\!\sqcup S_{a_3,a_4,a_5,a_6}(N) &- S_{a_1,a_2\wedge a_3,a_4,a_5,a_6}(N) - S_{a_1,a_3,a_2\wedge a_4,a_5,a_6}(N) - S_{a_1,a_3,a_4,a_2\wedge a_5,a_6}(N) \\ &- S_{a_1,a_3,a_4,a_5,a_2\wedge a_6}(N) - S_{a_3,a_1,a_2\wedge a_4,a_5,a_6}(N) - S_{a_3,a_1,a_4,a_2\wedge a_5,a_6}(N) \\ &- S_{a_3,a_1,a_4,a_5,a_2\wedge a_6}(N) - S_{a_3,a_4,a_1,a_2\wedge a_5,a_6}(N) - S_{a_3,a_4,a_1,a_5,a_2\wedge a_6}(N) \\ &- S_{a_3,a_4,a_5,a_1\wedge a_6,a_2}(N) - S_{a_3,a_4,a_5,a_1,a_2\wedge a_6}(N) - S_{a_3,a_4,a_1\wedge a_5,a_2,a_6}(N) \\ &- S_{a_3,a_4,a_1\wedge a_5,a_6,a_2}(N) - S_{a_3,a_1\wedge a_4,a_2,a_5,a_6}(N) - S_{a_3,a_1\wedge a_4,a_5,a_2,a_6}(N) \\ &- S_{a_3,a_1\wedge a_4,a_5,a_6,a_2}(N) - S_{a_1\wedge a_3,a_2,a_4,a_5,a_6}(N) - S_{a_1\wedge a_3,a_4,a_2,a_5,a_6}(N) \\ &- S_{a_1\wedge a_3,a_4,a_5,a_2,a_6}(N) - S_{a_1\wedge a_3,a_4,a_5,a_6,a_2}(N) + S_{a_3,a_4,a_1\wedge a_5,a_2\wedge a_6}(N) \\ &+ S_{a_3,a_1\wedge a_4,a_2\wedge a_5,a_6}(N) + S_{a_3,a_1\wedge a_4,a_5,a_2\wedge a_6}(N) + S_{a_3,a_4,a_1\wedge a_5,a_2\wedge a_6}(N) \\ &+ S_{a_1\wedge a_3,a_2\wedge a_4,a_5,a_6}(N) + S_{a_1\wedge a_3,a_4,a_2\wedge a_5,a_6}(N) + S_{a_1\wedge a_3,a_4,a_5,a_2\wedge a_6}(N) \\ &- S_{a_1,a_2}(N)S_{a_3,a_4,a_5,a_6}(N) = 0 \end{aligned} \tag{2.24}$$

ALLOW FOR ANY INDEX PERMUTATION.

HOW MANY OF THESE EQ. ARE INDEPENDENT ?

BASIC SUMS – # PERM. – # IND. EQS.

$$
\begin{aligned}
S_{a_1,a_2,a_3}(N) \sqcup S_{a_4,a_5,a_6}(N) \; - \; & S_{a_1,a_2,a_3\wedge a_4,a_5,a_6}(N) - S_{a_1,a_2,a_4,a_3\wedge a_5,a_6}(N) - S_{a_1,a_2,a_4,a_5,a_3\wedge a_6}(N) \\
- \; & S_{a_1,a_4,a_2,a_3\wedge a_5,a_6}(N) - S_{a_1,a_4,a_2,a_5,a_3\wedge a_6}(N) - S_{a_1,a_4,a_5,a_2\wedge a_6,a_3}(N) \\
- \; & S_{a_1,a_4,a_5,a_2,a_3\wedge a_6}(N) - S_{a_1,a_4,a_2\wedge a_5,a_3,a_6}(N) - S_{a_1,a_4,a_2\wedge a_5,a_6,a_3}(N) \\
- \; & S_{a_1,a_2\wedge a_4,a_3,a_5,a_6}(N) - S_{a_1,a_2\wedge a_4,a_5,a_3,a_6}(N) - S_{a_1,a_2\wedge a_4,a_5,a_6,a_3}(N) \\
- \; & S_{a_4,a_5,a_1\wedge a_6,a_2,a_3}(N) - S_{a_4,a_5,a_1,a_2\wedge a_6,a_3}(N) - S_{a_4,a_5,a_1,a_2,a_3\wedge a_6}(N) \\
- \; & S_{a_4,a_1,a_5,a_2\wedge a_6,a_3}(N) - S_{a_4,a_1,a_5,a_2,a_3\wedge a_6}(N) - S_{a_4,a_1,a_2,a_3\wedge a_5,a_6}(N) \\
- \; & S_{a_4,a_1,a_2,a_5,a_3\wedge a_6}(N) - S_{a_4,a_1,a_2\wedge a_5,a_6,a_3}(N) - S_{a_4,a_1,a_2\wedge a_5,a_3,a_6}(N) \\
- \; & S_{a_4,a_1\wedge a_5,a_6,a_2,a_3}(N) - S_{a_4,a_1\wedge a_5,a_2,a_6,a_3}(N) - S_{a_4,a_1\wedge a_5,a_2,a_3,a_6}(N) \\
- \; & S_{a_1\wedge a_4,a_2,a_3,a_5,a_6}(N) - S_{a_1\wedge a_4,a_2,a_5,a_3,a_6}(N) - S_{a_1\wedge a_4,a_2,a_5,a_6,a_3}(N) \\
- \; & S_{a_1\wedge a_4,a_5,a_6,a_2,a_3}(N) - S_{a_1\wedge a_4,a_5,a_2,a_6,a_3}(N) - S_{a_1\wedge a_4,a_5,a_2,a_3,a_6}(N) \\
+ \; & S_{a_1,a_4,a_2\wedge a_5,a_3\wedge a_6}(N) + S_{a_1,a_2\wedge a_4,a_3\wedge a_5,a_6}(N) + S_{a_1,a_2\wedge a_4,a_5,a_3\wedge a_6}(N) \\
+ \; & S_{a_4,a_1,a_2\wedge a_5,a_3\wedge a_6}(N) + S_{a_4,a_1\wedge a_5,a_2\wedge a_6,a_3}(N) + S_{a_4,a_1\wedge a_5,a_2,a_3\wedge a_6}(N) \\
+ \; & S_{a_1\wedge a_4,a_2,a_3\wedge a_5,a_6}(N) + S_{a_1\wedge a_4,a_2,a_5,a_3\wedge a_6}(N) + S_{a_1\wedge a_4,a_5,a_2\wedge a_6,a_3}(N) \\
+ \; & S_{a_1\wedge a_4,a_5,a_2,a_3\wedge a_6}(N) + S_{a_1\wedge a_4,a_2\wedge a_5,a_3,a_6}(N) + S_{a_1\wedge a_4,a_2\wedge a_5,a_6,a_3}(N) \\
- \; & S_{a_1\wedge a_4,a_2\wedge a_5,a_3\wedge a_6}(N) - S_{a_1,a_2,a_3}(N) S_{a_4,a_5,a_6}(N) = 0 \, . \qquad (2.25)
\end{aligned}
$$

W	MATRICES	
3	$6 \times 6 \leq$	
4	$24 \times 48 \leq$	
5	$120 \times 240 \leq$	
6	$720 \times 2160 \leq$	$\leftarrow$ 1 CPU day (2GHz, 2GBYTE)

$\longrightarrow$ NUMBER OF BASIS SUMS

& BASIS SUMS (EXPL. FORM.)

$\longrightarrow$ ALL RELATIONS.

DEPTH = 3

Index Set	Number	Dep. Sums of Depth 3	min. Weight	Fraction of fund. Sums
$\{a,a,a\}$	1	1	3	0
$\{a,a,b\}$	3	2	3	1/3
$\{a,b,c\}$	6	4	4	1/3

DEPTH = 4

Index Set	Number	Dep. Sums of Depth 4	min. Weight	Fraction of fund. Sums
$\{a,a,a,a\}$	1	1	4	0
$\{a,a,a,b\}$	4	3	4	1/4
$\{a,a,b,b\}$	6	5	4	1/6
$\{a,a,b,c\}$	12	9	5	1/4
$\{a,b,c,d\}$	24	18	6	1/4

DEPTH = 5

Index Set	Number	Dep. Sums of Depth 5	min. Weight	Fraction of fund. Sums
$\{a,a,a,a,a\}$	1	1	5	0
$\{a,a,a,a,b\}$	5	4	5	1/5
$\{a,a,a,b,b\}$	10	8	5	1/5
$\{a,a,a,b,c\}$	20	16	6	1/5
$\{a,a,b,b,c\}$	30	24	6	1/5
$\{a,a,b,c,d\}$	60	48	7	1/5
$\{a,b,c,d,e\}$	120	96	9	1/5

DEPTH = 6

Index Set	Number	Rel.1	Rel.2	Rel.3	Rel.1,2	Rel.1,2,3	min. Weight	Frac. of fund. Sums
$\{a,a,a,a,a,a\}$	1	1	1	1	1	1	6	0
$\{a,a,a,a,a,b\}$	6	5	5	5	5	5	6	1/6
$\{a,a,a,a,b,b\}$	15	11	9	7	12	13	6	2/15
$\{a,a,a,b,b,b\}$	20	14	12	8	16	17	6	3/20
$\{a,a,a,a,b,c\}$	30	22	18	12	24	25	7	1/6
$\{a,a,a,b,b,c\}$	60	41	35	23	47	50	7	1/6
$\{a,a,b,b,c,c\}$	90	60	52	36	70	76	8	7/45
$\{a,a,a,b,c,d\}$	120	81	70	45	94	100	8	1/6
$\{a,a,b,b,c,d\}$	180	118	104	67	140	150	8	1/6
$\{a,a,b,c,d,e\}$	360	232	208	132	280	300	10	1/6
$\{a,b,c,d,e,f\}$	720	455	416	261	560	600	12	1/6

THE ALGEBRAIC RELATIONS REDUCE THE NUMBER OF MELLIN TRANSFORMS TO ~~24~~ 23 (OUT OF 80):

$$
\begin{array}{cccc}
\dfrac{\log(1+x)}{x+1} & \dfrac{\log^2(1+x)-\log^2(2)}{x-1} & \dfrac{\log^2(1+x)}{x+1} & \dfrac{\mathrm{Li}_2(x)}{x+1} \\[2ex]
\dfrac{\mathrm{Li}_2(x)-\zeta(2)}{x-1} & \dfrac{\mathrm{Li}_2(-x)}{x+1} & \dfrac{\mathrm{Li}_2(-x)+\zeta(2)/2}{x-1} & \rightarrow \dfrac{\log(x)\mathrm{Li}_2(x)}{x+1} \\[2ex]
\rightarrow \dfrac{\log(x)\mathrm{Li}_2(x)}{x-1} & \dfrac{\mathrm{Li}_3(x)}{x+1} & \cancel{\dfrac{\mathrm{Li}_3(x)-\zeta(3)}{x-1}} & \dfrac{\mathrm{Li}_3(-x)}{x+1} \\[2ex]
\dfrac{\mathrm{Li}_3(-x)-3\zeta(3)/4}{x-1} & \dfrac{S_{1,2}(x)}{x+1} & \dfrac{S_{1,2}(x)-\zeta(3)}{x-1} & \dfrac{S_{1,2}(-x)-\zeta(3)/8}{x-1} \\[2ex]
\dfrac{S_{1,2}(-x)}{x+1} & \dfrac{S_{1,2}(x^2)}{x+1} & \dfrac{S_{1,2}(x^2)-\zeta(3)}{x-1} & \log(1-x)\dfrac{\mathrm{Li}_2(-x)}{x+1} \\[2ex]
\dfrac{\log(1+x)-\log(2)}{x-1}\mathrm{Li}_2(x) & \dfrac{\log(1+x)-\log(2)}{x-1}\mathrm{Li}_2(-x) & \dfrac{\log(1-x)\mathrm{Li}_2(x)}{1+x} & \dfrac{\log(1+x)}{1+x}\mathrm{Li}_2(x)
\end{array}
$$

$$
\rightarrow \frac{\log(x)\log^2(1+x)}{1-x}, \quad \frac{1}{1+x}\left[2\mathrm{Li}_3\left(\frac{1-x}{2}\right) - \ln(1-x)\,\mathrm{Li}_2\left(\frac{1-x}{2}\right)\right] \tag{191}
$$

⟶ IN 2-LOOP PHYSICAL PROCESSES EVEN A LOWER NUMBER OF BASIC TRANSFORMS IS GOING TO OCCUR!

⟶ ANY 2-LOOP QUANTITY ($m \to 0$) CAN BE REPRESENTED AS A MELLIN POLYNOMIAL OF THE ABOVE FUNCTIONS.

6. Theory of Words

(LOTHAIRE
REUTENAUER)

CAN WE COUNT THE BASIS SIMPLER ?

YES.

INTRODUCE FREE LIE ALGEBRAS & THE THEORY OF CODES INTO PARTICLE PHYSICS.

EVERYTHING GOES THROUGH THE INDEX SET.

$\mathfrak{A} = \{a, b, c, d, \dots\}$ ALPHABET

$a < b < c < d < \dots$ ORDERED

$\mathfrak{A}^*(\mathfrak{A})$ SET OF WORDS OVER $\mathfrak{A}$

$W = a_1 \cdot a_2 \cdot a_{27} \, a_3 \cdot a_3 \dots a_{532}$ WORD

↑
NON-COMMUTATIVE PRODUCT.

$W = p \cdot x \cdot s$

↑ PREFIX ↑ SUFFIX

DEFINITION:

A LYNDON WORD IS SMALLER THAN ALL ITS SUFFIXES.

THEOREM [RADFORD, 1979]

THE SHUFFLE ALGEBRA $K\langle\mathfrak{A}\rangle$ IS FREELY GENERATED BY THE LYNDON WORDS.

$\longrightarrow$ I.E. THE NUMBER OF LYNDON WORDS IS THE NUMBER OF BASIC ELEMENTS.

EXERCISES:

$\{\underbrace{a, a, \dots, a}, b\}$ aaa..ab 1 LYNDON WORD

n PERMUTATIONS

$$\frac{n_{basic}}{n_{all}} = \frac{1}{n}$$

$n \equiv$ DEPTH (OF THE SUMS).

$\{a, a, a, b, b, b\}$

aaabbb
aababb
aabbab

3 LYNDON WORDS

$$\frac{n_{basic}}{n_{all}} = \frac{3}{20} < \frac{1}{6}$$

CAN ONE DERIVE A FORMULA ON THESE RELATIONS ?

(... DIG THE MATHEMATICAL LITERATURE.)

! E. WITT (HH) (1937): JOURN. REINE & ANGEW. MATHEMATIK.

"TREUE DARSTELLUNG LIESCHER RINGE"

$$l_n(n_1, \ldots, n_q) = \frac{1}{n} \sum_{d|n} \mu(d) \frac{\left(\frac{n}{d}\right)!}{\left(\frac{n_1}{d}\right)! \ldots \left(\frac{n_q}{d}\right)!}$$

with $\sum_i n_i = n$.

$$l_6(\{a,a,a,b,b,b\}) = \frac{1}{6}\left[\mu(1)\frac{6!}{3!3!} + \mu(3)\frac{2!}{1!1!}\right] = 3$$

$$n_6(\{a,a,a,b,b,b\}) = \frac{6!}{3!3!} = \frac{4\cdot5\cdot6}{2\cdot3} = 20$$

$$\frac{l_6(\{\ldots\})}{n_6(\{\ldots\})} = \frac{3}{20} < \frac{1}{6} \qquad ; \mu(3) = -1.$$

SUM OVER THE 2nd WITT FORMULA.

↓

Weight	# Sums	Cum. # Sums	# Basic Sums	Cum. # Basic Sums	Cum. Fraction
1	2	2	0	0	0.0
2	6	8	1	1	0.1250
3	18	26	6	7	0.2692
4	54	80	16	23	0.2875
5	162	242	46	69	0.2851
6	486	728	114	183	0.2513

7. Deeper Relations

CAN ONE REDUCE THE BASIS FURTHER ?

PRESENT RESULTS ONLY FINISHED FOR $O(\alpha^2)$, I.E. DEPTH = 4 SUMS ($w=4$).

FIG.

- NO SYSTEMATIC MATHEMATICAL THEORY YET.

THE NUMBER OF LYNDON WORDS $l_n(\{a_1, \dots a\}) = 0$ $\left(\sum_{d|n} \mu(d) = 0\right)$.

$\longrightarrow$ DO NOT COUNT SINGLE HARMONIC SUMS, POLYNOMIALS OR RAT. FCT'S IN N.

$$\boxed{M[\ln^l(x) f(x)](N) = \frac{\partial^l}{(\partial N)^l} M[f(x)](N)}$$

IF $M[f(x)](N)$ IS KNOWN, ANY DERIVATIVE IS KNOWN (EASILY CALCULATED).

$\psi(N)$ KNOWN $\longrightarrow$ $\psi^{(k)}(N)$ $\forall k$, KNOWN. etc.

23 FCTS $\longrightarrow$ 20 FCTS $\quad w \leq 4$.

RELATIONS BETWEEN MELLIN TRANSFORMS LEAD TO A FURTHER REDUCTION.

$\longrightarrow$ EXPLICIT CALCULATIONS.

WORK IN PROGRESS.

$\boxed{\text{THE LORD IS MERCY.}}$

FEYNMAN DIAGRAM CALCULATIONS SEEM NOT TO PRODUCE ALL POSSIBLE SUMS.

<u>$O(\alpha^2)$</u>

$$\frac{1}{\varepsilon} \to \quad \frac{\mathrm{Li}_2(x)}{1+x} \qquad \ln(x)\frac{\mathrm{Li}_2(x)}{1+x} \qquad \frac{\mathrm{Li}_3(x)}{1+x} \qquad \frac{S_{1,2}(x)}{1+x}$$

$$\left(\frac{\mathrm{Li}_2(x)}{1-x}\right)_+ \qquad \ln(x)\frac{\mathrm{Li}_2(x)}{1-x} \qquad \left(\frac{\mathrm{Li}_3(x)}{1-x}\right)_+ \qquad \left(\frac{S_{1,2}(x)}{1-x}\right)_+$$

JB, S. MOCH

MASSLESS QCD @ 2 LOOPS DEPENDS ON ESSENTIALLY **5** FUNCTIONS FOR ANOM. DIMS. & WILSON COEFFICIENTS

REDUCTION: $77 \longrightarrow 5$

THANKS TO MELLIN SYMMETRY.

W		ALGEBRA	FURTHER MATH.	
5	162 $\to$	46 $\longrightarrow$	? PROBABLY	14 '11/03
6	486 $\to$	114 $\longrightarrow$	?	≤ 10

ANALYTIC CONTINUATION

$$M[f(x)](N) = \int_0^1 dx\, x^{N-1} f(x) \qquad N \in \mathbb{N} \ \text{even or odd}$$

$$N \longrightarrow \mathbb{C}$$

WHERE ARE SINGULARITIES ? SINGLE POLES

$Re(N_s) \leqslant n_1$ fixed HARMONIC SUMS $\rightarrow$ N_s integer

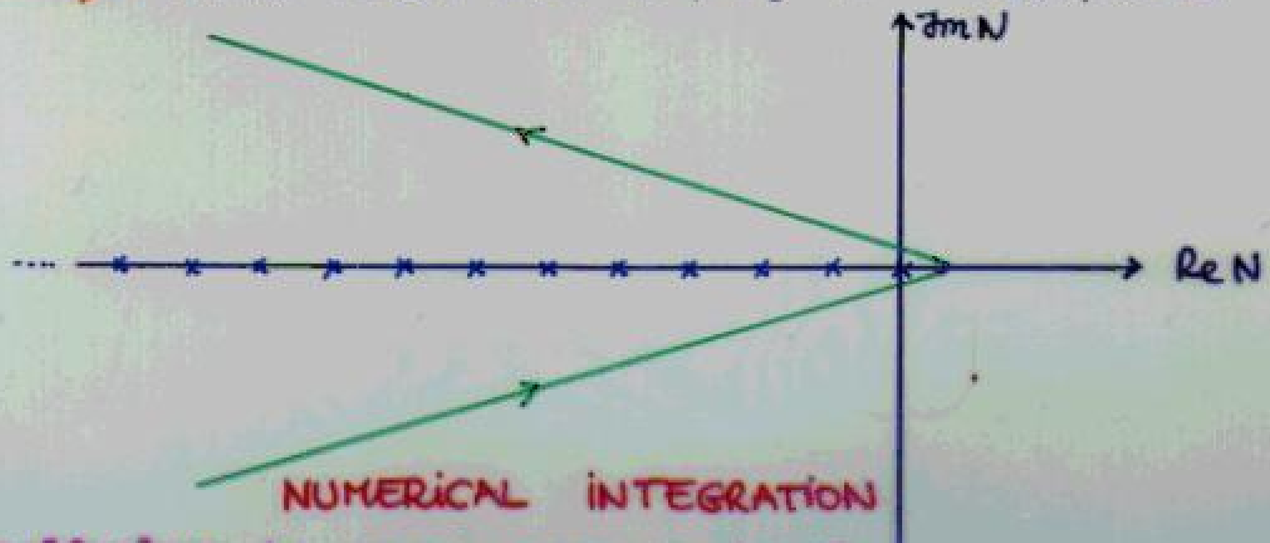
fixed order PT $Im(N_s) = 0$

REPRESENTATION: (NIELSEN, MELLIN ~ 1905)

- STIRLING-LIKE ASYMPTOTIC REPRESENTATIONS.
- USE RECURSION RELATIONS:

$$S_{ab_1\ldots b_p}(N+1) - S_{ab_1\ldots b_p}(N) = \frac{[\mathrm{sign}(a)]^N}{N^{|a|}} S_{b_1\ldots b_p}(N)$$

$\longrightarrow$ MOVE FROM ANY $N \neq N_s$ TO $Re(N) \gg 1$



$M[f(x)](N)$ IS MEROMORPHIC IN $\mathbb{C}$.

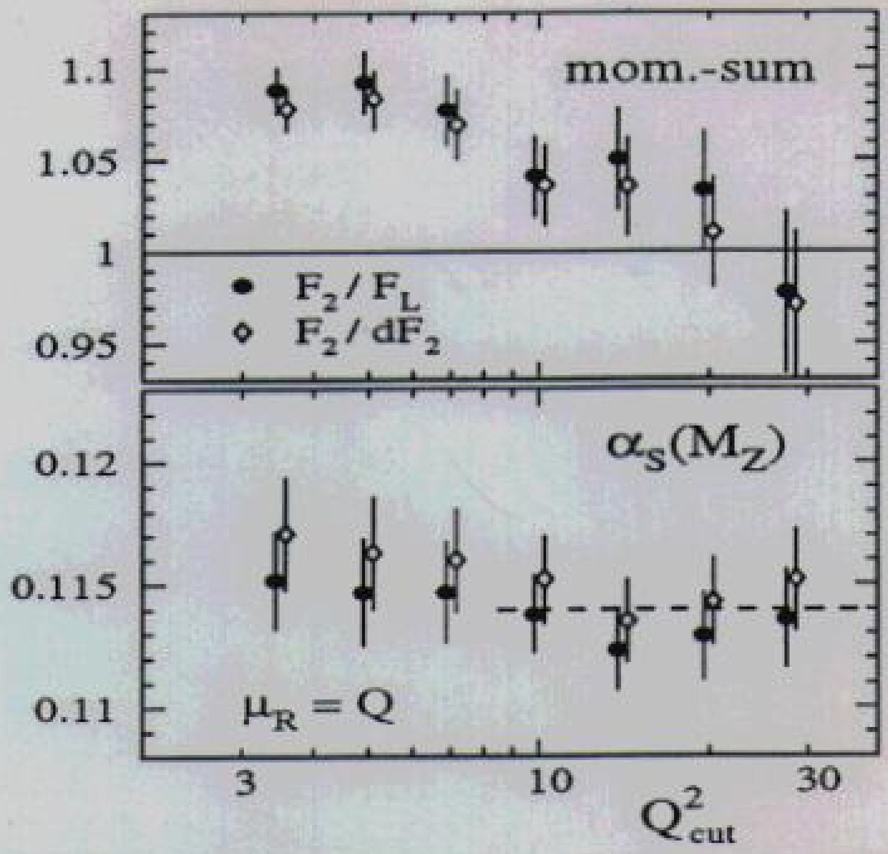


Figure 1. The dependence of the fit results for the energy-momentum sum and for $\alpha_s(M_Z)$ on the Q^2-cut imposed in addition to $W^2 > 10$ GeV^2.

JB, VOGT